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SOS for Systems with Multiple Delays: Part 1. H_∞-Optimal Control

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***Abstract*— We propose an LMI-based solution to the problem of H_∞-optimal state-feedback control of systems with multiple state delays. This result is based on a generalization of the LMI framework to infinite-dimensional systems using the recently developed PQRS framework. The H_∞ norm bounds are certified using Lyapunov-Krasovskii functionals and do not rely on discretization. The algorithms are scalable to large numbers of states and delays and accurate to at least 4 decimal places when compared with Padé-based methods. We include efficient implementations of the proposed controllers for real-time control and provide a user-friendly interface available online via Code Ocean.**

I. Introduction

In this paper we consider systems of the form

$$\dot{x}(t) = A_0 x(t) + \sum_i A_i x(t-\tau_i) + B_1 w(t) + B_2 u(t)$$
$$y(t) = C_0 x(t) + \sum_i C_i x(t-\tau_i) + D_1 w(t) + D_2 u(t) \quad (1)$$

where $w(t) \in \mathbb{R}^m$ is the disturbance input, $u(t) \in \mathbb{R}^p$ is the controlled input, $y(t) \in \mathbb{R}^q$ is the regulated output, $x(t)$ are the state variables and $\tau_i > 0$ for $i \in [1, \cdots, K]$ are the delays ordered by increasing magnitude. We assume $x(s) = 0$ for $s \in [-\tau_K, 0]$. Our goal is to construct a controller of the form

$$u(t) = K_0 x(t) + \sum_{i=1}^K K_{1i} x(t-\tau_i) + \sum_{i=1}^K \int_{-\tau_i}^0 K_{2i}(s) x(t+s) ds$$

where the K_{2i} are polynomial and which minimizes $\gamma := \sup_{w \in L_2} \frac{\|y\|_{L_2}}{\|w\|_{L_2}}$.

Our controller synthesis conditions: 1) Are expressed as LMIs; 2) Are not conservative in any significant sense; 3) Are prima facie provable in that they are certified using Lyapunov-Krasovskii functionals; 4) Are scalable to large numbers of states and delays; 5) Have an efficient real-time implementation; and 6) Are publicly available for verification via Code Ocean.

However, our controllers require knowledge of the history of the state variables and must be coupled with state-estimators if the system has partial sensor measurements or input delay. The estimator synthesis problem is solved in Part 2 of this paper [1]. In addition, these controllers are not currently suitable when the delays are unknown or time-varying.

The result is based on an generalization of a well-known LMI for optimal control of Ordinary Differential Equations. Specifically, when $A_i = 0$ and $C_i = 0$ for $i > 0$, in [2], it was shown that if there exist $P > 0$ and Z such that

$$\begin{bmatrix} PA^T + AP + Z^T B_2^T + B_2 Z & B_1 & PC_1^T + Z^T D_2^T \\ B_1^T & -\gamma I & D_1^T \\ C_1 P + D_2 Z & D_1 & -\gamma I \end{bmatrix} < 0$$

then for $u(t) = ZP^{-1}x(t)$, $\|y\|_{L_2} \le \gamma \|w\|_{L_2}$. In Section II, Theorem 1, we show that an operator-valued version of this LMI is also valid for distributed parameter systems. In Theorem 1, however, the system matrices are replaced by operators and the variables are likewise operators. In Section III, we introduce the PQRS framework which allows us to use polynomials to parameterize our operator variables $\mathcal{P}$ and $\mathcal{Z}$ as in Equations (4) and (5), respectively. PQRS operators are defined for multiple delays in Eqn (4) and for a single delay have the form

$$\left(\mathcal{P}_{\{P,Q,S,R\}} \begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_{-\tau}^0 Q(s)\phi(s) ds \\ \tau Q(s)^T x + \tau S(s)\phi_i(s) + \int_{-\tau}^0 R(s,\theta)\phi(\theta)\, d\theta. \end{bmatrix}$$

If the matrix-valued functions $\{P, Q, R, S\}$ are polynomial, then positivity of such a PQRS operator can be enforced using LMIs as described in Section IV, Theorem 6. In Theorem 4, we show that the optimal controller synthesis conditions introduced in Theorem 1 can be represented in the PQRS framework and hence enforced using LMIs. Finally, in Theorem 10, we summarize the results as an LMI for optimal control of multi-delay systems. In Section VIII, we describe a Matlab toolbox which dramatically simplifies the implementation of these results. This implementation, along with an efficient and easy-to-use simulation and validation code for user-defined multi-delay systems are available on Code Ocean [3]. The results are applied to several numerical examples and compared with controllers designed using a high-order Padé approximation. The results are shown to be accurate to at least 4 significant figures as measured by the minimal achievable closed-loop H_∞ norm. Furthermore, the algorithms are scalable in that they can be implemented on a desktop computer when the number of delays times the number of states is less than 50.

Finally, we note that there are existing results on controller synthesis for time-delay systems. First, there are approaches such as the use of Padé approximations which reduce the delayed system to an ODE [4]. These approaches show convergence to the true system as the order of approximation increases [5]. However, for any given level of discretization, the H_∞ bounds are not provable. Although analysis may be used a-posteriori to obtain provable bounds (at some

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cost in accuracy), in practice this is rarely, if ever, done. By contrast, every controller we design has a provable, highly accurate H_∞ norm bound. This critique also applies to operator discretization techniques such as in [6]. Alternatively, Lyapunov-Krasovskii-based synthesis conditions, such as in [7], [8], [9] have provable bounds but are either heuristic, significantly conservative, or both - as is shown in Section VIII.

A. Notation

Shorthand notation used throughout this paper includes the Hilbert spaces $L_2^m[X] := L_2(X;\mathbb{R}^m)$ of square integrable functions from X to $\mathbb{R}^m$ and $W_2^m[X] := W^{1,2}(X;\mathbb{R}^m) = H^1(X;\mathbb{R}^m) = \{x \, : \, x, \dot{x} \in L_2^m[X]\}$. We use L_2^m and W_2^m when domains are clear from context. We also use the extensions $L_2^{n\times m}[X] := L_2(X;\mathbb{R}^{n\times m})$ and $W_2^{n\times m}[X] := W^{1,2}(X;\mathbb{R}^{n\times m})$ for matrix-valued functions. $S^n \subset \mathbb{R}^{n\times n}$ denotes the symmetric matrices. We say an operator $\mathcal{P} : Z \to Z$ is positive on a subset X of Hilbert space Z if $\langle x, \mathcal{P}x\rangle_Z \geq 0$ for all $x \in X$. $\mathcal{P}$ is coercive on X if $\langle x, \mathcal{P}x\rangle_Z \geq \epsilon \|x\|_Z^2$ for some $\epsilon > 0$ and for all $x \in X$. Given an operator $\mathcal{P} : Z \to Z$ and a set $X \subset Z$, we use the shorthand $\mathcal{P}(X)$ to denote the image of $\mathcal{P}$ on subset X. $I_n \in \mathbb{S}^n$ denotes the identity matrix. $0_{n\times m} \in \mathbb{R}^{n\times m}$ is the matrix of zeros with shorthand $0_n := 0_{n\times n}$. We will occasionally denote the intervals $T_i := [-\tau_i, 0]$. For a natural number, $K \in \mathbb{N}$, we adopt the index shorthand notation where $i \in [K]$ denotes $i = 1, \cdots, K$. The symmetric completion of a matrix is denoted $*^T$.

