Optimal Control Strategies for Systems with Input Delay using the PIE Framework

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Abstract: The Partial Integral Equation (PIE) framework provides a unified algebraic representation for use in analysis, control, and estimation of infinite-dimensional systems. However, the presence of input delays results in a PIE representation with dependence on the derivative of the control input, $\dot{u}$. This dependence complicates the problem of optimal state-feedback control for systems with input delay – resulting in a bilinear optimization problem. In this paper, we present two strategies for convexification of the H_∞-optimal state-feedback control problem for systems with input delay. In the first strategy, we use a generalization of Young's inequality to formulate a convex optimization problem, albeit with some conservatism. In the second strategy, we filter the actuator signal – introducing additional dynamics, but resulting in a convex optimization problem without conservatism. We compare these two optimal control strategies on four example problems, solving the optimization problem using the latest release of the PIETOOLS software package for analysis, control and simulation of PIEs.

Keywords: Input Delay, Optimal Control, LMIs, Delay Systems, Partial Integral Equations

1. INTRODUCTION

We revisit the classic problem of optimal static state-feedback control of a set of linear ordinary differential equations with delay in the actuation – a formulation defined in Eqn. (1). It is well-known that delays in the input can destabilize an otherwise stable closed-loop system, even if the open-loop dynamics are stable.

Perhaps the most common approach to control in the presence of input delay is to use a predictor which uses a model of the system to predict where the state will lie when the actuation signal is applied. This approach is typified by the Smith predictor (Smith [1959]) and when the model is known perfectly it can be shown that under certain conditions the interconnection of a stabilizing state-feedback controller and Smith predictor will be stable. However, it is also know that such an approach is sensitive to errors in the system model (Laughlin et al. [1987]). As a result, there have been many attempts to find state-feedback controllers which are robustly stabilizing in the presence of input delay. A representative sampling of this work can be found in Krstic [2009], Zhang et al. [2005], Yue [2004], Moon et al. [2001], Cheres et al. [1990], Yue [2004], Li et al. [1999], Du et al. [2005, 2010, 2005], and Liu et al. [2012].

However, if we restrict our consideration to the question of *optimal* control (as opposed to stabilizing control) in the presence of input delay, the literature becomes relatively sparse. Examples of optimal control in the presence of input delay include Du et al. [2005], Carravetta et al. [2010], Cacace et al. [2016], Basin and Rodriguez-Gonzalez [2005, 2006]. In addition, closely related to the problem of optimal control is the question of eigenvalue assignment, which was considered in Furtat et al. [2017]. In none of these works, however, do we find anything approaching a necessary and sufficient condition – implying all controllers obtained from such methods will be rather sub-optimal.

Typically, the conservatism in approaches to analysis of time-delay systems (ignoring the question of control) is a result of the two factors: 1) The use of a restrictive class of Lyapunov functions or 2) the use of conservative inequalities in bounding the derivative of the Lyapunov function. Of course, for optimal control, the problem becomes significantly harder in that we are simultaneously searching for a Lyapunov function and a set of controller gains - resulting in a bilinear optimization problem. While bilinearity in the controller synthesis problem for ODEs is typically resolved using a dual representation combined with a variable substitution, the application of this approach to time-delay systems has been more limited – See Peet [2019].

One class of systems for which we do have a relatively well-defined notion of duality is the class of Partial Integral Equations (PIEs). The study of PIEs is motivated by the need for an algebraic representation of infinite-dimensional systems, wherein the analysis, simulation, and control techniques developed for state-space ODEs can be generalized to systems of PDEs and systems with delay. Specifically, it was shown in Peet [2021] that almost every linear optimal control and estimation problem involving time-delays can be reformulated as a question of optimal control and estimation of an associated PIE representation. Furthermore, because of the algebraic parameterization of PIEs, duality results have been proposed in Shivakumar et al. [2020b] which yield a dual PIE representation which

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retains the input-output properties of the primal PIE and hence the original PDE or delayed system. Furthermore, for certain classes of PIEs, it has been shown that the question of H_∞-optimal state-feedback control of that class of PIE can be formulated as a convex optimization problem – convertible to a Linear Matrix Inequality (LMI) using parsers such as PIETOOLS (See Shivakumar et al. [2021]). As a result, the problem of optimal state-feedback control with state-delay has been more or less solved. Unfortunately, the extension of these techniques to systems with input delay is unresolved.

The PIE representation of the H_∞-optimal state-feedback control problem with input delay does not fall into the class of PIEs for which we have an exact convex formulation of the problem. Specifically, the $\mathcal{T}_u$ term in Eqns. (3) is non-zero. The goal of this paper, then, is to evaluate two proposed methods for convexification of the problem of optimal control with input delay. We will then compare these two approaches as applied to several numerical examples to determine which is superior.

In the first approach, we apply a duality result and use a conservative convex formulation of the resulting optimization problem – See Shivakumar and Peet [2022]. While this approach is expected to be sub-optimal (relying on Young's inequality), it does not require us to filter the input or alter the input-delay problem in any way.

