Declaring War on Boundary Conditions: A Control-Oriented Framework for PDEs

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Why do we have Boundary Conditions (BC's)???

Laplace Equation:

$$(\Delta u)(s) = 0$$

Heat Equation:

$$\dot{u}(t,s) = (\Delta u)(t,s)$$

Boundary Conditions:

$$u(t,s) = 0 \qquad \forall s \in \Gamma$$

Who Came up with BCs, anyway?

Semigroup Correction:

$$u \in D(\mathcal{A}) := \{u \in H^2 \ : \ u(0) = 0, u(1) = 0\}$$

Question: Why do we have BCs?
Answer: To make the solution unique.
Q: Are BCs part of the state?
A: No!
Q: Why do we need them?
A: Otherwise solution not unique.
Q: Are all PDE solns sort of the same?
A: No!
Q: Can BCs change the dynamics?
A: Yes!

Euler-Bernoulli Beam:

$$\mathbf{u}_t = \underbrace{\begin{bmatrix} 0 & -c \\ 1 & 0 \end{bmatrix}}_{=A_2\ (A_0=A_1=0)} \mathbf{u}_{ss}$$

State Space: $u \in H_2^2$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u(0) \\ u(L) \\ u_s(0) \\ u_s(L) \end{bmatrix} = 0$$

Looking For A Universal Formulation

Dynamics are usually expressed in the Primal State $\mathbf{x}_p \in X_p$:

$$\mathbf{x}_p \in L^2_{n_1} \times H^1_{n_2} \times H^2_{n_3} := X_p$$

$$\begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}_t = A_0(s) \underbrace{\begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}}_{\mathbf{x}_p} + A_1(s) \begin{bmatrix} x_2(t,s) \\ x_3(t,s) \end{bmatrix}_s + A_2(s) \begin{bmatrix} x_3(t,s) \end{bmatrix}_{ss}$$

Boundary Conditions:

$$B \begin{bmatrix} x_2(0) \\ x_2(L) \\ x_3(0) \\ x_3(L) \\ x_{3s}(0) \\ x_{3s}(L) \end{bmatrix} = 0, \quad \text{rank}(B) = n_2 + 2n_3$$

Euler-Bernoulli Beam:

$$\mathbf{u}_t = \underbrace{\begin{bmatrix} 0 & -c \\ 1 & 0 \end{bmatrix}}_{=A_2\ (A_0=A_1=0)} \mathbf{u}_{ss}$$

State Space: $u \in H^2_2$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u(0) \\ u(L) \\ u_s(0) \\ u_s(L) \end{bmatrix} = 0$$

Illustration 1: The Euler-Bernoulli Beam

Consider a simple cantilevered E-B beam:

$$u_{tt}(t,s) = -c u_{ssss}(t,s), \qquad \text{where} \quad u(0) = u_s(0) = u_{ss}(L) = u_{sss}(L) = 0$$

Step 1: Eliminate the u_{tt} term (let $u_1 = u_t$)
Step 2: Eliminate u_{ssss} (let $u_2 = u_{ss}$)

$$\dot{u}_1 = u_{tt} = -c u_{ssss} = -c u_{2ss}, \qquad \dot{u}_2 = u_{tss} = u_{1ss}.$$

Universal Formulation:

$$\mathbf{x}_t = \underbrace{\begin{bmatrix} 0 & -c \\ 1 & 0 \end{bmatrix}}_{A_2} \mathbf{x}_{ss}$$

where $A_0 = A_1 = 0$, $n_3 = 2$, and $n_1 = n_2 = 0$.
Boundary Conditions:

$$u_{ss}(L) = u_2(L) = 0 \quad \text{and} \quad u_{sss}(L) = u_{2s}(L) = 0.$$

Insufficient BCs! - $rank(B) = 2$. Differentiate BCs in time to get:

$$u_t(0) = u_1(0) = 0 \quad \text{and} \quad u_{ts}(0) = u_{1s}(0) = 0.$$

This yields $rank(B) = 4$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_1(L) \\ u_2(L) \\ u_{1s}(0) \\ u_{2s}(0) \\ u_{1s}(L) \\ u_{2s}(L) \end{bmatrix} = 0.$$

Conclusion: The E-B beam is exp. stable for any $c > 0$ w/r to u_t and u_{ss}.

The BCs strongly influence the dynamics!

Extreme Example: $D(\mathcal{A}) = \{\mathbf{u} \in H^2 \ : \ \mathbf{u}(0) = w_1(t), \ \mathbf{u}_s(0) = w_2(t)\}$

$$\dot{\mathbf{u}}(t,s) = \mathbf{u}(t,s), \qquad \mathbf{u}(t,0) = w_1(t), \quad \mathbf{u}_s(t,0) = w_2(t)$$

By the Fundamental Theorem of Calculus:

$$\begin{aligned}\mathbf{u}(s) &= s\mathbf{u}(0) + \mathbf{u}_s(0) + \int_0^s (s-\eta)\mathbf{u}_{ss}(\eta)d\eta \\ &= sw_1(t) + w_2(t) + \int_0^s (s-\eta)\mathbf{u}_{ss}(\eta)d\eta\end{aligned}$$

Now rewrite the dynamics in terms of $\mathbf{u}_{ss}$:

Time-Delay System:

$$\begin{aligned}\dot{x}(t) &= -x(t) + u(t,-\tau) \\ \mathbf{u}_t(t,s) &= \mathbf{u}_s(t,s), \quad u(t,0) = x(t)\end{aligned}$$

or completely eliminate BCs:

$$\int_0^s \dot{\mathbf{u}}_s(t,\eta)d\eta = \mathbf{u}_s(t,s) + \int_0^\tau \mathbf{u}_s(t,\eta)d\eta$$

$$\dot{\mathbf{u}}(t,s) = sw_1(t) + w_2(t) + \int_0^s (s-\eta)\mathbf{u}_{ss}(t,\eta)d\eta$$

Conclusion: The BCs *fundamentally alter* the structure of the dynamics!

What is the Fundamental State? (BCs force us to choose $\mathbf{x}_f = \mathbf{u}_{ss}$)

Problems with the Primal State

Simplify the dynamics

$$\dot{\mathbf{x}}(t,s) = A_0(s)\mathbf{x} + A_1(s)\mathbf{x}_s + A_2(s)\mathbf{x}_{ss}$$

Define a Lyapunov Function:

$$V(\mathbf{x}) = \int_0^L \mathbf{x}(s)^T M(s)\mathbf{x}(s)ds$$

Then $V(x) > 0$ if $M(s) \geq 0$ for all s. However,

$$\dot{V}(\mathbf{x}) = \int_0^L \begin{bmatrix}\mathbf{x}\\ \mathbf{x}_s\\ \mathbf{x}_{ss}\end{bmatrix}(s)^T \underbrace{\begin{bmatrix} A_0(s)^T M(s) + M(s)A_0(s) & M(s)A_1(s) & M(s)A_2(s) \\ A_1(s)^T M(s) & 0 & 0 \\ A_2(s)^T M(s) & 0 & 0 \end{bmatrix}}_{D(s)} \begin{bmatrix}\mathbf{x}\\ \mathbf{x}_s\\ \mathbf{x}_{ss}\end{bmatrix}(s)ds$$

Problem: $D(s) \not< 0$ for ANY choice of A_i! Why?

Answer: $\mathbf{x}, \mathbf{x}_s, \mathbf{x}_{ss}$ are not independent states!

Old Solution: IBP, Poincaré, Bessel, Jensen, Wirtinger, Agmaon, Young, et c.

New Solution: Express the dynamics using the **Fundamental State**

The **Fundamental State:** is the *minimal* part of $\mathbf{x}$ which is needed to define the dynamics

Requirements: No Partial Derivatives!
Then What?

$$\mathcal{P}_{\{G_0,G_1,G_2\}}\dot{\mathbf{x}}_f(t) = \mathcal{P}_{\{H_0,H_1,H_2\}}\mathbf{x}_f(t) + \mathcal{P}_{\{J,0,0\}}w(t)$$

$$\mathbf{x}_f(t,s) := \begin{bmatrix} x_1(t,s) \\ x_{2s}(t,s) \\ x_{3ss}(t,s) \end{bmatrix}$$

where

$$\left(\mathcal{P}_{\{N_0,N_1,N_2\}}\mathbf{x}\right)(s) := N_0(s)\mathbf{x}(s)ds + \int_a^s N_1(s,\theta)\mathbf{x}(\theta)d\theta + \int_s^b N_2(s,\theta)\mathbf{x}(\theta)d\theta$$

$\mathcal{P}_{\{N_0,N_1,N_2\}}$ **Operators Inherit the Properties of Matrices!** Closed under:

- Composition, Addition, Scalar Multiplication, Transpose

Previous Example: $\dot{\mathbf{u}}(t,s) = \mathbf{u}(t,s)$

$$\int_0^s (s-\eta)\dot{\mathbf{u}}_{ss}(t,\eta)d\eta = \int_0^s (s-\eta)\mathbf{u}_{ss}(t,\eta)d\eta + s(w_1(t)-\dot{w}_1(t)) + (w_2(t)-\dot{w}_2(t))$$

$$\mathcal{P}_{\{0,s-\eta,0\}}\dot{\mathbf{u}}(t) = \mathcal{P}_{\{0,s-\eta,0\}}\mathbf{u}(t) + \mathcal{P}_{\{[s \quad 1],0,0\}}\begin{bmatrix} w_1(t)-\dot{w}_1(t) \\ w_2(t)-\dot{w}_2(t) \end{bmatrix}$$

Why are PIEs better than PDEs?

The N_0, N_1, N_2 Algebra (Integration Behaves Better than Differentiation!!!)