II. An Convex Formulation of the Controller Synthesis Problem for Distributed Parameter Systems

Consider the generic distributed-parameter system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t), \quad \mathbf{x}(0) = 0, \\ y(t) &= \mathcal{C}\mathbf{x}(t) + \mathcal{D}_1 w(t) + \mathcal{D}_2 u(t), \end{aligned} \tag{2}$$

where $\mathcal{A} : X \to Z$, $\mathcal{B}_1 : \mathbb{R}^m \to Z$, $\mathcal{B}_2 : U \to Z$, $\mathcal{C} : X \to \mathbb{R}^q$, $\mathcal{D}_1 : \mathbb{R}^m \to \mathbb{R}^q$, and $\mathcal{D}_2 : U \to \mathbb{R}^q$.

We begin with the following mathematical result on duality, which is a reduced version of Theorem 3 in [10].

Theorem 1: Suppose $\mathcal{P}$ is a bounded, coercive linear operator $\mathcal{P} : X \to X$ with $\mathcal{P}(X) = X$ and which is self-adjoint with respect to the Z inner product. Then $\mathcal{P}^{-1}$: exists; is bounded; is self-adjoint; $\mathcal{P}^{-1} : X \to X$; and $\mathcal{P}^{-1}$ is coercive.

Using Theorem 1, we give a convex formulation of the H_∞ optimal full-state feedback controller synthesis problem. This result combines: a) a relatively simple extension of the Schur complement Lemma to infinite dimensions; with b) the dual synthesis condition in [10]. We note that the ODE equivalent of this theorem is necessary and sufficient and the proof structure can be credited with, e.g. [2].

Theorem 2: Suppose there exists an $\epsilon > 0$, an operator $\mathcal{P} : Z \to Z$ which satisfies the conditions of Theorem 1, and an operator $\mathcal{Z} : X \to U$ such that

$$\begin{aligned} &\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z \\ &+ \langle \mathbf{z}, \mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z}\rangle_Z \leq \gamma w^T w - v^T(\mathcal{C}\mathcal{P}\mathbf{z}) - (\mathcal{C}\mathcal{P}\mathbf{z})^T v \\ &- v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) - (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v - v^T(\mathcal{D}_1 w) - (\mathcal{D}_1 w)^T v \\ &+ \gamma \|v\|^2 - \epsilon \|z\|_Z^2 \end{aligned}$$

for all $\mathbf{z} \in X$, $w \in \mathbb{R}^m$, and $v \in \mathbb{R}^q$. Then for any $w \in L_2$, if $\mathbf{x}(t)$ and $y(t)$ satisfy $\mathbf{x}(t) \in X$ and

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (\mathcal{A} + \mathcal{B}_2\mathcal{Z}\mathcal{P}^{-1})\mathbf{x}(t) + \mathcal{B}_1 w(t) \\ y(t) &= (\mathcal{C} + \mathcal{D}_2\mathcal{Z}\mathcal{P}^{-1})\mathbf{x}(t) + \mathcal{D}_1 w(t) \end{aligned} \tag{3}$$

for all $t \geq 0$, then $\|y\|_{L_2} \leq \gamma \|w\|_{L_2}$.

Proof: By Theorem 1 $\mathcal{P}^{-1}$: exists; is bounded; is self-adjoint; $\mathcal{P}^{-1} : X \to X$; and is coercive.

For $w \in L_2$, let $\mathbf{x}(t)$ and $y(t)$ be a solution of

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (\mathcal{A} + \mathcal{B}_2\mathcal{Z}\mathcal{P}^{-1})\mathbf{x}(t) + \mathcal{B}_1 w(t) \\ y(t) &= (\mathcal{C} + \mathcal{D}_2\mathcal{Z}\mathcal{P}^{-1})\mathbf{x}(t) + \mathcal{D}_1 w(t) \end{aligned}$$

such that $\mathbf{x}(t) \in X$ for any finite t.

Define the storage function $V(t) = \left\langle \mathbf{x}(t), \mathcal{P}^{-1}\mathbf{x}(t)\right\rangle_Z$. Then $V(t) \geq \delta \|\mathbf{x}(t)\|_Z^2$ for some $\delta > 0$. Define $\mathbf{z}(t) = \mathcal{P}^{-1}\mathbf{x}(t) \in X$. Differentiating the storage function in time, we obtain

$$\begin{aligned} \dot{V}(t) &= \left\langle \mathbf{x}(t), \mathcal{P}^{-1}(\mathcal{A}\mathbf{x}(t) + \mathcal{B}_2\mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t) + \mathcal{B}_1 w(t))\right\rangle_Z \\ &\quad + \left\langle \mathcal{P}^{-1}(\mathcal{A}\mathbf{x}(t) + \mathcal{B}_2\mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t) + \mathcal{B}_1 w(t)), \mathbf{x}(t)\right\rangle_Z \\ &= \left\langle \mathcal{P}^{-1}\mathbf{x}(t), \mathcal{A}\mathbf{x}(t)\right\rangle_Z + \left\langle \mathcal{P}^{-1}\mathbf{x}(t), \mathcal{B}_2\mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t)\right\rangle_Z \\ &\quad + \left\langle \mathcal{P}^{-1}\mathbf{x}(t), \mathcal{B}_1 w(t)\right\rangle_Z + \left\langle \mathcal{A}\mathbf{x}(t), \mathcal{P}^{-1}\mathbf{x}(t)\right\rangle_Z \\ &\quad + \left\langle \mathcal{B}_2\mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t), \mathcal{P}^{-1}\mathbf{x}(t)\right\rangle_Z + \left\langle \mathcal{B}_1 w(t), \mathcal{P}^{-1}\mathbf{x}(t)\right\rangle_Z \\ &= \langle \mathbf{z}(t), \mathcal{A}\mathcal{P}\mathbf{z}(t)\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}(t), \mathbf{z}(t)\rangle_Z + \langle \mathbf{z}(t), \mathcal{B}_1 w(t)\rangle_Z \\ &\quad + \langle \mathcal{A}\mathcal{P}\mathbf{z}(t), \mathbf{z}(t)\rangle_Z + \langle \mathbf{z}(t), \mathcal{B}_2\mathcal{Z}\mathbf{z}(t)\rangle_Z + \langle \mathcal{B}_1 w(t), \mathbf{z}(t)\rangle_Z \\ &\leq \gamma w(t)^T w(t) - v(t)^T(\mathcal{C}\mathcal{P}\mathbf{z}(t)) - (\mathcal{C}\mathcal{P}\mathbf{z}(t))^T v(t) \\ &\quad - v(t)^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}(t)) - (\mathcal{D}_2\mathcal{Z}\mathbf{z}(t))^T v(t) - v(t)^T(\mathcal{D}_1 w(t)) \\ &\quad - (\mathcal{D}_1 w(t))^T v(t) + \gamma \|v(t)\|^2 - \epsilon \|\mathbf{z}(t)\|_Z^2 \\ &= \gamma w(t)^T w(t) - v(t)^T \left((\mathcal{C} + \mathcal{D}_2\mathcal{Z}\mathcal{P}^{-1})\mathbf{x}(t) + \mathcal{D}_1 w(t)\right) \\ &\quad - \left((\mathcal{C} + \mathcal{D}_2\mathcal{Z}\mathcal{P}^{-1})\mathbf{x}(t) + \mathcal{D}_1 w(t)\right)^T v(t) + \gamma \|v(t)\|^2 \\ &\quad - \epsilon \|\mathbf{z}(t)\|_Z^2 \\ &= \gamma w(t)^T w(t) - v(t)^T y(t) - y(t)^T v(t) + \gamma \|v(t)\|^2 \\ &\quad - \epsilon \|\mathbf{z}(t)\|_Z^2 \end{aligned}$$

for any $v(t) \in \mathbb{R}^q$ and all $t \geq 0$. Choose $v(t) = \frac{1}{\gamma} y(t)$ and we get

$$\begin{aligned} \dot{V}(t) &\leq \gamma \|w(t)\|^2 - \frac{2}{\gamma} \|y(t)\|^2 + \frac{1}{\gamma} \|y(t)\|^2 - \epsilon \|\mathbf{z}(t)\|_Z^2 \\ &= \gamma \|w(t)\|^2 - \frac{1}{\gamma} \|y(t)\|^2 - \epsilon \|\mathbf{z}(t)\|_Z^2 . \end{aligned}$$

Since $\mathcal{P}$ is bounded, there exists a $\sigma > 0$ such that

$$V(t) = \left\langle \mathbf{x}(t), \mathcal{P}^{-1}\mathbf{x}(t)\right\rangle_Z = \langle \mathbf{z}(t), \mathcal{P}\mathbf{z}(t)\rangle_Z \leq \sigma \|\mathbf{z}(t)\|_Z^2 .$$

We conclude, therefore, that

$$\dot{V}(t) \leq -\frac{\epsilon}{\sigma} V(t) + \gamma \|w(t)\|^2 - \frac{1}{\gamma} \|y(t)\|^2 .$$

Integrating the inequality forward in time, and using $V(0) = 0$, we obtain

$$\frac{1}{\gamma} \|y\|_{L_2}^2 \leq \gamma \|w\|_{L_2}^2$$

which concludes the proof.

