Extension of the Partial Integral Equation Representation to GPDE Input-Output Systems

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***Abstract*—Partial Integral Equations (PIEs) are an alternative representation of systems governed by Partial Differential Equations (PDEs). PIEs have advantages over PDEs in that they are defined by integral (not differential) operators and do not include boundary conditions or continuity constraints on the solution – a convenience when computing system properties, designing controllers, or performing simulation. In prior work, PIE representations were proposed for a class of 2^{nd}-order PDEs in a single spatial variable. In this paper, we extend the PIE representation to a more general class of PDE systems including, e.g., higher-order spatial derivatives (N^{th}-order), PDEs with inputs and outputs, PDEs coupled with ODEs, PDEs with distributed input and boundary effects, and boundary conditions which combine boundary values with inputs and integrals of the state. First, we propose a unified parameterization of PDE systems, which we refer to as a Generalized PDE (GPDE). Given a PDE system in GPDE form, we next propose formulae that take the GPDE parameters and construct the Partial Integral (PI) operators that define an associated PIE system, including a unitary map that converts solutions of the PIE to solutions of the GPDE model. This map is then used to show that the GPDE model and associated PIE have equivalent system properties, including well-posedness, internal stability, and input-output behaviour. These representations, conversions, and mappings are implemented in an open-source software package and illustrated through several examples: including beams, mixing problems, entropy modeling, and wave equations. Finally, we demonstrate the significance of the PIE representation by solving analysis, simulation, and control problems for several representative GPDE models.**

***Index Terms*—PDEs, Optimization, LMIs**

I. Introduction

Although Partial Differential Equations (PDEs) have been used to model spatially-distributed physical phenomena since the time of Newton and Leibniz, the central importance of boundary conditions (BCs) when defining a PDE model was not formally recognized until the time of Dirichlet (See [4] for a survey of the history of PDEs and BCs). However, even with the inclusion of BCs, a PDE model is not complete without a restriction on 'continuity' of the solution – spatial derivatives and boundary values must be suitably well-defined. The mathematical formalism for a continuity restriction was only established in the middle of the 20th century by Sergei Sobolev, defining what are now termed Sobolev spaces, and allowing for the use of generalized functions or distributions to define weak solutions.

When the PDE, BCs, and continuity constraints are combined, we obtain what can be called a 'PDE model' – a system defined by three types of constraints, none of which is individually sufficient but which, when combined, yield a well-posed map from an initial state to a unique solution. In the latter half of the 20th century, this map and its continuity properties were formalized and generalized by the notion of a C_0-semigroup, with the BCs and continuity constraints of the PDE system (now including delay systems and PDEs coupled with ODEs) being defined as the 'domain of the infinitesimal generator' (See, e.g. [5], [10]). Today, as a consequence of almost 300 years of careful study and mathematical progress, we may conclude that a well-posed PDE model is necessarily defined by three constraints: a) the differential equation, or 'PDE', which constrains the spatio-temporal evolution of the solutions inside the domain, c) the continuity condition, which ensures that the solutions have sufficient regularity for the BCs to be well-defined; and c) the BCs, which may constrain the limit values or other properties of the solutions as permitted by the regularity guaranteed by the continuity constraints.

A. The Challenge of using a 3-Constraint PDE Model

As described above, the natural representation of phenomena such as diffusion would seem to be a three constraint PDE model – given the historical context and the clear physical interpretation of spatial derivatives and BCs. However, as described below, when considering computational methods for the analysis, control, and simulation of spatially distributed phenomena, the use of a three constraint PDE model is inconvenient. The most significant inconveniences are as follows:

1) **Non-Algebraic Structure** All computation is fundamentally algebraic – consisting primarily of a sequence of addition and multiplication operations. The PDE model formulation, however, is defined by spatial differentiation and evaluation of limit points (Dirac operations). Neither differentiation nor Dirac operators can be embedded in a *-algebra of bounded linear operators on a Hilbert space [21]. The unbounded nature of the differential and Dirac operators complicates both simulation and analysis – resulting either in ill-conditioned ODE representations or a lack of the algebraic structure needed for parameterization and optimization.
2) **No Universality** Computational methods are traditionally centered on the 'PDE' part of the 'PDE model', and are designed for a fixed set of BCs and continuity constraints. This means every change in boundary condition or continuity constraint requires a change in the algorithm, with such changes being ad-hoc and requiring significant mathematical analysis. As a result, there are no generic/universal algorithms

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for analysis, control, and simulation of PDEs.