The second approach is to use a simple filter on the input signal, thus converting the input delay to a state delay. This filter may represent actuator dynamics or may be introduced artificially. Whatever the source, conversion of the input delay to a state delay allows us to apply the results in Shivakumar and Peet [2022] without any of the conservatism associated with the use of Young's inequality.

Having defined these two approaches, we apply them to four standard numerical examples. These numerical results indicate that the suboptimal approach without filtering approach typically (but not consistently) results in smaller closed-loop L_2-gain bounds than the exact condition with filtering. We end the paper by concluding that both the filtered and non-filtered controller synthesis conditions allow us to obtain state-feedback controllers for systems with input delay which significantly outperform any currently known class of controllers.

2. PROBLEM FORMULATION

In this paper, we consider the problem of H_∞ optimal control of a state-space system of Ordinary Differential Equations (ODEs) subject to input delay. For simplicity, we use a single delay and do not consider model uncertainty, state delay, disturbance delay, or neutral-type systems – although such features can be integrated into the modelling framework presented here. Specifically, we use the standard 9-matrix state-space formulation of the optimal control framework, with 3 additional matrices which model the effect of the delayed input.

$$\begin{bmatrix}\dot{x}(t)\\ z(t)\\ y(t)\end{bmatrix} = \begin{bmatrix}A_0 & B_1 & B_2\\ C_{10} & D_{11} & D_{12}\\ C_{20} & D_{21} & D_{22}\end{bmatrix}\begin{bmatrix}x(t)\\ w(t)\\ u(t)\end{bmatrix} + \begin{bmatrix}B_{2i}\\ D_{12i}\\ D_{22i}\end{bmatrix} u(t-\tau) \quad (1)$$

Here $x(t) \in \mathbb{R}^n$ is the internal state, $w(t) \in \mathbb{R}^{n_w}$ is the exogenous disturbance, $u(t) \in \mathbb{R}^{n_u}$ is the controlled input, $y(t) \in \mathbb{R}^{n_y}$ is the sensed output, $z(t) \in \mathbb{R}^{n_z}$ is the output to be regulated, and $\tau > 0$ is the input delay.

We focus on the problem of static state-feedback, so that

$$u(t) = K_1 x(t) + \int_{-\tau}^{0} K_2(s)\partial_s x(t+s)ds \quad (2)$$

which is a slight generalization of the more typical class of state-feedback controllers of the form

$$u(t) = K_1 x(t) + \int_{-\tau}^{0} K_2(s)x(t+s)ds.$$

The goal is to find the smallest $\gamma > 0$ such that for any $w, z \in L_2$ which satisfy Eqns. (1)-(2) for some x and u, we have that $\|z\|_{L_2} \leq \gamma \|w\|_{L_2}$.

3. PARTIAL INTEGRAL EQUATIONS

Our approach to optimal control uses Partial Integral Equations (PIEs) to represent the system of equations defined in (1). Specifically, the PIE formulation of the optimal control framework is defined by 11 Partial Integral (PI) operators and has the form

$$\begin{aligned}&\mathcal{T}\dot{\mathbf{x}}(t) + \mathcal{T}_w\dot{w}(t) + \mathcal{T}_u\dot{u}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t)\\ &z(t) = \mathcal{C}_1\mathbf{x}(t) + \mathcal{D}_{11}w(t) + \mathcal{D}_{12}u(t),\\ &y(t) = \mathcal{C}_2\mathbf{x}(t) + \mathcal{D}_{21}w(t) + \mathcal{D}_{22}u(t),\end{aligned} \quad (3)$$

where the operators $\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u, \mathcal{A}, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_{ij}$ are PI operators (denoted $\mathcal{P} \in \Pi_{p,q}^{n,m}$) of the form

$$\left(\mathcal{P}\begin{bmatrix}P, & Q_1\\ Q_2, & \{R_i\}\end{bmatrix}\begin{bmatrix}x\\ \mathbf{\Phi}\end{bmatrix}\right)(s) := \begin{bmatrix}Px + \int_{-1}^{0} Q_1(s)\mathbf{\Phi}(s)ds\\ Q_2(s)x + \left(\mathcal{P}_{\{R_i\}}\mathbf{\Phi}\right)(s)\end{bmatrix}$$

and where

$$\begin{aligned}&\left(\mathcal{P}_{\{R_i\}}\mathbf{\Phi}\right)(s) :=\\ &R_0(s)\mathbf{\Phi}(s) + \int_{-1}^{s} R_1(s,\theta)\mathbf{\Phi}(\theta)d\theta + \int_{s}^{0} R_2(s,\theta)\mathbf{\Phi}(\theta)d\theta.\end{aligned}$$

As has been shown in Peet [2021], for every system of linear delay-differential equations (DDEs), there exists an associated PIE the solution to which yields the solution of the DDE and for which internal and input-output stability properties are equivalent. In addition, an H_∞-optimal state-feedback controller for the PIE yields an H_∞-optimal state-feedback controller for the DDE of the form given in Eqn. (2). Furthermore, analytic expressions for construction of the associated PIE were given in Peet [2021]. In the following section, we apply these formulae to the problem of optimal control of a system of ODEs with input delay.