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III. Theorem 2 Applied to Multi-Delay Systems

Theorem 2 gives a convex formulation of the controller synthesis problem for a general class of distributed-parameter systems. In this section and the next, we apply Theorem 2 to the case of systems with multiple delays. Specifically, we consider solutions to the system of equations given by Equation (1). First, we express System (1) in the abstract form of (2). Following the mathematical formalism developed in [10], we define the inner-product space $Z_{m,n,K} := \{\mathbb{R}^m \times L_2^n[T_1] \times \cdots \times L_2^n[T_K]\}$ and for $\{x, \phi_1, \cdots, \phi_K\} \in Z_{m,n,K}$, we define the following shorthand notation

$$\begin{bmatrix} x \\ \phi_i \end{bmatrix} := \{x, \phi_1, \cdots, \phi_K\},$$

which allows us to simplify expression of the inner product on $Z_{m,n,K}$, which we define to be

$$\left\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} = \tau_K y^T x + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \psi_i(s)^T \phi_i(s) ds.$$

When $m = n$, we simplify the notation using $Z_{n,K} := Z_{n,n,K}$. The state-space for System (1) is defined as

$$X := \left\{ \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{n,K} \;:\; \begin{smallmatrix} \phi_i \in W_2^n[T_i] \text{ and} \\ \phi_i(0)=x \text{ for all } i\in[K] \end{smallmatrix} \right\}.$$

We now represent the infinitesimal generator, $\mathcal{A} : X \to Z_{n,K}$, of Eqn. (1) as

$$\mathcal{A} \begin{bmatrix} x \\ \phi_i \end{bmatrix} (s) := \begin{bmatrix} A_0 x + \sum_{i=1}^{K} A_i \phi_i(-\tau_i) \\ \dot{\phi}_i(s) \end{bmatrix}.$$

Furthermore, $\mathcal{B}_1 : \mathbb{R}^m \to Z_{n,K}$, $\mathcal{B}_2 : \mathbb{R}^p \to Z_{n,K}$, $\mathcal{D}_1 : \mathbb{R}^m \to \mathbb{R}^q$, $\mathcal{D}_2 : \mathbb{R}^p \to \mathbb{R}^q$, and $\mathcal{C} : Z_{n,K} \to \mathbb{R}^p$ are defined as

$$(\mathcal{B}_1 w)(s) := \begin{bmatrix} B_1 w \\ 0 \end{bmatrix}, \quad (\mathcal{B}_2 u)(s) := \begin{bmatrix} B_2 u \\ 0 \end{bmatrix},$$

$$\left(\mathcal{C} \begin{bmatrix} \psi \\ \phi_i \end{bmatrix} \right) := \left[C_0 \psi + \textstyle\sum_i C_i \phi_i(-\tau_i) \right],$$

$$\mathcal{D}_1 w := D_1 w, \quad \mathcal{D}_2 u := D_2 u.$$

Having defined these operators, we note that for any solution $x(t)$ of Eqn. (1), using the above notation if we define

$$(\mathbf{x}(t))(s) = \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_{2i}(t) \end{bmatrix} (s) = \begin{bmatrix} x(t) \\ x(t+s) \end{bmatrix}$$

then $\mathbf{x}$ satisfies Eqn. (2) using the operator definitions given above. The converse statement is also true.

A. The PQRS Parametrization of Operators

We now introduce a class of operators $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} : Z_{m,n,K} \to Z_{m,n,K}$, parameterized by matrix P and matrix-valued functions $Q_i \in W_2^{m\times n}[T_i]$, $S_i \in W_2^{n\times n}[T_i]$, $R_{ij} \in W_2^{n\times n}[T_i \times T_j]$ as

$$\left(\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right)(s) := \tag{4}$$
$$\begin{bmatrix} Px + \sum_{i=1}^{K} \int_{-\tau_i}^{0} Q_i(s)\phi_i(s)ds \\ \tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) + \sum_{j=1}^{K} \int_{-\tau_j}^{0} R_{ij}(s,\theta)\phi_j(\theta)\, d\theta. \end{bmatrix}$$

For this class of operators, the following Lemma combines Lemmas 3 and 4 in [10] and gives conditions under which $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ satisfies the conditions of Theorem 1.

Lemma 3: Suppose that $S_i \in W_2^{n\times n}[T_i]$, $R_{ij} \in W_2^{n\times n}[T_i \times T_j]$ and $S_i(s) = S_i(s)^T$, $R_{ij}(s,\theta) = R_{ji}(\theta,s)^T$, $P = \tau_K Q_i(0)^T + \tau_K S_i(0)$ and $Q_j(s) = R_{ij}(0,s)$ for all $i,j \in [K]$. Further suppose $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is coercive on $Z_{n,K}$. Then $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$: is a self-adjoint bounded linear operator with respect to the inner product defined on $Z_{n,K}$; maps $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} : X \to X$; and $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}(X) = X$.

Starting in Section IV, we will assume Q_i, S_i, and R_{ij} are polynomial and give LMI conditions for positivity of operators of the form $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$.

B. The Controller Synthesis Problem for Systems with Delay

Theorem 2 gives a convex formulation of the controller synthesis problem, where the data is the 6 operators $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{C}$, $\mathcal{D}_1$, and $\mathcal{D}_2$ and the variables are the operators $\mathcal{P}$ and $\mathcal{Z}$. For multi-delay systems, we have defined the 6 operators and parameterized the decision variable $\mathcal{P}$ using $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$. We now likewise parameterize the decision variable $\mathcal{Z} : Z_{n,k} \to \mathbb{R}^p$ using matrices Z_0, Z_{1i} and functions Z_{2i} as

$$\left(\mathcal{Z} \begin{bmatrix} \psi \\ \phi_i \end{bmatrix} \right) := \left[Z_0 \psi + \textstyle\sum_i Z_{1i} \phi_i(-\tau_i) + \sum_i \int_{-\tau_i}^{0} Z_{2i}(s)\phi_i(s) ds \right]. \tag{5}$$

The following theorem gives convex constraints on the variables P, Q_i, S_i, R_{ij}, Z_0, Z_{1i} and Z_{2i} under which Theorem 2 is satisfied when $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{C}$, $\mathcal{D}_1$, and $\mathcal{D}_2$ are as defined above.