To illustrate, consider the problem of computing the evolution of a PDE model from a given initial condition. Specifically, consider a simple transport equation $u_t = u_s$ and construct a finite-difference approximation of $u_s = \frac{u(s_{i+1})-u(s_i)}{\Delta_s}$ – yielding an finite-dimensional representation $\dot{x}(t) = \frac{1}{\Delta_s}Ax(t)$, where $x_i = u(s_i)$, $\Delta_s = s_{i+1} - s_i$ is uniform, and A is a bi-diagonal matrix of ± 1 entries. In an ideal simulation we would desire $\Delta_s \to 0$ – which implies that an ideal ODE representation of the transport equation would have all infinitely large coefficients. Of course, we can avoid many problems associated with discretization by constructing an explicit basis for the domain of the infinitesimal generator (bases which staisfy the continuity constraints and BCs) and projecting our solution onto this basis – an approach used in Galerkin projection. The problem, however, is that every change in the set of BCs and continuity constraints necessitates a change in the basis functions. Such changes require significant ad-hoc analysis – an obstacle to design of general/universal simulation tools.

Having illustrated the disadvantages of the three constraint PDE representation in the context of simulation, let us also consider the problem of computational analysis and control of a PDE model. For simplicity, consider the very stable heat equation $u_t = u_{ss}$ with zero BCs, e.g. $u(t,0) = u_s(t,1) = 0$, and propose an energy metric (Lyapunov function) of the form $V(u) = \int_0^1 u(s)^2 ds$. This energy metric is uniformly decreasing with time – thus proving the stability of the PDE model. The challenge, however, is to use computation to prove this fact. By parameterizing positive operators using positive matrices, optimization-based methods for stability analysis can easily recognize that $V(u) = \langle u, u\rangle_{L_2}$ and hence V is a positive form [19] (i.e., a valid candidate Lyapunov function). However, the algorithm must also verify that $\dot{V}(u(t)) \leq 0$ for all solutions $u(t) \in W_2$ satisfying the PDE model. Unfortunately, if we differentiate $V(u(t))$ in time along solutions of the PDE model we obtain $\dot{V}(u(t)) = 2\left\langle u(t), \partial_s^2 u(t)\right\rangle = 2\int_0^1 u(t,s)u_{ss}(t,s)ds$. Because differentiation is not embedded in a *-algebra, we cannot simply parameterize a cone of positive quadratic forms involving differential operators, e.g., $\langle \partial_s u, \partial_s u\rangle$. Moreover, since the derivative operator is unbounded, the functions u and u_{ss} are independent until the continuity constraints and BCs are enforced. However, accounting for the continuity and BCs is an ad-hoc process, using integration-by-parts or inequalities such as Wirtinger or Poincare. While such ad-hoc methods have been used to generate computational stability tests for specific classes of PDE models (See LMI-methods in [2], [8], [11], [12], [17], [29], backstepping methods in [1], [14], [15], [20], [27], [32], late-lumping methods in [16] and port-Hamiltonian methods in [30]), there exists no universal approach to computational analysis and control of PDE models.

To summarize, while the representation of spatially-distributed systems using the three constraint PDE model has significant history and is the natural modeling framework, the presence of unbounded operators, continuity constraints, and BCs poses significant challenges to the development of a universal computational framework for analysis, control, and simulation. As will be shown in the following subsection, however, these limitations are primarily an artifact of the PDE modeling approach, are not inherent to spatially distributed systems, and can be remedied by using an alternative modeling framework defined by Partial Integral Equations (PIEs).

B. The Partial Integral Equation (PIE) Framework

The PIE framework is an approach to the modelling of spatially distributed systems. PIE models can be considered a generalization of the integro-differential systems which have been used to model phenomena such as elasticity, mechanical fracture, etc. [3], [13]. Unlike a PDE model, wherein the state (e.g., $u(t) \in W_2$) is differentiated, consistent with continuity constraints, the state of a PIE model is the highest spatial derivative (e.g., $u_{ss}(t) \in L_2$) of the PDE model and this state is integrated in space in order to obtain the evolution equation. Consequently, a PIE model is defined by a single integro-differential equation, is parameterized by the *-algebra of Partial Integral (PI) operators, and can be used to represent almost any well-posed PDE model.