3.1 Linear PI Inequalities (LPIs)

One of the advantages of the PIE representation is that the class of PI operators forms a $*$-algebra of bounded linear operators, being closed under composition, concatenation, addition, and adjoint. Furthermore, PI operators can be represented using polynomials which can, in turn, be represented using vectors and matrices. This mathematical structure enables one to generalize most matrix operations to PI operators and such operations can be computed efficiently using software packages such as PIETOOLS (See Shivakumar et al. [2020a]).

In addition, positive matrices can be used to parameterize positive PI operators. This allows us to define a class of convex optimization problems with linear objectives and linear operator inequality constraints. Such problems are referred to as Linear PI Inequalities (LPIs) and the PIETOOLS software package includes a parser for conversion of LPIs to LMIs, which can then be solved efficiently using SDP solvers such as SeDuMi (Sturm [1999]) or Mosek (Andersen and Andersen [2000]). Details of this parser can be found in Shivakumar et al. [2021] and will be used to solve the LPIs formulated in Theorems 2 and 4.

4. INPUT DELAY FORMULATION A

As discussed in the preceding section, a system of ODEs with input delay admits an associated PIE of the form given in (3). This associated PIE representation can be constructed as follows.

Lemma 1. Suppose the operators $\mathcal{A}, \mathcal{T}, \mathcal{T}_w, \mathcal{T}_u, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_{ij}$ are defined as

$$\mathcal{A} = \mathcal{P}\begin{bmatrix} A_0, & -B_{2d} \\ 0, & \left\{\frac{1}{\tau} I_{n_u}, 0, 0\right\}\end{bmatrix}, \qquad \mathcal{T} = \mathcal{P}\begin{bmatrix} I, & 0 \\ 0, & \left\{0, 0, -I_{n_u}\right\}\end{bmatrix}, \quad (4)$$

$$\mathcal{B}_1 = \mathcal{P}\begin{bmatrix} B_1, \emptyset \\ 0, \{\emptyset\}\end{bmatrix}, \qquad \mathcal{B}_2 = \mathcal{P}\begin{bmatrix} B_2 + B_{2d}, \emptyset \\ 0, \quad \{\emptyset\}\end{bmatrix},$$

$$\mathcal{T}_w = \mathcal{P}\begin{bmatrix} 0, \emptyset \\ 0, \{\emptyset\}\end{bmatrix}, \qquad \mathcal{T}_u = \mathcal{P}\begin{bmatrix} 0, \ \emptyset \\ I_{n_u}, \{\emptyset\}\end{bmatrix},$$

$$\mathcal{C}_1 = \mathcal{P}\begin{bmatrix} C_{10}, -D_{12d} \\ \emptyset, \quad \{\emptyset\}\end{bmatrix}, \qquad \mathcal{C}_2 = \mathcal{P}\begin{bmatrix} C_{20}, -D_{22d} \\ \emptyset, \quad \{\emptyset\}\end{bmatrix},$$

$$\mathcal{D}_{11} = \mathcal{P}\begin{bmatrix} D_{11}, \emptyset \\ \emptyset, \{\emptyset\}\end{bmatrix}, \qquad \mathcal{D}_{12} = \mathcal{P}\begin{bmatrix} D_{12} + D_{12d}, \emptyset \\ \emptyset, \quad \{\emptyset\}\end{bmatrix},$$

$$\mathcal{D}_{21} = \mathcal{P}\begin{bmatrix} D_{21}, \emptyset \\ \emptyset, \{\emptyset\}\end{bmatrix}, \qquad \mathcal{D}_{22} = \mathcal{P}\begin{bmatrix} D_{22} + D_{22d}, \emptyset \\ \emptyset, \quad \{\emptyset\}\end{bmatrix}.$$

Then for any x_0, $w, u \in W_{2e}$ with $w(0) = 0$, and $u(0) = 0$, we have that $z, y, x \in L_{2e}$ satisfy the ODE with input delay in Eqns. (1) with inputs w, u and initial condition $x(s) = 0$ for $s \leq 0$ if and only if z, y and $\mathbf{x}(t, s) = \begin{bmatrix} x(t) \\ \partial_s u(t + \tau s)\end{bmatrix}$ satisfy the PIE in Eqns. (3) with inputs w, u and initial condition $\mathbf{x}(0) = 0$.

Proof. To apply the formulae given in Peet [2021], we must first represent the ODE with input delay in the form of a differential-difference equation (DDF). Although formulae exist for direct conversion of a DDE to PIE, such formulae fail to account for the low-dimensional nature of the input delay channel.

To represent Eqns. (1) as a DDF, the primary task it to identify the delay channel, $r(t)$. In our case, this is trivially $r(t) = u(t)$. This yields the following DDF representation.

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \\ r(t)\end{bmatrix} = \begin{bmatrix} A_0 & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \\ 0 & 0 & I_{n_u}\end{bmatrix}\begin{bmatrix} x(t) \\ w(t) \\ u(t)\end{bmatrix} + \begin{bmatrix} B_{2d} \\ D_{12d} \\ D_{22d} \\ 0\end{bmatrix} v(t) \qquad (5)$$

$$v(t) = r(t - \tau).$$

Clearly, since $u(t) = 0$ for $t \leq 0$, we have that x, z, y satisfy Eqn. (1) with inputs w, u and initial condition $x(0) = 0$ if and only if x, z, y, $r = u$ and $v(t) = u(t - \tau)$ satisfy the DDF with inputs w, u for initial condition $x(0) = 0$ and $v(s) = 0$ for $s \leq 0$.