Theorem 4: Suppose that there exist $S_i \in W_2^{n\times n}[T_i]$, $R_{ij} \in W_2^{n\times n}[T_i \times T_j]$ and $S_i(s) = S_i(s)^T$ such that $R_{ij}(s,\theta) = R_{ji}(\theta,s)^T$, $P = \tau_K Q_i(0)^T + \tau_K S_i(0)$ and $Q_j(s) = R_{ij}(0,s)$ for all $i,j \in [K]$, and matrices $Z_0 \in \mathbb{R}^{p\times n}$, $Z_{1i} \in \mathbb{R}^{p\times n}$ and $Z_{2i} \in W_2^{p\times n}[T_i]$ such that $\langle \mathbf{x}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \mathbf{x} \rangle_{Z_{n,K}} \geq \epsilon \|\mathbf{x}\|^2$ for all $\mathbf{x} \in Z_{n,K}$ and

$$\left\langle \begin{bmatrix} \begin{bmatrix} v \\ w \\ y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D,E_i,\dot{S}_i,G_{ij}\}} \begin{bmatrix} \begin{bmatrix} v \\ w \\ y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle_{Z_{q+m+n(K+1),n,K}} \leq -\epsilon \left\| \begin{bmatrix} y_1 \\ \phi_i \end{bmatrix} \right\|_{Z_{n,K}}^2$$

for all $y_1 \in \mathbb{R}^n$ and $\begin{bmatrix} \begin{bmatrix} v \\ w \\ y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \in Z_{q+m+n(K+1),n,K}$ where

$$D = \begin{bmatrix} -\frac{\gamma}{\tau_K} I & \frac{1}{\tau_K} D_1 & L_1 & L_{21} & \dots & L_{2K} \\ *^T & -\frac{\gamma}{\tau_K} I & B_1^T & 0 & \dots & 0 \\ *^T & *^T & L_0 + L_0^T & L_{31} & \dots & L_{3K} \\ *^T & *^T & *^T & -S_1(-\tau_1) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ *^T & *^T & *^T & *^T & \dots & -S_k(-\tau_K) \end{bmatrix}$$

$$L_0 := A_0 P + \sum_{i=1}^K \left(\tau_K A_i Q_i(-\tau_i)^T + \frac{1}{2} S_i(0) \right) + B_2 Z_0,$$

$$L_1 := \frac{1}{\tau_K} C_0 P + \sum_i C_i Q_i(-\tau_i)^T + \frac{1}{\tau_K} D_2 Z_0$$

$$L_{2i} := C_i S_i(-\tau_i) + \frac{1}{\tau_K} D_2 Z_{1i}, \quad L_{3i} := \tau_K A_i S_i(-\tau_i) + B_2 Z_{1i}$$

$$E_i(s) = \frac{1}{\tau_K} \cdot \begin{bmatrix} C_0 Q_i(s) + \sum_j C_j R_{ji}(-\tau_j, s) + D_2 Z_{2i}(s) \\ 0 \\ \tau_K \left(A_0 Q_i(s) + \dot{Q}_i(s) + \sum_{j=1}^K A_j R_{ji}(-\tau_j, s) + B_2 Z_{2i}(s) \right) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$G_{ij}(s,\theta) := \frac{\partial}{\partial s} R_{ij}(s,\theta) + \frac{\partial}{\partial \theta} R_{ji}(s,\theta)^T, \quad i,j \in [K].$$

Then if

$$u(t) = \mathcal{Z} \mathcal{P}^{-1}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x(t) \\ x(t+s) \end{bmatrix}$$

where $\mathcal{Z}$ is defined in Eqn. (5) then for any $w \in L_2$, if $x(t)$ and $y(t)$ satisfy Eqn. (1), $\|y\|_{L_2} \le \gamma \|w\|_{L_2}$.

Proof: For any $w \in L_2$, using the definitions of $u(t)$, and $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{C}$, $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{Z}$ given above, $y(t)$ and $x(t)$ satisfy Eqn. (1) if and only if $y(t)$ and $\mathbf{x}(t) := \begin{bmatrix} x(t) \\ x(t+s) \end{bmatrix}$ satisfy Eqn. (2). Therefore, $\|y\|_{L_2} \le \gamma \|w\|_{L_2}$ if

$$\begin{aligned} &\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z} \rangle_Z + \langle \mathcal{B}_2 \mathcal{Z}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_2 \mathcal{Z}\mathbf{z} \rangle_Z \\ &\quad + \langle \mathbf{z}, \mathcal{B}_1 w \rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z} \rangle_Z \\ &\le \gamma w^T w - v^T (\mathcal{C}\mathcal{P}\mathbf{z}) - (\mathcal{C}\mathcal{P}\mathbf{z}^T v - v^T (\mathcal{D}_2 \mathcal{Z}\mathbf{z}) - (\mathcal{D}_2 \mathcal{Z}\mathbf{z})^T v \\ &\quad - v^T (D_1 w) - (D_1 w)^T v + \gamma \|v\|^2 - \epsilon \|z\|_Z^2 \end{aligned}$$

for all $\mathbf{z} \in X$, $w \in \mathbb{R}^m$, and $v \in \mathbb{R}^q$. The rest of the proof is lengthy but straightforward. We simply show that if we define

$$f := \begin{bmatrix} \mathbf{z}_{2,1}(-\tau_1)^T & \cdots & \mathbf{z}_{2,K}(-\tau_K)^T \end{bmatrix}^T,$$
$$h := \begin{bmatrix} v^T & w^T & \mathbf{z}_1^T & f^T \end{bmatrix}^T.$$

then

$$\begin{aligned} &\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z} \rangle_Z + \langle \mathcal{B}_2 \mathcal{Z}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_2 \mathcal{Z}\mathbf{z} \rangle_Z \\ &+ \langle \mathbf{z}, \mathcal{B}_1 w \rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z} \rangle_Z - \gamma w^T w + v^T (\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v \\ &+ v^T (\mathcal{D}_2 \mathcal{Z}\mathbf{z}) + (\mathcal{D}_2 \mathcal{Z}\mathbf{z})^T v + v^T (D_1 w) + (D_1 w)^T v - \gamma v^T v \\ &= \left\langle \begin{bmatrix} h \\ \mathbf{z}_{2i} \end{bmatrix}, \mathcal{P}_{\{D,E_i,\dot{S}_i,G_{ij}\}} \begin{bmatrix} h \\ \mathbf{z}_{2i} \end{bmatrix} \right\rangle_{Z_{m_0,n,K}} \le -\epsilon \left\| \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_{2i} \end{bmatrix} \right\|^2_{Z_{n,K}} \\ &= -\epsilon \|\mathbf{z}\|^2_{Z_{n,K}}. \end{aligned} \tag{6}$$

where for convenience and efficiency of presentation, we denote $m_0 := q + m + n(K+1)$.

It may also be helpful to note that the quadratic form defined by a $\mathcal{P}_{\{D,E_i,F_i,G_{ij}\}}$ operator expands out as

$$\begin{aligned} &\left\langle \begin{bmatrix} h \\ \mathbf{z}_{2i} \end{bmatrix}, \mathcal{P}_{\{D,E_i,F_i,G_{ij}\}} \begin{bmatrix} h \\ \mathbf{z}_{2i} \end{bmatrix} \right\rangle_{Z_{m_0,n,K}} \\ &= \tau_K h^T D h + \tau_K \sum_{i=1}^K \int_{-\tau_i}^0 h^T E_i(s) \mathbf{z}_{2i}(s) ds \\ &\quad + \tau_K \sum_i \int_{-\tau_i}^0 \mathbf{z}_{2i}(s)^T E_i(s)^T h ds \\ &\quad + \tau_K \sum_i \int_{-\tau_i}^0 \mathbf{z}_{2i}(s)^T F_i(s) \mathbf{z}_{2i}(s) ds \\ &\quad + \sum_{ij} \int_{-\tau_i}^0 \int_{-\tau_j}^0 \mathbf{z}_{2i}(s)^T G_{ij}(s,\theta) \mathbf{z}_{2j}(\theta) \, d\theta ds. \end{aligned} \tag{7}$$