The simplest form of PIE, in which we ignore ODEs, inputs, and outputs, is defined by two Partial Integral (PI) operators, $\mathcal{T}, \mathcal{A} : L_2 \to L_2$ as $\mathcal{T}\dot{\mathbf{v}}(t) = \mathcal{A}\mathbf{v}(t)$, where the state, $\mathbf{v}(t) \in L_2$ admits no continuity constraints or BCs. An operator $\mathcal{P}$ is said to be a 3-PI operator, denoted $\mathcal{P} \in \Pi_3$ if there exist $R_0 \in L_\infty$ and separable functions R_1, R_2 such that

$$(\mathcal{P}\mathbf{u})(s) = R_0(s)\mathbf{u}(s) + \int_a^s R_1(s,\theta)\mathbf{u}(\theta)\,d\theta + \int_s^b R_2(s,\theta)\mathbf{u}(\theta)\,d\theta.$$

To illustrate a simple PIE, consider a PDE model of the heat equation, $u_t = u_{ss}$ with BCs $u(t,0) = u_s(t,1) = 0$, continuity constraint $u \in W_2$ and initial condition $u(0,\cdot) = u_0 \in W_2$. A PIE representation of this PDE model is given by

$$\int_0^s \theta\, v_t(t,\theta)\,d\theta + \int_s^1 s\, v_t(t,\theta)\,d\theta = -v(s,t) \qquad (1)$$

with initial condition $v(0,\cdot) = \partial_s^2 u_0 \in L_2$. In this case, $\mathcal{T} \in \Pi_3$ is parameterized by $R_1(s,\theta) = -\theta, R_2(s,\theta) = -s$ with $R_0 = 0$, while $\mathcal{A} \in \Pi_3$ is parameterized by $R_0(s) = I$ with $R_1 = R_2 = 0$. The solution to the PIE yields a solution to the PDE model as $u(t) = \mathcal{T}v(t)$, so that $u(t,s) = -\int_0^s \theta\, v(t,\theta)d\theta - \int_s^1 s\, v(t,\theta)d\theta$.

B.1 Properties of PIEs and PI Operators

The distinguishing feature of the class of PIE models is its parameterization using the *-algebras of PI operators (Π_i and Π_i^p). In contrast to differential and Dirac operators, **PI operators have the following properties:**

1) **Algebraic Structure** The set of PI operators is a subspace of $\mathcal{L}(L_2)$ – the space of bounded linear operators on the Hilbert space L_2. PI operators form *-algebras, denoted Π_i, being closed under addition, composition, and transposition (See [25, Appendix H]). In addition, Π_3 and Π_4 are unital algebras – implying that these operators inherit most of the properties of matrices, including operations that preserve positivity.
2) **Parameterization by Polynomials** The subspaces of Π_i with polynomial parameters also form a *-subalgebra, denoted Π_i^p. PIEs which represent PDE models are typically pa-

rameterized by operators in Π_i^p. Because polynomials admit a linear parameterization using coefficient vectors, and because multiplication, addition, and integration reduce to algebraic operations on these coefficient vectors, the complexity of computing operations involving operators in Π_i^p is negligable.

3) **Computation via PIETOOLS** Most matrix operations defined in Matlab have a Π_i^p equivalent which is easy to compute. These operations have been embedded into an `opvar` class in the MATLAB toolbox PIETOOLS [26]. This toolbox also allows one to solve Linear PI Inequality Optimization (LPIs) problems (a natural extension of the class of Linear Matrix Inequality (LMI) optimization problems).