Using the parameterization of a DDF given in Eqn. (3) in Peet [2021], we may extract the following hitherto undefined DDF parameters.

$$\begin{bmatrix} B_v \\ D_{1v} \\ D_{2v} \\ D_{rv1}\end{bmatrix} = \begin{bmatrix} B_{2d} \\ D_{12d} \\ D_{22d} \\ 0\end{bmatrix}, \ C_{v1} = I_{n_u}, \ C_{vd1} = 0,$$

$$[C_{r1} \ B_{r11} \ B_{r21}] = [0 \ 0 \ I_{n_u}].$$

We now apply these parameters to the formulae in Eqns. (10) in Peet [2021] to obtain the following.

$$\hat{C}_{vi} = I_{n_u}, \qquad D_I = I_{n_u}, \quad C_{Ii}(s) = -I_{n_u}$$

$$[\mathbf{T}_0 \ \mathbf{T}_1 \ \mathbf{T}_2] = [0 \ 0 \ I_{n_u}], \quad [C_{vx} \ D_{vw} \ D_{vu}] = [0 \ 0 \ I_{n_u}]$$

$$\mathbf{T}_a(s, \theta) = 0, \qquad \mathbf{T}_b(s, \theta) = -I_{n_u}, \quad I_\tau = \frac{1}{\tau} I_{n_u},$$

$$\begin{bmatrix} \mathbf{A}(s) \\ \mathbf{C}_{11}(s) \\ \mathbf{C}_{21}(s)\end{bmatrix} = -\begin{bmatrix} B_{2d} \\ D_{12d} \\ D_{22d}\end{bmatrix},$$

$$\begin{bmatrix} \mathbf{A}_0 & \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{C}_{10} & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_{20} & \mathbf{D}_{21} & \mathbf{D}_{22}\end{bmatrix} = \begin{bmatrix} A_0 & B_1 & B_2 \\ C_{10} & D_{11} & D_{12} \\ C_{20} & D_{21} & D_{22}\end{bmatrix} + \begin{bmatrix} 0 & 0 & B_{2d} \\ 0 & 0 & D_{12d} \\ 0 & 0 & D_{22d}\end{bmatrix}$$

Finally, we apply these parameters to Eqns. (10) in Peet [2021] to obtain the PIE operators given in the theorem statement. Now, if we apply Lemmas 3 and 4 in Peet [2021] we find that x, z, y, $r = u$ and $v(t) = u(t - \tau)$ satisfy the DDF with inputs w, u for initial condition $x(0) = 0$ and $v(s) = 0$ for $s \leq 0$ if and only if z, y and

$$\mathbf{x}(t, s) = \begin{bmatrix} x(t) \\ \partial_s \phi(t, s)\end{bmatrix} = \begin{bmatrix} x(t) \\ \partial_s r(t + \tau s)\end{bmatrix} = \begin{bmatrix} x(t) \\ \partial_s u(t + \tau s)\end{bmatrix}$$

satisfies the PIE with inputs w, u and initial condition $\mathbf{x}(0) = 0$.

5. H_∞-OPTIMAL CONTROL W/O PRE-FILTERING

Although we have established a PIE representation of the optimal control framework with input delay, we have yet to establish a numerical approach to solving the problem of optimal state-feedback control of such a PIE representation. Although LPI formulations of the problem of $\mathcal{H}_\infty$-optimal state-feedback have been proposed in Shivakumar et al. [2020b], these results were restricted to the case where $\mathcal{T}_u = \mathcal{T}_w = 0$. Unfortunately, the PIE associated with the optimal control formulation of the system with input delay, as defined in Lemma 1, does not satisfy this restriction. Specifically, while $\mathcal{T}_w = 0$, we have

$$\mathcal{T}_u = \mathcal{P}\begin{bmatrix} 0, \ \emptyset \\ I_{n_u}, \{\emptyset\}\end{bmatrix}$$

which is clearly non-zero. Recently, however, in Shivakumar and Peet [2022] it has been shown that by application of Young's inequality, the problem of $\mathcal{H}_\infty$-optimal control of PIEs with $\mathcal{T}_u \neq 0$ may be tightened to the following formulation of the $\mathcal{H}_\infty$-suboptimal control problem.