Property 1: Composition

$$\mathcal{P}_{\{R_0,R_1,R_2\}} = \mathcal{P}_{\{B_0,B_1,B_2\}}\mathcal{P}_{\{N_0,N_1,N_2\}}$$

where

$$R_0(s) = B_0(s)N_0(s)$$

$$R_1(s,\theta) = B_0(s)N_1(s,\theta) + B_1(s,\theta)N_0(\theta) + \int_a^\theta B_1(s,\xi)N_2(\xi,\theta)d\xi + \int_\theta^s B_1(s,\xi)N_1(\xi,\theta)d\xi + \int_s^b B_2(s,\xi)N_1(\xi,\theta)d\xi$$

$$R_2(s,\theta) = B_0(s)N_2(s,\theta) + B_2(s,\theta)N_0(\theta) + \int_a^s B_1(s,\xi)N_2(\xi,\theta)d\xi + \int_s^\theta B_2(s,\xi)N_2(\xi,\theta)d\xi + \int_\theta^b B_2(s,\xi)N_1(\xi,\theta)d\xi$$

$$\{R_0, R_1, R_2\} = \{B_0, B_1, B_2\} \times \{N_0, N_1, N_2\}$$

Property 2: Transpose

$$\langle \mathbf{x}, \mathcal{P}_{\{\hat{N}_0,\hat{N}_1,\hat{N}_2\}}\mathbf{y}\rangle_{L_2} = \langle \mathcal{P}_{\{N_0,N_1,N_2\}}\mathbf{x}, \mathbf{y}\rangle_{L_2}$$

where

$$\hat{N}_0(s) = N_0(s)^T, \quad \hat{N}_1(s,\eta) = N_2(\eta,s)^T, \quad \hat{N}_2(s,\eta) = N_1(\eta,s)^T$$

$$\{\hat{R}_0, \hat{R}_1, \hat{R}_2\} = \{R_0, R_1, R_2\}^*$$

Conversion Between PIE and PDE States

For simplicity, only consider x_3.

Define the Primal State, $\mathbf{x}_p$ and Fundamental State, $\mathbf{x}_f$ as

$$\mathbf{x}_p(t,s) := \left[x(t,s)\right], \qquad \mathbf{x}_f(t,s) = \left[x_{ss}(t,s)\right] \in L_2^n, \quad x_{bf} = \begin{bmatrix} x(0) \\ x(L) \\ x_s(0) \\ x_s(L) \end{bmatrix}, \quad x_{bs} = \begin{bmatrix} x(0) \\ x_s(0) \end{bmatrix}$$

Question: How to Convert? First note that

$$x_s(s) = x_s(0) + \int_a^s \mathbf{x}_{ss}(\eta)d\eta \qquad = \begin{bmatrix} 0 & I \end{bmatrix} x_{bs} + \mathcal{P}_{\{0,I,0\}}\mathbf{x}_{ss}$$

$$x(s) = x(0) + s x_s(0) + \int_0^s (s-\eta)\mathbf{x}_{ss}(\eta)d\eta \qquad = \begin{bmatrix} I & s \end{bmatrix} x_{bs} + \mathcal{P}_{\{0,s-\eta,0\}}\mathbf{x}_{ss}$$

This implies that ANY boundary condition can be represented as

$$Bx_{bf} = B\left(Kx_{bs} + \mathcal{P}_{\{0,T_1,T_2\}}\mathbf{x}_{ss}\right) = 0$$

For some fixed T_1, T_2. Hence

$$BKx_{bs} = -B\mathcal{P}_{\{0,T_1,T_2\}}\mathbf{x}_{ss}$$

Hence we can solve for x_{bs} in terms of $\mathbf{x}_{ss}$

$$x_{bs} = -(BK)^{-1}B\mathcal{P}_{\{0,T_1,T_2\}}\mathbf{x}_{ss}$$

Conclusion: Given $\mathbf{x}_{ss}$, we can reconstruct $\mathbf{x}$!

$$\mathbf{x} = \mathcal{P}_{\{0,G_1,G_2\}}\mathbf{x}_{ss}, \qquad \mathbf{x}_s = \mathcal{P}_{\{0,G_3,G_4\}}\mathbf{x}_{ss}$$

Conversion Between PDE and PIE

Converting from PDE state to PIE state

$$\mathbf{x}_p(t,s) := \begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}, \qquad \mathbf{x}_f(t,s) := \begin{bmatrix} x_1(t,s) \\ x_{2s}(t,s) \\ x_{3ss}(t,s) \end{bmatrix}, \qquad \begin{bmatrix} x_2(a) \\ x_2(b) \\ x_3(a) \\ x_3(b) \\ x_{3s}(a) \\ x_{3s}(b) \end{bmatrix} = 0$$

Part 1: Fundamental Theorem of Calculus in Selected BCs

$$\mathbf{x}_p(s) = K(s) \begin{bmatrix} x_2(a) \\ x_3(a) \\ x_{3s}(a) \end{bmatrix} + (\mathcal{P}_{\{L_0,L_1,0\}}\mathbf{x}_f)(s).$$

Part 2: Convert Given BCs to Selected BCs

$$B \begin{bmatrix} x_2(a) \\ x_2(b) \\ x_3(a) \\ x_3(b) \\ x_{3s}(a) \\ x_{3s}(b) \end{bmatrix} = BT \begin{bmatrix} x_2(a) \\ x_3(a) \\ x_{3s}(a) \end{bmatrix} + B\mathcal{P}_{\{0,Q,Q\}}\mathbf{x}_f = 0 \quad \text{or} \quad \begin{bmatrix} x_2(a) \\ x_3(a) \\ x_{3s}(a) \end{bmatrix} = -(BT)^{-1}B\mathcal{P}_{\{0,Q,Q\}}\mathbf{x}_f.$$