Having motivated the PI algebra, we summarize the **benefits of using PIE models** in place of equivalent PDE models:

1) **Known map from PDE model to PIE model** For the large class of well-posed linear PDE models defined in this paper, we have explicit formulae for construction of an associated PIE model, including the map from PIE solution to PDE solution. In addition, most PDE models map to PIE models parameterized by PI operators with polynomial parameters.
2) **State-Space Structure** Because PIE models are parameterized by the PI *-algebra of bounded linear operators on L_2, PIEs inherit many of the benefits of the state-space representation of linear ODEs. This implies that many numerical methods designed for analysis, control, and simulation of ODEs in state-space form may be extended to PIEs. Specifically, many LMIs for analysis and control of ODEs have been extended to PIEs, including stability analysis [18], L_2-gain analysis [23], $\mathcal{H}_\infty$-optimal estimation [7], $\mathcal{H}_\infty$-optimal control [24], and robust stability/performance [6], [31].
3) **Universal Methods** A PIE model is defined by a single differential equation with no further constraints on the state, such as BCs or continuity constraints. This allows us to develop universal algorithms for analysis, control, and simulation which apply to any well-posed PIE model. Examples of such algorithms can be found in PIETOOLS [26].

To illustrate these advantages, consider again the problem of proving stability of the heat equation (with state u). Using the PIE representation of the heat equation (with state v) in Eqn. (1), proving stability is now much simpler. Specifically, consider the standard energy metric/Lyapunov function $V = \langle u,u\rangle_{L_2} = \langle \mathcal{T}v,\mathcal{T}v\rangle_{L_2}$ and differentiate in time to obtain

$$\begin{aligned}\dot V(v(t)) &= \langle \mathcal{T}\dot v(t),\mathcal{T}v(t)\rangle_{L_2} + \langle \mathcal{T}v(t),\mathcal{T}\dot v(t)\rangle_{L_2}\\ &= \langle v(t),(\mathcal{T}+\mathcal{T}^*)v(t)\rangle_{L_2} = \langle v(t),\mathcal{D}v(t)\rangle_{L_2}\end{aligned}$$

where $\mathcal{D} \in \Pi_3^p$ is parameterized by $R_1(s,\theta) = -2\theta$, $R_2(s,\theta) = -2s$ and $R_0 = 0$. We may now use convex optimization to find the PI operator $Q \in \Pi_3^p$ such that $\mathcal{D} = -\mathcal{Q}^*\mathcal{Q}$. In this case Q is parameterized by $R_1 = \sqrt{2}$, $R_2 = 0$ and $R_0 = 0$. This proves that $\dot V(v) = \langle v,\mathcal{D}v\rangle = -\langle Qv,\mathcal{Q}v\rangle \le 0$.

C. Contribution of this Paper

Because the PIE representation is unified, any algorithm or method designed for analysis, control, or simulation of PIE models can be applied to any system which admits such a representation. The impact of such algorithms and methods, therefore, can be increased by expanding the class of PDE models for which there exists an equivalent PIE model representation. Unfortunately, however, the class of PDE models for which there exist PDE-PIE conversion formulae is still rather limited. To demonstrate, consider the following two systems:

1) Entropy evolution of 1D thermoelastic rod (c.f. [9]):

$$\dot\eta(s,t) = \eta_{ss}(s,t), \quad \eta(0,t) = \eta(1,t) = -\int_0^1 \eta(s,t)ds.$$

2) Octopus-inspired soft robot arm (c.f. [28]):

$$\begin{aligned}&\ddot x(t) = -\mathbf{x}_{sss}(t,0) + d_0(t), \quad \mathbf{x}(t,0) = x(t),\\ &\ddddot{\mathbf{x}}(t,s) = -\ddot{\mathbf{x}}(t,s) + \ddot{\mathbf{x}}_s(t,s) - \mathbf{x}_{ssss}(t,s) + d_1(t,s),\\ &\mathbf{x}_s(t,0) = 0,\ \mathbf{x}_{ss}(t,1) = d_2(t),\ \mathbf{x}_{sss}(t,1) = u(t).\end{aligned}$$

The first PDE model has integral terms at the boundary, whereas the second model is a 4^{th}-order PDE model coupled with an ODE and has input signals forcing both the generator and the BCs. At present, however, the class of PDE models with known PIE conversion formulae does not include: generators with spatial derivatives of order higher than 2 or driven by boundary values, PDEs with inputs and outputs, PDEs coupled with ODEs, or BCs that combine boundary values with inputs and integrals of the state (formulae for input-output PDEs and ODEs coupled with PDEs appear in conference format [23]).