Theorem 2. Let $\mathcal{T}_w = 0$ and suppose there exist $\epsilon > 0, \gamma > 0$, $\mathcal{P} : \Pi^{n_x, n_u}_{n_x, n_u}$ and $\mathcal{Z} \in \Pi^{n_u, 0}_{n_x, n_u}$ such that $\mathcal{P} = \mathcal{P}^*$, $\mathcal{P} \geq \epsilon I_{m+n}$ and $\mathcal{H} \leq 0$ where

$$\mathcal{H} := \begin{bmatrix} -\gamma & 0 & 0 & 0 & \mathcal{D}_1^* & \mathcal{B}_1^* \\ 0 & -\mathcal{P} & 0 & 0 & (\mathcal{D}_{12}\mathcal{Z})^* & 0 \\ 0 & 0 & -\mathcal{P} & 0 & 0 & \sqrt{2}(\mathcal{T}_u\mathcal{Z})^* \\ 0 & 0 & 0 & -\mathcal{P} & 0 & (\mathcal{B}_2\mathcal{Z})^* \\ \mathcal{D}_1 & \mathcal{D}_{12}\mathcal{Z} & 0 & 0 & -\gamma & \mathcal{H}_{12} \\ \mathcal{B}_1 & 0 & \sqrt{2}\mathcal{T}_u\mathcal{Z} & \mathcal{B}_2\mathcal{Z} & \mathcal{H}_{12}^* & \mathcal{D}_{22}\end{bmatrix}$$

$$\mathcal{H}_{12} = \mathcal{C}_1\mathcal{P}\mathcal{T}^* + \mathcal{C}_1\mathcal{Z}^*\mathcal{T}_u^* + \mathcal{D}_{12}\mathcal{Z}\mathcal{T}^*$$

$$\mathcal{H}_{22} = (\mathcal{T}\mathcal{P}\mathcal{A}^* + +\mathcal{T}_u\mathcal{Z}\mathcal{A}^* + \mathcal{T}\mathcal{Z}^*\mathcal{B}_2^*) + (\mathcal{T}\mathcal{P}\mathcal{A}^* + +\mathcal{T}_u\mathcal{Z}\mathcal{A}^* + \mathcal{T}\mathcal{Z}^*\mathcal{B}_2^*)^* \qquad (6)$$

Let $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$. Then:

(1) For any $w \in L_2$, if $z, w, u, \mathbf{x}$ satisfy the PIE in Eqns. (3) for initial condition $\mathbf{x}(0) = 0$ where $u(t) = \mathcal{K}\mathbf{x}(t)$ we have that $\|z\|_{L_2} \leq \gamma \|w\|_{L_2}$.

(2) For any $w \in L_2$, if z, w, u, x satisfy the DDE in Eqns. (1) for initial condition $x(0) = 0$ where $u(t) = \mathcal{K}\begin{bmatrix} x(t) \\ u_s(t + \cdot * \tau) \end{bmatrix}$ for $t \geq 0$ we have $\|z\|_{L_2} \leq \gamma \|w\|_{L_2}$.

Proof. The first statement, which provides an L_2-gain bound on the PIE in Eqns. (3), follows directly from Shivakumar and Peet [2022]. The second statement follows from the first statement combined with Lemma 1.

While the LPI provided in Theorem 2 allows us to minimize an upper bound on the minimum achievable L_2-gain of the closed loop system with static state-feedback and input delay, such a bound is likely conservative, as discussed in Shivakumar and Peet [2022]. For this reason, we turn to an alternative formulation of the problem of optimal control with input delay for which we can solve the problem directly and which is possibly better-posed.

6. INPUT DELAY FORMULATION B

Our second approach to controller synthesis with input delay converts the problem to one of optimal control with state-delay by adding a filter between the input signal and the plant. The delay is then applied to the output of this filter. This filter may be chosen ad-hoc or may model the dynamics of the actuator.

Specifically, implementation of a state-feedback controller requires some form of actuation and while we often assume this actuator is static (with no internal dynamics), in reality most actuators are driven by electrical signals which are then converted to force, torque, etc. Indeed, this conversion of electrical stimulus to force or torque often requires sophisticated dynamic models, including those in Demerdash and Nehl [1980] and Zaccarian [2012].

For simplicity, we assume that the filter or actuator dynamics have the form

$$C\dot{x}_c(t) = -Rx_c(t) + Lu(t) \tag{7}$$

where the meaning of the constants will depend on the application. This approach has the advantage that if we assume the dynamics of the filter are relatively fast, we then obtain the pseudo-equilibrium $x_c(t) = R^{-1}Lu(t)$ so that if $R^{-1}L \cong I$, the output of the filter is simply the desired actuation signal – implying that weights on the input signal $u(t)$ translate to weights on the actuation signal $x_c(t)$. This means that bounds on achievable L_2 gains from Formulation A can be compared directly to achievable gains from Formulation B.

The delay is then applied to the output of the filter which influences the plant as $\dot{x}(t) = Ax(t) + B_1w(t) + B_2u(t) + B_{2d}x_c(t-\tau)$, yielding a modified form of the optimal control framework (taking $C = I_{n_x}$ for simplicity).

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -R & 0 & 0 & L \\ 0 & A_0 & B_1 & B_2 \\ 0 & C_{10} & D_{11} & D_{12} \\ 0 & C_{20} & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x_c(t) \\ x(t) \\ w(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_{2d} \\ D_{12d} \\ D_{22d} \end{bmatrix} x_c(t-\tau) \tag{8}$$

We now obtain a revised version of Lemma 1 as follows.