Our task, therefore, is simply to write all the terms we find in (6) in the form of Equation (7) for an appropriate choice of matrix D and functions E_i, F_i, and G_{ij}. Fortunately, the most complicated part of this operation has already been completed. Indeed, from Theorem 5 in [10], we have the first two terms can be represented as

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z} \rangle_{Z_{n,K}} + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z} \rangle_{Z_{n,K}} = \left\langle \begin{bmatrix} h \\ \mathbf{z}_{2i} \end{bmatrix}, \mathcal{D} \begin{bmatrix} h \\ \mathbf{z}_{2i} \end{bmatrix} \right\rangle_{Z_{m_0,n,K}},$$

where $\mathcal{D} := \mathcal{P}_{\{D_1,E_{1i},\dot{S}_i,G_{ij}\}}$ (Do not confuse this D_1 with the D_1 in Eqn. (2)) and

$$D_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & C_0 + C_0^T & C_1 & \dots & C_k \\ 0 & 0 & C_1^T & -S_1(-\tau_1) & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & C_{k^T} & 0 & 0 & -S_k(-\tau_K) \end{bmatrix},$$

$$C_0 := A_0 P + \tau_K \sum_{i=1}^K (A_i Q_i(-\tau_i)^T + \frac{1}{2} S_i(0)),$$

$$C_i := \tau_K A_i S_i(-\tau_i), \quad i \in [K]$$

$$E_{1i}(s) := \begin{bmatrix} 0 & 0 & B_i(s)^T & 0 & \cdots & 0 \end{bmatrix}^T, \quad i \in [K]$$

$$B_i(s) := A_0 Q_i(s) + \dot{Q}_i(s) + \sum_{j=1}^K A_j R_{ji}(-\tau_j, s), \quad i \in [K]$$

$$G_{ij}(s,\theta) := \frac{\partial}{\partial s} R_{ij}(s,\theta) + \frac{\partial}{\partial \theta} R_{ji}(s,\theta)^T, \quad i,j \in [K].$$

Having already dealt with the most difficult terms, we now start with the easiest. Using the definitions of $\mathcal{B}_1$ and $\mathcal{D}_1$, it is relatively easy to show that $\langle \mathbf{z}, \mathcal{B}_1 w\rangle_Z = \tau_K z_1^T B_1 w$ and hence

$$\langle \mathbf{z}, \mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z}\rangle_Z - \gamma w^T w + v^T D_1 w + (D_1 w)^T v - \gamma v^T v = \left\langle \begin{bmatrix} h \\ \mathbf{z}_{2i}\end{bmatrix}, \mathcal{P}_{\{D_0,0,0,0\}} \begin{bmatrix} h \\ \mathbf{z}_{2i}\end{bmatrix}\right\rangle_{Z_{m_0,n,K}}$$

where

$$D_0 = \frac{1}{\tau_K}\begin{bmatrix} -\gamma I & D_1 & 0 & 0 & \dots & 0 \\ D_1^T & -\gamma I & \tau_K B_1^T & 0 & \dots & 0 \\ 0 & \tau_K B_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Next we have that

$$\begin{aligned} &2v^T(\mathcal{C}\mathcal{P}\mathbf{z}) \\ &= 2v^T \tau_K \Bigg[\left(\frac{1}{\tau_K} C_0 P + \sum_i C_i Q_i(-\tau_i)^T\right)\mathbf{z}_1 + \sum_i C_i S_i(-\tau_i)\mathbf{z}_{2i}(-\tau_i) \\ &\quad + \frac{1}{\tau_K}\sum_{i=1}^K \int_{-\tau_i}^0 \left(C_0 Q_i(s) + \sum_j C_j R_{ji}(-\tau_j, s)\right)\mathbf{z}_{2i}(s)ds\Bigg] \\ &= \tau_K h^T \underbrace{\frac{1}{\tau_K}\begin{bmatrix} 0_{q+m} & \Pi_2 \\ \Pi_2^T & 0_{n(K+1)}\end{bmatrix}}_{D_2} h \\ &+ 2\tau_K \sum_{i=1}^K \int_{-\tau_i}^0 h^T \underbrace{\frac{1}{\tau_K}\begin{bmatrix} C_0 Q_i(s) + \sum_j C_j R_{ji}(-\tau_j, s) \\ 0_{n\times(q+N(K+1))}\end{bmatrix}}_{E_{2i}(s)} \mathbf{z}_{2i}(s)ds. \end{aligned}$$

where

$$\Pi_1 = C_0 P + \sum_i \tau_K C_i Q_i(-\tau_i)^T$$

$$\Pi_2 = \begin{bmatrix}\Pi_1 & \tau_K C_1 S_1(-\tau_1) & \cdots & \tau_K C_K S_K(-\tau_K)\end{bmatrix}$$

We therefore conclude that

$$v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v = \left\langle \begin{bmatrix} h \\ \mathbf{z}_{2i}\end{bmatrix}, \mathcal{P}_{\{D_2,E_{2i},0,0\}} \begin{bmatrix} h \\ \mathbf{z}_{2i}\end{bmatrix}\right\rangle_{Z_{m_0,n,K}}.$$

Finally, we have

$$\begin{aligned} &2\langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z + 2v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) \\ &= 2\tau_K \mathbf{z}_1^T\Big[B_2 Z_0 \mathbf{z}_1 + \sum_i B_2 Z_{1i}\mathbf{z}_{2i}(-\tau_i) \\ &\qquad + \sum_i \int_{-\tau_i}^0 B_2 Z_{2i}(s)\mathbf{z}_{2i}(s)ds\Big] \\ &+ 2v^T\Big[D_2 Z_0 \mathbf{z}_1 + \sum_i D_2 Z_{1i}\mathbf{z}_{2i}(-\tau_i) \\ &\qquad + \sum_i \int_{-\tau_i}^0 D_2 Z_{2i}(s)\mathbf{z}_{2i}(s)ds\Big] \\ &= \left\langle \begin{bmatrix} h \\ \mathbf{z}_{2i}\end{bmatrix}, \mathcal{P}_{\{D_3,E_{3i},0,0\}} \begin{bmatrix} h \\ \mathbf{z}_{2i}\end{bmatrix}\right\rangle_{Z_{m_0,n,K}} \end{aligned}$$

where

$$D_3 = \begin{bmatrix} 0 & *^T & *^T & *^T & \dots & *^T \\ 0 & 0 & *^T & *^T & \dots & *^T \\ (\frac{1}{\tau_K}D_2 Z_0)^T & 0 & B_2 Z_0 + Z_0^T B_2^T & *^T & \dots & *^T \\ (\frac{1}{\tau_K}D_2 Z_{11})^T & 0 & (B_2 Z_{11})^T & 0 & \dots & *^T \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (\frac{1}{\tau_K}D_2 Z_{1K})^T & 0 & (B_2 Z_{1K})^T & 0 & \dots & 0 \end{bmatrix}$$

$$E_{3i}(s) = \frac{1}{\tau_K}\begin{bmatrix}(D_2 Z_{2i}(s))^T & 0 & (\tau_K B_2 Z_{2i}(s))^T & 0 & \cdots & 0\end{bmatrix}^T.$$

Summing all the terms we have

$$D = D_0 + D_1 + D_2 + D_3$$

and

$$E_i(s) = E_{1i}(s) + E_{2i}(s) + E_{3i}(s).$$

We conclude, therefore, that for any $\mathbf{z} \in X$,

$$\begin{aligned} &\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z \\ &+ \langle \mathbf{z}, \mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z}\rangle_Z - \gamma w^T w + v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v \\ &+ v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v \\ &= \left\langle \begin{bmatrix} h \\ \mathbf{z}_{2i}\end{bmatrix}, \mathcal{P}_{\{D,E_i,\dot{S}_i,G_{ij}\}} \begin{bmatrix} h \\ \mathbf{z}_{2i}\end{bmatrix}\right\rangle_{Z_{m_0,n,K}} \\ &\le -\epsilon \left\| \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_{2i}\end{bmatrix}\right\|^2_{Z_{n,K}} = -\epsilon \|\mathbf{z}\|^2_{Z_{n,K}}. \end{aligned}$$

Thus, by Lemma 3 and Theorem 2, we have that for any $w \in L_2$, if $x(t)$ and $y(t)$ satisfy Eqn. (1), $\|y\|_{L_2} \le \gamma \|w\|_{L_2}$. ∎

Theorem 4 provides a convex formulation of the controller synthesis problem for systems with multiple delays. However, the theorem does not provide a way to enforce the operator inequalities or reconstruct the optimal controller. In Section IV we will review how the operator inequalities can be represented using LMIs. In Sections V and VI, we discuss how to invert operators of the $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ class and reconstruct the controller gains in a numerically reliable manner.