Part 3: Substitute

$$\mathbf{x}_p = \mathcal{P}_{\{G_0,G_1,G_2\}}\mathbf{x}_f$$

where

$$G_0(s) = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad G_1(s,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (s-\theta)I \end{bmatrix} + G_2(s,\theta), \qquad G_2(s,\theta) = -K(s)(BT)^{-1}BQ(s,\theta)$$

$$T = \begin{bmatrix} I & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & (b-a)I \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix}, \quad Q(s,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (b-\theta)I \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad K(s) = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & (s-a) \end{bmatrix}$$

Converting a PDE to a PIE

We may now replace $Bb_{bf} = 0$ and

$$\dot{\mathbf{x}}_p = A_0(s)\mathbf{x}_p + A_1(s)\begin{bmatrix} x_2(t,s) \\ x_3(t,s) \end{bmatrix}_s + A_2(s)\left[x_3(t,s)\right]_{ss}$$

with the more fundamental version:

$$\dot{\mathbf{x}}_p(t) = \mathcal{P}_{\{H_0,H_1,H_2\}}\mathbf{x}_f(t) \qquad \mathbf{x}_p(t,s) := \begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}, \mathbf{x}_f(t,s) := \begin{bmatrix} x_1(t,s) \\ x_{2s}(t,s) \\ x_{3ss}(t,s) \end{bmatrix}$$

Where: A_0, A_1, A_2 and B come from problem definition and

$$H_0(s) = A_0(s)G_0(s) + A_1(s)G_3(s) + A_{20}(s)$$

$$H_1(s,\theta) = A_0(s)G_1(s,\theta) + A_1(s)G_4(s,\theta),$$

$$H_2(s,\theta) = A_0(s)G_2(s,\theta) + A_1(s)G_5(s,\theta), \qquad A_{20}(s) = \begin{bmatrix} 0 & 0 & A_2(s) \end{bmatrix}$$

$$G_0(s) = L_0, \quad G_1(s,\theta) = L_1(s,\theta) + G_2(s,\theta), \quad G_2(s,\theta) = -K(s)(BT)^{-1}BQ(s,\theta)$$

$$G_3(s) = F_0, \quad G_4(s,\theta) = F_1 + L_1(s,\theta) + G_5(s,\theta), \quad G_5(s,\theta) = -V(BT)^{-1}BQ(s,\theta)$$

where

$$T = \begin{bmatrix} I & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & (b-a)I \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix}, \quad Q(s,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (b-\theta)I \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad K(s) = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & (s-a) \end{bmatrix}, \quad L_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$L_1(s,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (s-\theta)I \end{bmatrix}, \quad F_0 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

Recall Example: $\dot{\mathbf{u}}(t,s) = sw_1(t) + w_2(t) + \int_0^s (s-\eta)\mathbf{u}_{ss}(t,\eta)d\eta$

Lyapunov (Energy) Stability - Converting an LMI to a LOI

LOI Stability Condition: $\mathbf{x}_p = \mathcal{P}_{\{G_0,G_1,G_2\}}\mathbf{x}_f$

$$\dot{\mathbf{x}}_p(t) = \mathcal{P}_{\{H_0,H_1,H_2\}}\mathbf{x}_f(t)$$

We now propose a Lyapunov function of the form

$$V(\mathbf{x}_p) = \langle \mathbf{x}_p, \mathcal{P}_{\{N_0,N_1,N_2\}}\mathbf{x}_p \rangle$$

The time-derivative of the Lyapunov function is

$$\begin{aligned}\dot{V}(\mathbf{x}_p(t)) &= 2\langle \mathbf{x}_p, \mathcal{P}_{\{N_0,N_1,N_2\}}\dot{\mathbf{x}}_p \rangle \\ &= 2\langle \mathbf{x}_p, \mathcal{P}_{\{N_0,N_1,N_2\}}\mathcal{P}_{\{H_0,H_1,H_2\}}\mathbf{x}_f \rangle \\ &= 2\langle \mathcal{P}_{\{G_0,G_1,G_2\}}\mathbf{x}_f, \mathcal{P}_{\{N_0,N_1,N_2\}}\mathcal{P}_{\{H_0,H_1,H_2\}}\mathbf{x}_f \rangle \\ &= 2\langle \mathbf{x}_f, \mathcal{P}^*_{\{G_0,G_1,G_2\}}\mathcal{P}_{\{N_0,N_1,N_2\}}\mathcal{P}_{\{H_0,H_1,H_2\}}\mathbf{x}_f \rangle \\ &= \langle \mathbf{x}_f, \mathcal{P}_{\{K_0,K_1,K_2\}}\mathbf{x}_f \rangle + \langle \mathbf{x}_f, \mathcal{P}^*_{\{K_0,K_1,K_2\}}\mathbf{x}_f \rangle\end{aligned}$$

Stability Condition: $\mathcal{P}_{\{N_0,N_1,N_2\}} > 0$ and

$$\mathcal{P}_{\{K_0,K_1,K_2\}} + \mathcal{P}^*_{\{K_0,K_1,K_2\}} \le 0$$

LMI Equivalent: $x_p = Ex$

$$\dot{x}_p(t) = Ax(t)$$

$$V(x) = {x_p}^T P E x_p$$

$$\begin{aligned}\dot{V}(x_p) &= 2{x_p}^T P \dot{x}_p \\ &= 2x^T(E^T P A)x \\ &= x^T(E^T P A + A^T P E)x\end{aligned}$$

$$E^T P A + A^T P E < 0$$

Enforcing Positivity in the N_0, N_1, N_2 Framework

An LMI Condition

Theorem 1.