Lemma 3. Suppose the operators $\mathcal{A}, \mathcal{T}, \mathcal{T}_w, \mathcal{T}_u, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_{ij}$ are defined as

$$\mathcal{A} = \mathcal{P}\begin{bmatrix} \begin{bmatrix} -R & 0 \\ B_{2d} & A_0 \end{bmatrix}, -\begin{bmatrix} 0 \\ B_{2d} \end{bmatrix} \\ 0, \quad \{I_\tau, 0, 0\} \end{bmatrix}, \qquad \mathcal{T} = \mathcal{P}\begin{bmatrix} I_{n_x+n_u}, & 0 \\ \begin{bmatrix} I_{n_u} & 0 \end{bmatrix}, & \{0, 0, -I_{n_u}\} \end{bmatrix},$$

$$\mathcal{B}_{T_1} = \mathcal{P}\begin{bmatrix} 0, \emptyset \\ 0, \{\emptyset\} \end{bmatrix}, \mathcal{B}_{T_2} = \mathcal{P}\begin{bmatrix} 0, \emptyset \\ 0, \{\emptyset\} \end{bmatrix}, \mathcal{D}_{ij} = \mathcal{P}\begin{bmatrix} D_{ij}, \emptyset \\ \emptyset, \{\emptyset\} \end{bmatrix},$$

$$\mathcal{B}_1 = \mathcal{P}\begin{bmatrix} \begin{bmatrix} 0 \\ B_1 \end{bmatrix}, \emptyset \\ 0, \quad \{\emptyset\} \end{bmatrix}, \qquad \mathcal{B}_2 = \mathcal{P}\begin{bmatrix} \begin{bmatrix} L \\ B_2 \end{bmatrix}, \emptyset \\ 0, \quad \{\emptyset\} \end{bmatrix},$$

$$\mathcal{C}_1 = \mathcal{P}\begin{bmatrix} \begin{bmatrix} D_{12d} & C_{10} \end{bmatrix}, -D_{12d} \\ \emptyset, \quad \{\emptyset\} \end{bmatrix}, \qquad \mathcal{C}_2 = \mathcal{P}\begin{bmatrix} \begin{bmatrix} D_{22d} & C_{20} \end{bmatrix}, -D_{22d} \\ \emptyset, \quad \{\emptyset\} \end{bmatrix}.$$

Then for any $w, u \in L_{2e}$, we have that z, y, x and satisfy the ODE with input delay in Eqns. (8) with inputs w, u and initial condition $x(0) = 0$, $x_c(0) = 0$ if and only if $z, y \in L_{2e}$ and $\mathbf{x}(t, s) = \begin{bmatrix} x_c(t) \\ x(t) \\ \partial_s x_c(t + \tau s) \end{bmatrix}$ satisfy the PIE in Eqns. (3) with inputs w, u for initial condition $\mathbf{x}(0) = 0$.

Proof. As was the case in the proof of Lemma 1, we first represent Eqns. (1) as a DDF. In this case, however, the delay channel, $r(t)$ is given by $r(t) = x_c(t)$. This yields the following DDF representation.

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}(t) \\ z(t) \\ y(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} -R & 0 & 0 & L \\ 0 & A_0 & B_1 & B_2 \\ 0 & C_{10} & D_{11} & D_{12} \\ 0 & C_{20} & D_{21} & D_{22} \\ I_{n_u} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_c(t) \\ x(t) \\ w(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_{2d} \\ D_{12d} \\ D_{22d} \\ 0 \end{bmatrix} v(t)$$

$$v(t) = r(t - \tau)$$

Clearly, x, x_c, z, y satisfy Eqn. (8) with inputs w, u and initial condition $x(0) = 0$, $x_c(0) = 0$ if and only if $x, x_c, z, y, r = x_c$ and $v(t) = x_c(t-\tau)$ satisfy the DDF with inputs w, u for initial condition $x(0) = 0$ and $x_c(t) = 0$ for $t \leq 0$. Using the parameterization of a DDF given in Eqn. (3) in Peet [2021], we may extract the following hitherto undefined DDF parameters.

$$\begin{bmatrix} B_v \\ D_{1v} \\ D_{2v} \end{bmatrix} = \begin{bmatrix} 0 \\ B_{2d} \\ D_{12d} \\ D_{22d} \end{bmatrix}, \; C_{v1} = I_{n_u}, \; C_{vd1} = 0,$$

$$\begin{bmatrix} C_{r1} & B_{r11} & B_{r21} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I_{n_u} & 0 \end{bmatrix} & 0 & 0 \end{bmatrix}$$

We now apply these parameters to the formulae in Eqns. (10) in Peet [2021] to obtain the following.

$$\hat{C}_{vi} = I_{n_u}, \qquad D_I = I_{n_u}, \qquad C_{Ii}(s) = -I_{n_u} \tag{9}$$

$$\begin{bmatrix} \mathbf{T}_0 & \mathbf{T}_1 & \mathbf{T}_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I_{n_u} & 0 \end{bmatrix} & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} C_{vx} & D_{vw} & D_{vu} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I_{n_u} & 0 \end{bmatrix} & 0 & 0 \end{bmatrix},$$

$$\mathbf{T}_a(s, \theta) = 0, \qquad \mathbf{T}_b(s, \theta) = -I_{n_u}, \quad I_\tau = \frac{1}{\tau} I_{n_u},$$