IV. Enforcing Operator Inequalities in the $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ Framework

The problem of enforcing operator positivity on $Z_{m,n,K}$ in the $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ framework was solved in [10] by using a two-step approach. First, we construct an operator $\mathcal{P}_{\{\tilde{P},\tilde{Q},\tilde{S},\tilde{R}\}}$ whose positivity on $Z_{m,nK,1}$ is equivalent to positivity of the original operator on $Z_{m,n,K}$. Then, assuming that $\tilde{Q},\tilde{R},\tilde{S}$ are polynomials, we give an LMI condition on $\tilde{P}$ and the coefficients of $\tilde{Q},\tilde{R},\tilde{S}$ which ensures positivity of $\mathcal{P}_{\{\tilde{P},\tilde{Q},\tilde{S},\tilde{R}\}}$ on $Z_{m,nK,1}$. Because the transformation from $\{P,Q_i,S_i,R_{ij}\}$ to $\{\tilde{P},\tilde{Q},\tilde{S},\tilde{R}\}$ is linear, if Q_i, R_{ij}, S_i are polynomials, the result is an LMI constraint on the coefficients of these original polynomials. For ease of implementation, these two results are combined in single Matlab function which is described in Section VIII.

First, we give the following transformation. Specifically, we say that

$$\{\tilde{P},\tilde{Q},\tilde{S},\tilde{R}\} := \mathcal{L}_1(P,Q_i,S_i,R_{ij}) \tag{8}$$

if $a_i = \frac{\tau_i}{\tau_K}$, $\tilde{P} = P$ and

$$\tilde{Q}(s) := \begin{bmatrix}\sqrt{a_1}Q_1(a_1 s) & \cdots & \sqrt{a_K}Q_K(a_K s)\end{bmatrix}$$

$$\tilde{S}(s) := \begin{bmatrix} S_1(a_1 s) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & S_K(a_K s)\end{bmatrix}$$

$$\tilde{R}(s,\theta) :=$$

$$\begin{bmatrix} \sqrt{a_1 a_1}R_{11}(sa_1,\theta a_1) & \cdots & \sqrt{a_1 a_K}R_{1K}(sa_1,\theta a_K) \\ \vdots & \cdots & \vdots \\ \sqrt{a_K a_1}R_{K1}(sa_K,\theta a_1) & \cdots & \sqrt{a_K a_K}R_{KK}(sa_K,\theta a_K)\end{bmatrix}.$$

Then we have the following result [10].

Lemma 5: Let $\{\tilde{P},\tilde{Q},\tilde{S},\tilde{R}\} := \mathcal{L}_1(P,Q_i,S_i,R_{ij})$. Then

$$\left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} \geq \alpha \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|_{Z_{m,n,K}}$$

for all $\begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{m,n,K}$ if and only if

$$\left\langle \begin{bmatrix} x \\ \tilde{\phi} \end{bmatrix}, \mathcal{P}_{\{\tilde{P},\tilde{Q},\tilde{R},\tilde{S}\}} \begin{bmatrix} x \\ \tilde{\phi} \end{bmatrix} \right\rangle_{Z_{m,nK,1}} \geq \alpha \left\| \begin{bmatrix} \tilde{x} \\ \tilde{\phi} \end{bmatrix} \right\|_{Z_{m,nK,1}}$$

for all $\begin{bmatrix} \tilde{x} \\ \tilde{\phi} \end{bmatrix} \in Z_{m,nK,1}$.

To enforce positivity of $\mathcal{P}_{\{\tilde{P},\tilde{Q},\tilde{S},\tilde{R}\}}$ on $Z_{m,nK,1}$ as an LMI, we use the following result [10].

Theorem 6: For any functions $Y_1 : [-\tau_K, 0] \to \mathbb{R}^{m_1\times n}$ and $Y_2 : T_K \times T_K \to \mathbb{R}^{m_2 \times n}$, square integrable on T_K with $g(s) \geq 0$ for $s \in T_K$, suppose that

$$P = M_{11}\cdot \frac{1}{\tau_K}\int_{-\tau_K}^0 g(s)ds$$

$$Q(s) = \frac{1}{\tau_K}\left(g(s)M_{12}Y_1(s) + \int_{-\tau_K}^0 g(\eta)M_{13}Y_2(\eta,s)d\eta\right)$$

$$S(s) = \frac{1}{\tau_K}g(s)Y_1(s)^T M_{22}Y_1(s)$$

$$R(s,\theta) = g(s)Y_1(s)^T M_{23}Y_2(s,\theta) + g(\theta)Y_2(\theta,s)^T M_{32}Y_1(\theta) + \int_{-\tau_K}^0 g(\eta)Y_2(\eta,s)^T M_{33}Y_2(\eta,\theta)d\eta$$

where $M_{11} \in \mathbb{R}^{m\times m}$, $M_{22} \in \mathbb{R}^{m_1\times m_1}$, $M_{33} \in \mathbb{R}^{m_2\times m_2}$ and

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33}\end{bmatrix} \geq 0.$$

Then $\langle \mathbf{x}, \mathcal{P}_{\{P,Q,S,R\}}\mathbf{x}\rangle_{Z_{m,n,1}} \geq 0$ for all $\mathbf{x} \in Z_{m,n,1}$.

For notational convenience, we use $\{P,Q,S,R\} \in \Xi_{d,m,n}$ to denote the LMI constraints associated with Theorem 6 as

$$\Xi_{d,m,n} :=$$

$$\left\{\{P,Q,R,S\} : \begin{array}{c}\{P,Q,S,R\}=\{P_1,Q_1,S_1,R_1\}+\{P_2,Q_2,S_2,R_2\}, \\ \text{where } \{P_1,Q_1,S_1,R_1\} \text{ and } \{P_2,Q_2,S_2,R_2\} \text{ satisfy} \\ \text{Thm. 6 with } g=1 \text{ and } g=-s(s+\tau_K)\text{, respectively.}\end{array}\right\}$$

We now have the single unified result:

Corollary 7: Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, matrix $P \in \mathbb{R}^{m\times m}$, polynomials Q_i, S_i, R_{ij} for $i,j \in [K]$ such that

$$\mathcal{L}_1(P,Q_i,S_i,R_{ij}) \in \Xi_{d,m,nK}.$$

Then $\langle \mathbf{x}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}\mathbf{x}\rangle_{Z_{m,n,K}} \geq 0$ for all $\mathbf{x} \in Z_{m,n,K}$.

A more detailed discussion of these LMI-based methods can be found in [10].