The goal of this paper, then, is to extend the class of PDE models for which we have PDE to PIE conversion formulae to include the cases defined above. As with any extension of the PIE framework, we approach the incorporation of a new class of PDE models in three steps: (a) we propose a parametric representation of the PDE model class; (b) We define an appropriate state-space to be used in the corresponding PIE model; (c) We find a unitary transformation from the PIE state-space to the state space of the PDE model – proving equivalence of solutions and equivalence of stability properties. In this context, the three main contributions of the paper are:

1) *A unified class of PDE models*: We parameterize a class of linear PDE models we refer to as Generalized Partial Differential Equations (GPDEs) – See Section III. These GPDEs encompass: ODEs coupled with PDEs, N^{th}-order spatial derivatives, integrals of the state, control inputs and disturbances, and sensed and regulated outputs.
2) *Formulae to convert GPDE models to PIEs*: Given a sufficiently well-posed GPDE model, we give formulae for conversion to a PIE. These formulae are implemented in PIETOOLS, including a GUI for declaration and conversion of the GPDE model – Sections IV and V.
3) *Equivalence of GPDEs and PIEs*: We show the map between GPDE and PIE solution is unitary, implying a solution of a GPDE model yields a corresponding solution of the associated PIE and vice versa. We then prove input-output and internal stability of the GPDE model is equivalent to that of the associated PIE. See Section VI.

D. Approach and Organization of the paper

To conclude the introduction, we now summarize the organization of the paper. First, we define the algebra of PI operators in Section II-A and the class of PIE models, along with a definition of solution in Section II-B. Second, we introduce the class of GPDE systems in Section III – defining the interconnection structure of PDE and ODE subsystems and including a definition of solution for both the PDE subsystem and interconnected GPDE. Third, for a given GPDE, we propose a admissibility condition for well-posedness which guarantees the existence of an equivalent PIE representation. We then derive formulae for the PIE representation of the PDE subsystem, and we combine these formulae with the ODE subsystem to obtain the PIE representation of the interconnected GPDE model (Sections IV and V). Next, in Section VI, we show that the map between PIE and GPDE solutions is unitary – proving equivalence between GPDE and PIE models in terms of both existence of solutions and stability of these solutions. Finally, in Section VIII, we consider several specific GPDE models – using the PIE representation and PIETOOLS to simulate the GPDE and prove stability. For brevity, we include only an outline of most proofs and refer to Appendices included in the full version of this paper [25] for extended proofs and non-essential definitions.

II. Notation, PI operators and PIEs

In addition to denoting the empty set, $\emptyset$ is occasionally used to denote a matrix or matrix-valued function with either zero row or column dimension and whose non-zero dimension can be inferred from context. We denote by $0_{m,n} \in \mathbb{R}^{m\times n}$ the matrix of all zeros, $0_n := 0_{n,n}$, and $I_n \in \mathbb{R}^{n\times n}$ the identity matrix. We use 0 and I for these matrices when dimensions are clear from context. $\mathbb{R}_+$ is the set of non-negative real numbers. The set of k-times continuously differentiable n-dimensional vector-valued functions on the interval $[a, b]$ is denoted by $C^n_k[a, b]$. $L^n_2[a, b]$ is the Hilbert space of n-dimensional vector-valued Lebesgue square-integrable functions on the interval $[a, b]$ equipped with the standard inner product. $L^{m,n}_\infty[a, b]$ is the Banach space of $m \times n$-dimensional essentially bounded measurable matrix-valued functions on $[a, b]$ equipped with the essential supremum singular value norm.