For any functions $Z(s)$ and $Z(s,\theta)$, and $g(s) \ge 0$ for all $s \in [a,b]$

$$N_0(s) = g(s)Z(s)^T P_{11} Z(s)$$

$$N_1(s,\theta) = g(s)Z(s)^T P_{12} Z(s,\theta) + g(\theta)Z(\theta,s)^T P_{31} Z(\theta) + \int_a^\theta g(\nu)Z(\nu,s)^T P_{33} Z(\nu,\theta)d\nu + \int_\theta^s g(\nu)Z(\nu,s)^T P_{32} Z(\nu,\theta)d\nu + \int_s^L g(\nu)Z(\nu,s)^T P_{22} Z(\nu,\theta)d\nu$$

$$N_2(s,\theta) = g(s)Z(s)^T P_{13} Z(s,\theta) + g(\theta)Z(\theta,s)^T P_{21} Z(\theta) + \int_a^s g(\nu)Z(\nu,s)^T P_{33} Z(\nu,\theta)d\nu + \int_s^\theta g(\nu)Z(\nu,s)^T P_{23} Z(\nu,\theta)d\nu + \int_\theta^L g(\nu)Z(\nu,s)^T P_{22} Z(\nu,\theta)d\nu,$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \ge 0,$$

then $\mathcal{P}^*_{\{N_0,N_1,N_2\}} = \mathcal{P}_{\{N_0,N_1,N_2\}}$ and $\langle \mathbf{x}, \mathcal{P}_{\{N_0,N_1,N_2\}}\mathbf{x}\rangle_{L_2} \ge 0$ for all $\mathbf{x} \in L_2[a,b]$.

Proof: Let

$$\{Z_0, Z_1, Z_2\} := \left\{ \begin{bmatrix} \sqrt{g(s)}Z_{d1}(s) \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \sqrt{g(s)}Z_{d2}(s,\theta) \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \sqrt{g(s)}Z_{d2}(s,\theta) \end{bmatrix} \right\}$$

Then

$$\{N_0, N_1, N_2\} = \{Z_0, Z_1, Z_2\}^* \times \{P, 0, 0\} \times \{Z_0, Z_1, Z_2\}$$

Converting an LMI to an LOI:

The LMI to LOI conversion process:

Step 1: Write the dynamics

$$\dot{\mathbf{x}}_p(t) = \mathcal{A}\mathbf{x}_f(t) + \mathcal{B}w(t), \qquad y(t) = \mathcal{C}\mathbf{x}_f(t) + Dw(t), \qquad \mathbf{x}_p(t) = \mathcal{H}\mathbf{x}_f$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are in the $\{N_0, N_1, N_2\}$ algebra.

Step 2: Replace Matrices with Operators

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \le 0 \;\rightarrow\; \begin{bmatrix} u \\ v \\ \mathbf{x}_f \end{bmatrix}^T \begin{bmatrix} -\gamma I & D^T & \mathcal{B}^*\mathcal{P}\mathcal{H} \\ D & -\gamma I & \mathcal{C} \\ \mathcal{H}^*\mathcal{P}\mathcal{B} & \mathcal{C}^* & \mathcal{A}^*\mathcal{P}\mathcal{H} + \mathcal{H}^*\mathcal{P}\mathcal{A} \end{bmatrix} \begin{bmatrix} u \\ v \\ \mathbf{x}_f \end{bmatrix} \prec 0$$

Why Does This Work?:

- The conversion between primal and fundamental state is a $\{N_0, N_1, N_2\}$ operator.
- We express the dynamics as a $\{N_0, N_1, N_2\}$ operator.
- We express the Lyapunov Functions using a $\{N_0, N_1, N_2\}$ operator.
- $\{N_0, N_1, N_2\}$ operators are closed under composition, adjoint, and addition.
- We can parameterize $\{N_0, N_1, N_2\}$ operators using real numbers
- We can enforce positivity of $\{N_0, N_1, N_2\}$ operators.

An LMI for Stability of PDEs

A Matlab Toolbox

Notations and associated Matlab Functions:

$$\{N_0, N_1, N_2\} \in \Phi_d \qquad \rightarrow \qquad \mathcal{P}_{\{N_0,N_1,N_2\}} \geq 0$$

```
[prog, N0, N1, N2]= sosjointpos_mat_ker_semisep(prog,n,d,d,s,th,[a,b])
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$$\{N_0, N_1, N_2\} = \{T_0, T_1, T_2\} \times \{R_0, R_1, R_2\} \quad \rightarrow \quad \mathcal{P}_{\{N_0,N_1,N_2\}} = \mathcal{P}_{\{T_0,T_1,T_2\}}\mathcal{P}_{\{R_0,R_1,R_2\}}$$