$$\begin{bmatrix} \mathbf{A}(s) \\ \mathbf{C}_{11}(s) \\ \mathbf{C}_{21}(s) \end{bmatrix} = -\begin{bmatrix} 0 \\ B_{2d} \\ D_{12d} \\ D_{22d} \end{bmatrix}, \tag{10}$$

$$\begin{bmatrix} \mathbf{A}_0 & \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{C}_{10} & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_{20} & \mathbf{D}_{21} & \mathbf{D}_{22} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -R & 0 \\ B_{2d} & A_0 \end{bmatrix} & \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & \begin{bmatrix} L \\ B_2 \end{bmatrix} \\ \begin{bmatrix} D_{12d} & C_{10} \end{bmatrix} & D_{11} & D_{12} \\ \begin{bmatrix} D_{22d} & C_{20} \end{bmatrix} & D_{21} & D_{22} \end{bmatrix}$$

Finally, we apply these parameters to Eqns. (10) in Peet [2021] to obtain the PIE operators given in the theorem statement. Now, if we apply Lemmas 3 and 4 in Peet [2021] we find that $x, x_c, z, y, r = x_c$ and $v(t) = x_c(t - \tau)$ satisfy the DDF with input w, u for initial condition $x(0) = 0$, $x_c(t) = 0$ for $t \leq 0$ if and only if z, y and

$$\mathbf{x}(t,s)=\begin{bmatrix}x_c(t)\\x(t)\\\partial_s\phi(t,s)\end{bmatrix}=\begin{bmatrix}x_c(t)\\x(t)\\\partial_s r(t+\tau s)\end{bmatrix}=\begin{bmatrix}x_c(t)\\x(t)\\\partial_s x_c(t+\tau s)\end{bmatrix}$$

satisfies the PIE with initial condition $\mathbf{x}(0)=0$.

Clearly, the difference between the original input-delay model in (1) and that in (8) is that the input delay has been converted to a state delay. The advantage of this approach is that the PIE associated with Eqn. (8) satisfies the condition $\mathcal{T}_u=\mathcal{T}_w=0$ which implies that the problem of optimal controller synthesis can be formulated directly as an LPI. This LPI is defined in the following section.

7. H_∞-OPTIMAL CONTROL WITH PRE-FILTERING

Having now obtained a formulation of the optimal control framework with input delay for which the associated PIE admits $\mathcal{T}_u=\mathcal{T}_w=0$, we may now apply the results of Shivakumar et al. [2020a] and Shivakumar and Peet [2022] without the conservatism induced by the use of Young's inequality. This LPI is defined as follows.

Theorem 4. (Shivakumar et al. [2020a]). Let $\mathcal{T}_w=\mathcal{T}_u=0$ and suppose there exist $\epsilon>0,\gamma>0$, $\mathcal{P}:\Pi^{n_x+n_u,n_u}_{n_x+n_u,n_u}$ and $\mathcal{Z}\in\Pi^{n_u,0}_{n_x+n_u,n_u}$ such that $\mathcal{P}=\mathcal{P}^*$, $\mathcal{P}\geq\epsilon I_{n_x+n_u+n_u}$ and

$$\begin{bmatrix}-\gamma I & \mathcal{D}_{11}^* & (\mathcal{P}\mathcal{C}_1^*+\mathcal{Z}^*\mathcal{D}_{12}^*)\\ * & -\gamma I & \mathcal{B}_1\\ * & * & *+\mathcal{T}(\mathcal{A}\mathcal{P}+\mathcal{B}_2\mathcal{Z})^*\end{bmatrix}\prec 0. \quad (11)$$

Let $\mathcal{K}=\mathcal{Z}\mathcal{P}^{-1}$. Then:

(1) For any $w\in L_2$, if $z,w,u,\mathbf{x}$ satisfy the PIE in Eqns. (3) for initial condition $\mathbf{x}(0)=0$ where $u(t)=\mathcal{K}\mathbf{x}(t)$ we have that $\|z\|_{L_2}\leq\gamma\|w\|_{L_2}$.

(2) For any $w\in L_2$, if z,w,u,x satisfy the DDE in Eqns. (8) for initial condition $x(0)=0$, $x_c(0)=0$ where $u(t)=\mathcal{K}\begin{bmatrix}x(t)\\x_c(t)\\x_{c,s}(t+\cdot*\tau)\end{bmatrix}$ for $t\geq 0$ we have that $\|z\|_{L_2}\leq\gamma\|w\|_{L_2}$.

8. COMPARISON OF CONTROLLERS AS APPLIED TO NUMERICAL EXAMPLES

Having defined two approaches to the problem of optimal control with input delay, we now apply these approaches to four test cases and compare the performance as measured by achievable closed-loop L_2-gain bounds.

In each case, the LPIs obtained from Theorems 2 and 4 were implemented using the PIETOOLS parser as described in Shivakumar et al. [2021]. When a prefilter is applied, we specify $R=L=I$.