Normal font u or $u(t)$ typically implies that u or $u(t)$ is a scalar or finite-dimensional vector (e.g. $u(t) \in \mathbb{R}^n$), whereas the bold font, $\mathbf{x}$ or $\mathbf{x}(t)$, typically implies that $\mathbf{x}$ or $\mathbf{x}(t)$ is a scalar or vector-valued function (e.g. $\mathbf{u}(t) \in L^n_2[a, b]$). For a suitably differentiable function, $\mathbf{x}$, of spatial variable s, we use $\partial^j_s\mathbf{x}$ to denote the j-th order partial derivative $\frac{\partial^j \mathbf{x}}{\partial s^j}$. For a suitably differentiable function of time and possibly space, we denote $\dot{\mathbf{x}}(t) = \frac{\partial}{\partial t}\mathbf{x}(t)$. We use W^n_k to denote the Sobolev spaces $W^n_k[a, b] := \{\mathbf{u} \in L^n_2[a, b] \mid \partial^l_s\mathbf{u} \in L^n_2[a, b] \ \forall\ l \le k\}$ with inner product $\langle \mathbf{u}, \mathbf{v}\rangle_{W^n_k} = \sum_{i=0}^k \left\langle \partial^i_s\mathbf{u}, \partial^i_s\mathbf{v}\right\rangle_{L^n_2}$. Clearly, $W^n_0[a, b] = L^n_2[a, b]$. For given $n = \{n_0, \cdots, n_N\} \in \mathbb{N}^{N+1}$, we define the Cartesian product space $W^n := \prod_{i=0}^N W^{n_i}_i$ and for $\mathbf{u} = \{\mathbf{u}_0, \cdots, \mathbf{u}_N\} \in W^n$ and $\mathbf{v} = \{\mathbf{v}_0, \cdots, \mathbf{v}_N\} \in W^n$ we define the associated inner product as $\langle \mathbf{u}, \mathbf{v}\rangle_{W^n} = \sum_{i=0}^N \langle \mathbf{u}_i, \mathbf{v}_i\rangle_{W^{n_i}_i}$. We use $\mathbb{R}L^{m,n}_2[a, b]$ to denote the space $\mathbb{R}^m \times L^n_2[a, b]$ and for $\mathbf{x} = \begin{bmatrix} x_1 \\ \mathbf{x}_2\end{bmatrix} \in \mathbb{R}L^{m,n}_2$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ \mathbf{y}_2\end{bmatrix} \in \mathbb{R}L^{m,n}_2$, we define the associated inner product as

$$\left\langle \begin{bmatrix} x_1 \\ \mathbf{x}_2\end{bmatrix}, \begin{bmatrix} y_1 \\ \mathbf{y}_2\end{bmatrix} \right\rangle_{\mathbb{R}L^{m,n}_2} = x_1^T y_1 + \langle \mathbf{x}_2, \mathbf{y}_2\rangle_{L^n_2}.$$

Frequently, we omit the domain $[a, b]$ and simply write L^n_2, W^n_k, W^n, or $\mathbb{R}L^{m,n}_2$. For functions of time only ($L_2[\mathbb{R}_+]$ and $W_k[\mathbb{R}_+]$), we use the truncation operator

$$(P_T x)(t) := \begin{cases} x(t), & \text{if } t \le T \\ 0, & \text{otherwise.}\end{cases}$$

to denote the extended subspaces of such functions by $L_{2e}[\mathbb{R}_+]$ and $W_{ke}[\mathbb{R}_+]$ respectively as

$$L_{2e}[\mathbb{R}_+] := \{x \mid P_T x \in L_2[\mathbb{R}_+] \ \forall\ T \ge 0\},$$
$$W_{ke}[\mathbb{R}_+] := \{x \mid P_T x \in W_k[\mathbb{R}_+] \ \forall\ T \ge 0\}.$$

Finally, for normed spaces A, B, $\mathcal{L}(A, B)$ denotes the space of bounded linear operators from A to B equipped with the induced operator norm. $\mathcal{L}(A) := \mathcal{L}(A, A)$.

A. PI Operators: A $*$-algebra of bounded linear operators

The PI algebras are parameterized classes of operators on $\mathbb{R}L^{m,n}_2$ (the product space of $\mathbb{R}^m$ and L^n_2). The first of these is the algebra of 3-PI operators which map $L^n_2 \to L^n_2$.