```
[N0, N1, N2] = semisep_MN1N2_compose(T0,T1,T2,R0,R1,R2,s,th,[a,b])
```

$$\{N_0, N_1, N_2\} = \{T_0, T_1, T_2\}^* \quad \rightarrow \quad \mathcal{P}_{\{N_0,N_1,N_2\}} = \mathcal{P}^*_{\{T_0,T_1,T_2\}}$$

```
[N0, N1, N2] = semisep_MN1N2_transpose(T0,T1,T2,s,th)
```

Almost Complete Matlab Code:

```
pvar s th
[prog, G0, G1, G2]=...
[prog, H0, H1, H2]=...
prog = sosprogram([s th])
[prog, M, N1, N2]= sosjointpos_mat_ker_semisep(prog,n,d,d,s,th,II)
[J0, J1, J2] = semisep_MN1N2_compose(M+ep*I,N1,N2,G0,G1,G2,s,th,II)
[H0s, H1s, H2s] = semisep_MN1N2_transpose(H0,H1,H2,s,th)
[K0, K1, K2] = semisep_MN1N2_compose(H0s,H1s,H2s,J0,J1,J2,s,th,II)
[K0s, K1s, K2s] = semisep_MN1N2_transpose(K0,K1,K2,s,th)
[prog, [],N1e, N2e] = sosjointpos_mat_ker_semisep(prog,n,d+2,d+2,s,th,II)
[prog, [],gN1e, gN2e] = sosjointpos_mat_ker_semisep_psatz(prog,n,d+2,d+2,s,th,II)
[prog] = sosmateq(prog,K1+K1s+N1eq+gN1eq)
prog = sossolve(prog,pars)
```

Stability Conditions:

$$\{N_0, N_1, N_2\} \in \Phi_d$$

$$\{K_0, K_1, K_2\} = \{G_0, G_1, G_2\}^* \times \{N_0 + \varepsilon I, N_1, N_2\} \times \{H_0, H_1, H_2\}$$

$$-\{K_0, K_1, K_2\} - \{K_0, K_1, K_2\}^* \in \Phi_{d+2}$$

Verifying Inequalities

Poincare Inequality:

$$\int_0^1 \|x_s(s)\|^2 \leq C^2 \int_0^1 \|x_s(s)\|^2$$

$$V(\mathbf{x}) = \int_0^L \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \\ \mathbf{x}_{ss} \end{bmatrix} (s)^T \underbrace{\begin{bmatrix} -I & 0 & 0 \\ 0 & CI & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D(s)} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \\ \mathbf{x}_{ss} \end{bmatrix} (s) ds$$

For $u(0) = u(1) = 0$, $C_{min} = 1/\pi$
For $u(0) = u_s(0) = 0$, $C_{min} = .6366$

Testing for Accuracy

Example 1: Adapted from Valmorbida, 2014:

$$\dot{x}(t,s) = \lambda x(t,s) + x_{ss}(t,s) \qquad x(0) = x(1) = 0$$

Stable iff $\lambda < \pi^2 \cong 9.8696$. We prove stability for $\lambda = 9.8696$.

Example 2: From Valmorbida, 2016,

$$\dot{x}(t,s) = \lambda x(t,s) + x_{ss}(t,s) \qquad x(0) = 0, \quad x_s(1) = 0$$

Unstable for $\lambda > 2.467$. We prove stability for $\lambda = 2.467$.

Example 3: From Gahlawat, 2017:

$$\dot{x}(t,s) = (-.5s^3 + 1.3s^2 - 1.5s + .7 + \lambda)x(t,s) + (3s^2 - 2s)x_s(t,s) + (s^3 - s^2 + 2)x_{ss}(t,s)$$

with $x(0) = 0$ and $x_s(1) = 0$. Unstable for $\lambda > 4.65$. For $d = 1$, we prove stability for $\lambda = 4.65$.

Example 4: From Valmorbida,2014,

$$\dot{x}(t,s) = \begin{bmatrix} 1 & 1.5 \\ 5 & .2 \end{bmatrix} x(t,s) + R^{-1} x_{ss}(t,s), \qquad x(0) = x_s(1) = 0$$

With $d = 1$, we prove stability for $R = 2.93$ (improvement over $R = 2.45$).

Example 5: From Valmorbida,2016,

$$\dot{x}(t,s) = \begin{bmatrix} 0 & 0 & 0 \\ s & 0 & 0 \\ s^2 & -s^3 & 0 \end{bmatrix} x(t,s) + R^{-1} x_{ss}(t,s), \qquad x(0) = x_s(1) = 0$$

Using $d = 1$, we prove stability for $R = 21$ (and greater) with a computation time of $4.06s$.

Testing for Computational Complexity

Consider a simple n-dimensional diffusion equation

$$\dot{x}(t,s) = x(t,s) + x_{ss}(t,s)$$

where $x(t,s) \in \mathbb{R}^n$.

Computation Time:

n (# of states)	1	5	10	20
CPU sec	.54	37.4	745	31620

Illustration 2: The Timoschenko Beam

Consider a simple Timoschenko beam model:

$$\ddot{w} = \partial_s(w_s - \phi) \qquad = -\phi_s + w_{ss}$$

$$\ddot{\phi} = \phi_{ss} + (w_s - \phi) \qquad = -\phi + w_s + \phi_{ss}$$

with boundary conditions

$$\phi(0) = 0, \quad w(0) = 0, \quad \phi_s(L) = 0, \quad w_s(L) - \phi(L) = 0$$