8.1 Numerical Example 1

This system is adopted from Yue [2004], Moon et al. [2001], and Cheres et al. [1990] and is open-loop stable.

$$\dot{x}(t)=\begin{bmatrix}0&1\\-1.25&-3\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}w(t)+\begin{bmatrix}0\\1\end{bmatrix}u(t-\tau)$$
$$z(t)=\begin{bmatrix}0&1\\0&0\end{bmatrix}x(t)+\begin{bmatrix}0\\.1\end{bmatrix}u(t) \quad (12)$$

The minimum achievable closed-loop L_2-gain for 3 values of delay are listed in Table 1. This example is the only case where the filtered input consistently outperformed the sub-optimal controller.

Table 1. Closed Loop H_∞ gain of Eqn. (12).

$\tau\rightarrow$	1	2	3
$\gamma_{\min}$ w/o filter	.3286	.3333	.3333
$\gamma_{\min}$ w filter	.2718	.3103	.3270

Table 2. Closed Loop H_∞ gain of Eqn. (13).

$\tau\rightarrow$	1	2	3
$\gamma_{\min}$ w/o filter	1.0797	.4933	.4736
$\gamma_{\min}$ w filter	.2361	.4544	.6481

Table 3. Closed Loop H_∞ gain of Eqn. (14).

$\tau\rightarrow$	1	2	3
$\gamma_{\min}$ w/o filter	.7372	1.683	2.6044
$\gamma_{\min}$ w filter	.9813	2.2426	3.9781

8.2 Numerical Example 2

Adapted from Yue [2004], Li et al. [1999] and Du et al. [2005, 2010], this system is open-loop neutrally stable.

$$\dot{x}(t)=\begin{bmatrix}0&0\\1&-5\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}w(t)+\begin{bmatrix}1\\0\end{bmatrix}u(t-\tau)$$
$$z(t)=\begin{bmatrix}0&1\\0&0\end{bmatrix}x(t)+\begin{bmatrix}0\\.1\end{bmatrix}u(t) \quad (13)$$

The minimum achievable closed-loop L_2-gain for 3 values of delay are listed in Table 2. For comparison, in Li et al. [1999] a robust controller was found with $\gamma=1.56$ for $\tau=.24$ (a result with no weighting on the control effort). The corresponding gain with input-delayed controller was $\gamma=.0891$ and with filter was $\gamma=.0357$.

This example is interesting in that the suboptimal controller performs better at higher delay and for $\tau=3$ outperforms the filtered controller.

8.3 Numerical Example 3

Adapted from Liu et al. [2012] this system is open-loop unstable.

$$\dot{x}(t)=\begin{bmatrix}-0.8&-0.01\\1&0.1\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}w(t)+\begin{bmatrix}0.4\\0.1\end{bmatrix}u(t-\tau)$$
$$z(t)=\begin{bmatrix}0&1\\0&0\end{bmatrix}x(t)+\begin{bmatrix}0\\.1\end{bmatrix}u(t) \quad (14)$$

The minimum achievable closed-loop L_2-gain for 3 values of delay are listed in Table 3.

8.4 Numerical Example 4

Taken directly from Du et al. [2005] this sytem has 6 states and is open-loop stable.

$$\begin{bmatrix}I&0\\0&M\end{bmatrix}\dot{x}(t)=\begin{bmatrix}0&I\\-K&-C\end{bmatrix}x(t)+B_w w(t)+Bu(t-\tau)$$
$$z(t)=Cx(t)$$
$$M=\begin{bmatrix}1.1&0&0\\0&1.8&0\\0&0&1.6\end{bmatrix}\qquad C=\begin{bmatrix}1.2&-.6&0\\-.6&1.2&-.6\\0&-.6&.6\end{bmatrix}$$
$$K=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\qquad B=[0\ 0\ 0\ 1\ 0\ 0]^T$$
$$B_w=[0\ 0\ 0\ 0\ 0\ .1]^T\qquad C=\begin{bmatrix}.1&.1&.5&0&0&0\\0&0&0&.1&.1&.5\end{bmatrix} \quad (15)$$

The minimum achievable closed-loop L_2-gain for 3 values of delay are listed in Table 4. For comparison, at $\tau=.15$, Du et al. [2005] obtained an L_2-gain of .624 and for $\gamma=1$ the maximum allowable delay was .164. Note that

Table 4. Closed Loop H_∞ gain of Eqn. (15).

$\tau \rightarrow$	1	2	3
$\gamma_{\min}$ w/o filter	.0667	.1002	.1227
$\gamma_{\min}$ w filter	.0749	.1327	.2161

the very small closed-loop gains are partially a result of the failure to weight the control effort in the optimal control formulation.

9. CONCLUSION

In this paper, we have considered the problem of H_∞-optimal state feedback control with input delay. While the PIE framework has previously been used to design optimal state-feedback controllers for systems with state-delay, the question of optimal control with input delay is not included in the class of systems for which we have an equivalent convex formulation of the problem. To address this, we have proposed two approaches. In the first, we provide a sub-optimal, yet convex formulation of the controller synthesis problem. In the second, we add a pre-filter – essentially converting the input delay to a state-delay at the cost of introducing additional dynamics thereby slowing the response. We then perform numerical experiments to determine which approach results in smaller closed-loop L_2-gain bounds. The results seem to indicate that use of the suboptimal approach without filtering generally results in better performance than the sub-optimal control approach – possibly indicating that the sub-optimal approach is not very sub-optimal.

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