V. An Analytic Inverse of $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$

Having taken Q_i, R_{ij}, S_i to be polynomials and having given an LMI which enforces strict positivity of the operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$, we now give an analytical representation of the inverse of operators of this class. The inverse of $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is also of the form $\mathcal{P}_{\{\hat{P},\hat{Q}_i,\hat{S}_i,\hat{R}_{ij}\}}$ where expressions for the matrix $\hat{P}$ and functions $\hat{Q}_i, \hat{R}_{ij}, \hat{S}_i$ are given in the following theorem, which is a generalization of the result in [11] to the case of multiple delays. In this result, we first extract the coefficients of the polynomials Q_i and R_{ij} as $Q_i(s) = H_i Z(s)$ and $R_{ij}(s,\theta) = Z(s)^T\Gamma_{ij}Z(\theta)$ where $Z(s)$ is a vector of bases for vector-valued polynomials (typically a monomial basis). The theorem then gives an expression for the coefficients of $\hat{Q}_i$ and $\hat{R}_{ij}$ using a similar representation. Note that the results of the theorem are still valid even if the basis functions in $Z(s)$ are not monomials or even polynomials.

Theorem 8: Suppose that $Q_i(s) = H_i Z(s)$ and $R_{ij}(s,\theta) = Z(s)^T\Gamma_{ij}Z(\theta)$ and $\mathcal{P} := \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is a coercive operator where $\mathcal{P} : X \to X$ and $\mathcal{P} = \mathcal{P}^*$. Define

$$H = \begin{bmatrix} H_1 & \dots & H_K\end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix} \Gamma_{11} & \dots & \Gamma_{1K} \\ \vdots & & \vdots \\ \Gamma_{K,1} & \dots & \Gamma_{K,K}\end{bmatrix}.$$

Now let $K_i = \int_{-\tau_i}^0 Z(s)S_i(s)^{-1}Z(s)^T ds$,

$$K = \text{diag}(K_1,\cdots,K_K)$$

$$\hat{H} = P^{-1}H\left(KH^TP^{-1}H - I - K\Gamma\right)^{-1}$$

$$\hat{\Gamma} = -(\hat{H}^T H + \Gamma)(I + K\Gamma)^{-1}$$

$$\begin{bmatrix}\hat{H}_1 & \dots & \hat{H}_K\end{bmatrix} = \hat{H}, \qquad \begin{bmatrix} \hat{\Gamma}_{11} & \dots & \hat{\Gamma}_{1K} \\ \vdots & & \vdots \\ \hat{\Gamma}_{K,1} & \dots & \hat{\Gamma}_{K,K}\end{bmatrix} = \hat{\Gamma}.$$

Then if we define

$$\hat{P} = \left(I - \hat{H}KH^T\right)P^{-1}, \qquad \hat{Q}_i(s) = \hat{H}_i Z(s)S_i(s)^{-1}$$

$$\hat{S}_i(s) = S_i(s)^{-1}, \quad \hat{R}_{ij}(s,\theta) = \hat{S}_i(s)Z(s)^T\hat{\Gamma}_{ij}Z(\theta)\hat{S}_j(\theta),$$

then for $\hat{\mathcal{P}} := \mathcal{P}_{\left\{\hat{P},\frac{1}{\tau_K}\hat{Q}_i,\frac{1}{\tau_K^2}\hat{S}_i,\frac{1}{\tau_K}\hat{R}_{ij}\right\}}$, we have $\hat{\mathcal{P}} = \hat{\mathcal{P}}^*$, $\hat{\mathcal{P}} : X \to X$, and $\hat{\mathcal{P}}\mathcal{P}\mathbf{x} = \mathcal{P}\hat{\mathcal{P}}\mathbf{x} = \mathbf{x}$ for any $\mathbf{x} \in Z_{m,n,K}$.

Proof: See the extended version [12] of this paper for the proof. ∎

VI. Controller Reconstruction and Numerical Implementation

In this section, we reconstruct the controller using $\mathcal{Z}$ and $\mathcal{P}^{-1}$ and explain how this can be implemented numerically. First, we have the following obvious result.

Lemma 9: Suppose that $\mathcal{Z}$ is as defined in Theorem 4 and $\hat{\mathcal{P}}$ is as defined in Theorem 8. Then if $u(t) = \mathcal{Z}\hat{\mathcal{P}}\mathbf{x}(t)$,

$$u(t) = K_0 x(t) + \sum_i K_{1i} x(t-\tau_i) + \sum_i \int_{-\tau_i}^{0} K_{2i}(s) x(t+s) ds$$

where

$$K_0 = Z_0 \hat{P} + \sum_j \left(Z_{1j} \hat{Q}_j(-\tau_j)^T + \int_{-\tau_j}^{0} Z_{2j}(s) \hat{Q}_j(s)^T ds \right)$$

$$K_{1i} = \frac{1}{\tau_K} Z_{1i} \hat{S}_i(-\tau_i)$$

$$K_{2i}(s) = \frac{1}{\tau_K} \Bigg(Z_0 \hat{Q}_i(s) + Z_{2i}(s) \hat{S}_i(s) + \sum_{j=1}^{K} \Big(Z_{1j} \hat{R}_{ji}(-\tau_j, s) + \int_{\theta=-\tau_j}^{0} Z_{2j}(\theta) \hat{R}_{ji}(\theta, s) d\theta \Big) \Bigg)$$

Proof: The proof follows directly from the definitions. ∎

We conclude that given $\hat{P}, \hat{Q}_i, \hat{S}_i$ and $\hat{R}_{ij}$, it is possible to compute the controller gains K_0, K_{1i} and K_{2i}. In practice, however, if S is polynomial, then $\hat{S}_i(s) = S(s)^{-1}$ will be a rational matrix-valued function. This implies that $\hat{Q}_i$ and $\hat{R}_{ij}$ are likewise rational. Numerically, this step can be avoided, however, by using the reduced expressions found in the extended version of this paper [12].

VII. An LMI Formulation of the H_∞-Optimal Controller Synthesis Problem for Multi-Delay Systems

In this section, we combine all previous results to give a concise formulation of the controller synthesis problem in the LMI framework.

Theorem 10: For any $\gamma > 0$, suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, matrix $P \in \mathbb{R}^{n \times n}$, polynomials $S_i, Q_i \in W_2^{n\times n}[T_i]$, $R_{ij} \in W_2^{n\times n}[T_i \times T_j]$ for $i, j \in [K]$, matrices $Z_0, Z_{1i} \in \mathbb{R}^{p\times n}$ and polynomials $Z_{2i} \in W_2^{p\times n}[T_i]$ for $i \in [K]$ such that

$$\mathcal{L}_1(P - \epsilon I_n, Q_i, S_i - \epsilon I_n, R_{ij}) \in \Xi_{d,n,nK}$$
$$-\mathcal{L}_1(D + \epsilon \hat{I}, E_i, \dot{S}_i + \epsilon I_n, G_{ij}) \in \Xi_{d,q+m+n(K+1),nK},$$

where D, E_i, G_{ij} are as defined in Theorem 4, $\hat{I} = \mathrm{diag}(0_{q+m}, I_n, 0_{nK})$, and $\mathcal{L}_1$ is as defined in Eqn. (8). Furthermore, suppose P, Q_i, S_i, R_{ij} satisfy the conditions of Lemma 3. Let $u(t)$ be as defined in Lemma 9 where $\hat{P}$, $\hat{Q}_i$, $\hat{S}_i$ and $\hat{R}_{ij}$ are as defined in Theorem 8. Then for any $w \in L_2$, if $y(t)$ and $x(t)$ satisfy Equation (1), $\|y\|_{L_2} \le \gamma \|w\|_{L_2}$.