Step 1: Eliminate w_{tt} and ϕ_{tt} - $u_1 = w_t$ and $u_3 = \phi_t$.
Step 2: Use BCs to pick the state - $u_2 = w_s - \phi$ and $u_4 = \phi_s$.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_0} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}}_{\mathbf{x}_2} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_s$$

where $A_2 = []$ and $n_1 = n_3 = 0$ and $n_2 = 4$ - a purely "hyperbolic" form. We only need 4 BCs:

$$u_1(0) = 0, \quad u_3(0) = 0, \quad u_4(L) = 0, \quad u_2(L) = 0$$

This gives a B has row rank $n_2 = 4$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \\ u_4(0) \\ u_1(L) \\ u_2(L) \\ u_3(L) \\ u_4(L) \end{bmatrix} = 0$$

Stable! However, not exponentially stable ($\dot{V} \not< 0$) in all the given states.

Illustration 2b: The Timoschenko Beam revisited

Consider a modification - naively choose $u_2 = w_s$ and $u_4 = \phi$. This leads to

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A_0} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_s + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{A_2} u_{4ss}$$

where $n_1 = 0$, $n_2 = 3$, and $n_3 = 1$ and with 5 boundary conditions

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \\ u_1(L) \\ u_2(L) \\ u_3(L) \\ u_4(0) \\ u_4(L) \\ u_{4s}(0) \\ u_{4s}(L) \end{bmatrix} = 0.$$

NOT Stable in the given states!

However: If we add a damping term $-cu_{4t} = -cu_3$ to $\dot{u}_3$, then the only change is

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -c & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now Stable for any $c > 0$! Stability is sensitive to definition of states!

Illustration 3: The Tip-Damped Wave Equation

The simplest tip-damped wave equation is

$$u_{tt}(t,s) = u_{ss}(t,s) \qquad u(t,0) = 0 \qquad u_s(t,L) = -k u_t(t,L).$$

Guided by the boundary conditions, we choose

$$\begin{aligned} u_1(t,s) &= u_s(t,s) \\ u_2(t,s) &= u_t(t,s) \end{aligned}$$

This yields

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_s}_{x_2}$$

where $A_0 = 0$, $A_2 = []$ $n_1 = n_3 = 0$ and $n_2 = 2$. The BCs are now

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_1(L) \\ u_2(L) \end{bmatrix} = 0.$$

We prove exp. stability in the given states u_t, u_s for $k > 0$.

Illustration 4: Non-"Hyperbolic" Damped Wave Equation

Add u to the dynamics (stable for $a, k \neq 0$)

$$\begin{aligned} u_{tt}(t,s) &= u_{ss}(t,s) - 2au_t(t,s) - a^2 u(t,s) \qquad s \in [0,1] \\ \text{BCs:} \quad u(t,0) &= 0, \qquad u_s(t,1) = -ku_t(t,1) \end{aligned}$$

Must choose the variables $u_1 = u_t$ and $u_2 = u$. Yields the diffusive form:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \underbrace{\begin{bmatrix} -2a & -a^2 \\ 1 & 0 \end{bmatrix}}_{A_0} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{A_2} \underbrace{u_{2ss}}_{x_3}$$

where $A_1 = 0$, $n_1 = 0$, $n_2 = 1$, and $n_3 = 1$. The BCs on u_1 make us consider this a hyperbolic state!

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_1(L) \\ u_2(0) \\ u_2(L) \\ u_{2s}(0) \\ u_{2s}(L) \end{bmatrix} = 0.$$

Stable!, but not exponentially stable in the given state (confirmed analytically).

Distributed State Estimation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \lambda \begin{bmatrix}1 & 0.3\\ 0.1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_{1ss}\\ x_{2ss}\end{bmatrix} + \begin{bmatrix}s - s^2\\ 0\end{bmatrix} w(t),$$

$$y = \int_a^b \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(s)\\ x_2(s)\end{bmatrix} \mathsf{d}s, \quad z(t) = \int_a^b \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(s)\\ x_2(s)\end{bmatrix} \mathsf{d}s.$$

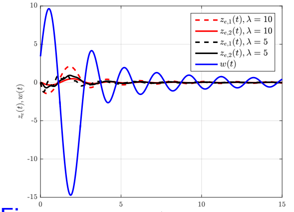


Figure: Time evolution of $z_e(t)$ and $w(t)$ for $\lambda = 5, 10$ where $w(t)$ is generated by damped sinusoidal functions.

1D Flexible Arm attached to Rigid Body

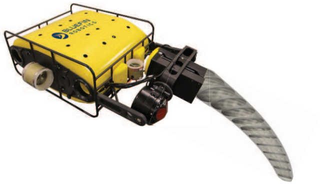
Position of Body: $z(t)$ **Deflection and Curvature:** $w(s,t), w_{ss}(s,t)$
Disturbance: $d(t), u(t)$ **Output:** $w_{ss}(s,t)$

$$\ddot{z}(t) = -F w_{sss}(0,t) + d(t),$$

$$\ddot{w}(s,t) = -\frac{EI}{\mu} w_{ssss}(s,t) + u(t),$$

$$w(0,t) = z(t), w_s(0,t) = 0,$$

$$w_{ss}(L,t) = 0, w_{sss}(L,t) = 0$$



Result: For $\frac{EI}{\mu} = 10$, we get $\|z\|^2_{L_2} \leq .8936\|u\|^2_{L_2}$.

Other Algebras

ODEs Coupled with PDEs:

Algebra of Operators on $\mathbb{R}^n \times L_2^m[a,b]$

$$\left(\mathcal{P}\left\{\begin{smallmatrix} P, & Q_1, & Q_2 \\ S, & R_1, & R_2 \end{smallmatrix}\right\} \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}\right)(s) :=$$

$$\begin{bmatrix} Px + \int_a^b Q_1(s)\mathbf{x}(s)ds \\ Q_2(s)x + S(s)\mathbf{x}(s) + \int_a^s R_1(s,\theta)\mathbf{x}(\theta)d\theta + \int_s^b R_2(s,\theta)\mathbf{x}(\theta)d\theta \end{bmatrix}.$$

Positivity using

$$P \geq 0 \quad \rightarrow \quad \left\{\begin{smallmatrix} I, & Z_1, & Z_2 \\ Z_3, & Z_4, & Z_5 \end{smallmatrix}\right\}^* \times \left\{\begin{smallmatrix} P_1, & P_2, & P_3 \\ P_4, & 0, & 0 \end{smallmatrix}\right\} \times \left\{\begin{smallmatrix} I, & Z_1, & Z_2 \\ Z_3, & Z_4, & Z_5 \end{smallmatrix}\right\} \succ 0$$

PDEs in 2 Spatial Dimensions:

Algebra of Operators on $L_2[[a,b] \times [c,d]]$

$$(\mathcal{P}\mathbf{u})(s) :=$$

$$N_0(x,y)\mathbf{u}(x,y) + \int_a^x \int_c^y N_1(x,y,s,\theta)\mathbf{u}(s,\theta)dsd\theta + \int_x^b \int_y^d N_1(x,y,s,\theta)\mathbf{u}(s,\theta)dsd\theta.$$

H_∞ Gain Analysis

Stable for $\lambda < 4.65$.

$$u_t(s,t) = A_0(s)u(s,t) + A_1(s)u_s(s,t) + A_2(s)u_{ss}(s,t) + w(t)$$
$$u(0,t) = 0 \qquad u_s(1,t) = 0$$

$$A_0(s) = (-0.5s^3 + 1.3s^2 - 1.5s + 0.7 + \lambda), \quad A_1(s) = (3s^2 - 2s), \quad A_2(s) = (s^3 - s^2 + 2)$$

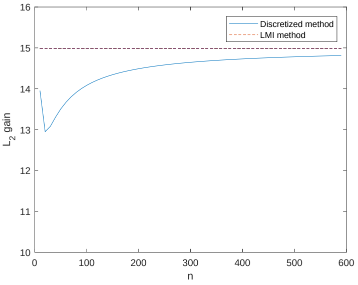


Figure: Comparison with Discretization ($d = 1$)

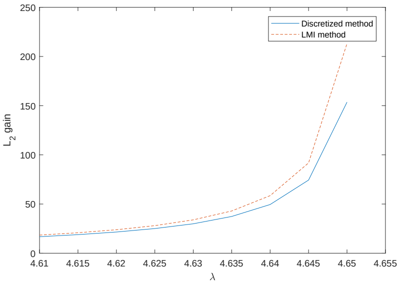


Figure: H_∞ Gain as a function of λ ($d = 1$)

Conclusion and Extensions (Thanks to ONR #N000014-17-1-2117)

$\mathcal{P}_{\{N_0,N_1,N_2\}}$ Framework extends LMI techniques to PDEs.

- $A^T P + PA < 0$ becomes

$$\underbrace{\mathcal{P}^*_{\{H_0,H_1,H_2\}}}_{A^T}\underbrace{\mathcal{P}_{\{N_0,N_1,N_2\}}}_{P}\mathcal{P}_{\{G_0,G_1,G_2\}}+\mathcal{P}^*_{\{G_0,G_1,G_2\}}\underbrace{\mathcal{P}_{\{N_0,N_1,N_2\}}}_{P}\underbrace{\mathcal{P}_{\{H_0,H_1,H_2\}}}_{A} \leq 0$$

Conclusions:

PROs:

- Computationally Efficient
- A more rational treatment of boundary conditions.
- No Conservatism (Almost N+S)
- Easily Extended to New Problems
  - e.g. higher order derivatives
  - e.g. distributed dynamics

CONs:

- Requires $n_2 + 2n_3$ BCs to be clearly specified
- PDE Must be Stable in all States

Extensions:

- Input-Output Properties (ACC, 2019)
  - H_∞ Gain
  - passivity
- ODEs coupled with PDEs (CDPS, 2019)
- Optimal Estimator Synthesis
- Optimal Controller Synthesis

Solvable (in order of difficulty)

- Extension to 3D
- Duality (Stability of $\mathcal{A}^*$)
- Inversion of the $\mathcal{P}_{\{N_0,N_1,N_2\}}$ Operator
  - Want an Analytic Formula