Proof: Define $\mathcal{P} := \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$. By assumption, $\mathcal{P}$ satisfies the conditions of Lemma 3. By Corollary 7, we have

$$\left\langle \mathbf{x}, \mathcal{P}_{\{P-\epsilon I_n, Q_i, S_i - \epsilon I_n, R_{ij}\}} \mathbf{x} \right\rangle_{Z_{n,K}} = \left\langle \mathbf{x}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \mathbf{x} \right\rangle_{Z_{n,K}} - \epsilon \|\mathbf{x}\|_{Z_{n,K}}^2 \ge 0$$

for all $\mathbf{x} \in Z_{n,K}$. Similarly, we have

$$\left\langle \begin{bmatrix} h \\ \mathbf{z}_{2i} \end{bmatrix}, \mathcal{P}_{\{D+\epsilon\hat{I}, E_i, \dot{S}_i + \epsilon I_n, G_{ij}\}} \begin{bmatrix} h \\ \mathbf{z}_{2i} \end{bmatrix} \right\rangle_{Z_{q+m+n(K+1),n,K}} = \left\langle \begin{bmatrix} h \\ \mathbf{z}_{2i} \end{bmatrix}, \mathcal{P}_{\{D, E_i, \dot{S}_i, G_{ij}\}} \begin{bmatrix} h \\ \mathbf{z}_{2i} \end{bmatrix} \right\rangle_{Z_{q+m+n(K+1),n,K}} + \epsilon \left\| \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_{2i} \end{bmatrix} \right\|_{Z_{n,K}}^2 \le 0.$$

for all $\mathbf{z}_1 \in \mathbb{R}^n$ and $\begin{bmatrix} h \\ \mathbf{z}_{2i} \end{bmatrix} \in Z_{q+m+n(K+1),n,K}$ where $h = \begin{bmatrix} v & w & \mathbf{z}_1 & f \end{bmatrix}$.

Furthermore, by Theorem 8 and Lemma 9, $u(t) = \mathcal{Z}\mathcal{P}^{-1} \begin{bmatrix} x(t) \\ x(t+s) \end{bmatrix}$ where $\mathcal{Z}$ is as defined in Eqn. (5). Therefore, by Theorem 4, if $y(t)$ and $x(t)$ satisfy Equation (1), $\|y\|_{L_2} \le \gamma \|w\|_{L_2}$ ∎

VIII. Numerical Testing, Validation and Practical Implementation

The algorithms described in this paper have been implemented in Matlab within the DelayTOOLs framework, which is based on SOSTOOLS and the pvar framework. Several supporting functions were described in [10] and these are sufficient to enforce the conditions of Theorem 10. For all examples, the computation time is in CPU seconds on an Intel i7-5960X 3.0GHz processor. This time corresponds to the interior-point (IPM) iteration in SeDuMi and does not account for preprocessing, postprocessing, or for the time spent on polynomial manipulations formulating the SDP using SOSTOOLS. Such polynomial manipulations can significantly exceed SDP computation time for small problems.

For simulation and practical use, some additional functionality has been added to facilitate calculation of controller gains and real-time implementation. The most significant new function introduced in this paper is `P_PQRS_Inverse_joint_sep_ndelay`, which takes the matrix P and polynomials Q_i, S_i, and R_{ij} and computes $\hat{P}$, $\hat{H}_i$, and $\hat{\Gamma}_{ij}$ as described in Theorem 8. In addition, the script `solver_ndelay_opt_control` combines all aspects of this paper and simulates the resulting controller in closed loop. For simulation, a fixed-step forward difference method is used, with a different set of states representing each delay channel. In the simulation results given below, 200 spatial discretization points are used for each delay channel. All these tools are available online for validation or download on Code Ocean [3].

A. Validation of H_∞ optimal controller synthesis

We now apply the controller synthesis algorithm to several problems. Unfortunately, there are very few challenging example problems in the literature. When these examples do exist, they are often trivial in the sense that the dynamics can be entirely eliminated by the controller - meaning only the control effort is to be minimized and the achievable norms do not change significantly with delay or other parameters. The problems listed below were found to be the most challenging as measured by either significant variation of the closed-loop

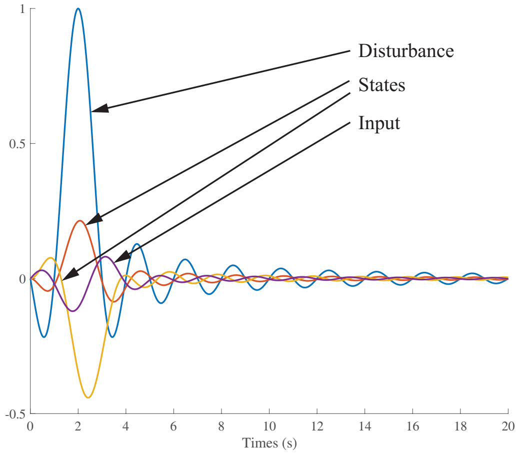

Fig. 1. Closed-loop system response to a sinc disturbance for Ex. B.3

norm with delay or the requirement for a degree of more than 1 to achieve optimal performance. In each case, the results are compared to existing results in the literature (when available) and to an H_∞ optimal controller designed for the ODE obtained by using a 10th order Padé approximation of the delay terms.

a) Example B.1:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -.9 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ .1 \end{bmatrix} u(t)$$

$\gamma_{\min}$	$d=1$	$d=2$	$d=3$	Padé	[7]	[8]
$\tau = .99$	.10001	.10001	.10001	.1000	.228	1.88
$\tau = 2$	1.438	1.353	1.332	1.339	∞	∞
CPU sec	.478	.879	2.48	2.78	N/A	N/A

b) Example B.3: This modified version is taken from [9] where B_2 was altered to make the problem more difficult and inputs/outputs were added). In [9], a stabilizing controller was found for maximum delays of $\tau_1 = .1934$ and $\tau_2 = .2387$. We found a controller for any τ_1 and τ_2 and the results listed are for $\tau_1 = 1$ and $\tau_2 = 2$. The closed-loop system response is illustrated in Fig. 1.

$$\dot{x}(t) = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} .6 & -.4 \\ 0 & 0 \end{bmatrix} x(t-\tau_1) + \begin{bmatrix} 0 & 0 \\ 0 & -.5 \end{bmatrix} x(t-\tau_2) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ .1 \end{bmatrix} u(t)$$

$\gamma_{\min}$	$d=1$	$d=2$	$d=3$	Padé
$\tau_1 = 1, \tau_2 = 2$	.6104	.6104	.6104	.6104
CPU sec	2.07	7.25	25.81	N/A

c) Example B.4:

$$\dot{x}(t) = -\sum_{i=1}^{K} \frac{x(t-i/K)}{K} + \mathbf{1}w(t) + \mathbf{1}u(t)$$

$$y(t) = \begin{bmatrix} \mathbf{1}^T \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$K \downarrow n \rightarrow$	1	2	3	5	10
1	.438	.172	.266	1.24	17.2
2	.269	.643	2.932	17.1	647.2
3	.627	2.634	10.736	91.43	5170.2
5	1.294	13.12	84.77.7	1877	65281
10	11.41	469.86	4439	57894	NA

TABLE I

CPU SEC INDEXED BY # OF STATES (n) AND # OF DELAYS (K)

In this example, we examine the computational complexity of the proposed algorithm. We use a n-D system with K delays, a single disturbance $w(t)$ and a single input $u(t)$. Here $\mathbf{1} \in \mathbb{R}^n$ is the vector of ones. The computation time is listed in Table I. As expected, these results indicate the synthesis problem is not significantly more complex that the stability test. The complexity scales as a function of nK and is possible on desktop computers when $nK < 50$.

IX. Conclusion

In this paper, we have shown how the problem of H_∞-optimal control of multi-delay systems can be reformulated as a convex optimization problem with operator variables. We have proposed a parametrization of positive operators using positive matrices and verified the resulting LMIs are accurate to several decimal places when measured by the minimal achievable closed-loop H_∞ norm bound. We have developed an analytic formula for the inverse of the proposed parameterized class of positive operators. Finally, we have demonstrated effective methods for real-time computation of the control inputs.

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