Definition 1 (Separable Function)**.** *We say $R : [a, b]^2 \to \mathbb{R}^{p\times q}$ is separable if there exist $r \in \mathbb{N}$, $F \in L^{r\times p}_\infty[a, b]$ and $G \in L^{r\times q}_\infty[a, b]$ such that $R(s, \theta) = F(s)^T G(\theta)$.*

Definition 2 (3-PI operators, Π_3)**.** *Given $R_0 \in L^{p\times q}_\infty[a, b]$ and separable functions $R_1, R_2 : [a, b]^2 \to \mathbb{R}^{p\times q}$, we define the operator $\mathcal{P}_{\{R_i\}}$ for $\mathbf{v} \in L_2[a, b]$ as*

$$\left(\mathcal{P}_{\{R_i\}}\mathbf{v}\right)(s) := R_0(s)\mathbf{v}(s) + \int_a^s R_1(s, \theta)\mathbf{v}(\theta)d\theta + \int_s^b R_2(s, \theta)\mathbf{v}(\theta)d\theta. \quad (2)$$

Furthermore, we say an operator, $\mathcal{P}$, is 3-PI of dimension $p\times q$, denoted $\mathcal{P} \in [\Pi_3]_{p,q} \subset \mathcal{L}(L^q_2, L^p_2)$, if there exist functions R_0 and separable functions R_1, R_2 such that $\mathcal{P} = \mathcal{P}_{\{R_i\}}$.

For any $p \in \mathbb{N}$, $[\Pi_3]_{p,p}$ is a *-algebra, being closed under addition, composition, scalar multiplication, and adjoint (See [25, Appendix H]). Closed-form expressions for the composition, adjoint, etc. of 3-PI operators in terms of the parameters R_i are also included in [25, Appendix H]. The algebra of 3-PI operators can be extended to $\mathcal{L}(\mathbb{R}L^{m,p}_2, \mathbb{R}L^{n,q}_2)$ as follows.

Definition 3 (4-PI operators)**.** *Given $P \in \mathbb{R}^{m\times n}$, $Q_1 \in L^{m\times q}_\infty$, $Q_2 \in L^{p\times n}_\infty$, and R_0, R_1, R_2 with $\mathcal{P}_{\{R_i\}} \in [\Pi_3]_{p,q}$, we say $\mathcal{P} = \mathcal{P}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right] \in \mathcal{L}(\mathbb{R}L^{m,p}_2, \mathbb{R}L^{n,q}_2)$ if*

$$\left(\mathcal{P}\begin{bmatrix} u \\ \mathbf{v}\end{bmatrix}\right)(s) := \begin{bmatrix} Pu + \int_a^b Q_1(\theta)\mathbf{v}(\theta)d\theta \\ Q_2(s)u + \left(\mathcal{P}_{\{R_i\}}\mathbf{v}\right)(s)\end{bmatrix}. \quad (3)$$

Furthermore, we say $\mathcal{P}$, is 4-PI, denoted $\mathcal{P} \in [\Pi_4]^{m,n}_{p,q}$, if there exist $P, Q_1, Q_1, R_0, R_1, R_2$ such that $\mathcal{P} = \mathcal{P}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]$.

Definition 4 (*-subalgebras of Π_i with polynomial parameters)**.** *We say $\mathcal{P} \in [\Pi^p_3]_{p,q}$ if there exist polynomials R_i of appropriate dimension such that $\mathcal{P} = \mathcal{P}_{\{R_i\}}$. We say $\mathcal{P} \in [\Pi^p_4]^{m,n}_{p,q}$ if there exist matrix P and polynomials Q_i, R_i of appropriate dimension such that $\mathcal{P} = \mathcal{P}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]$.*

Parametric Representation of Operations on Π_i: Algebraic operations on Π_i are defined by algebraic operations on the parameters which represent these operators. Specifically, corresponding to Π_3 and Π_4 let us associate the corresponding parameter spaces

$[\Gamma_3]_{p,q} := \{\{R_0, R_1, R_2\} : R_i \in L^{p\times q}_\infty, R_1, R_2 \text{ are separable}\}$,

$$[\Gamma_4]^{m,p}_{n,q} := \left\{ \left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] \;:\; \begin{array}{l} P \in \mathbb{R}^{m\times n},\, Q_1 \in L_\infty^{m\times q}, \\ Q_2 \in L_\infty^{p\times n},\, \{R_i\} \in [\Gamma_3]_{p,q} \end{array} \right\}.$$

Then if the parametric maps $\mathbf{P}^4_\times, \mathbf{P}^4_+ : [\Gamma_4]^{m,m}_{p,p} \times [\Gamma_4]^{m,m}_{p,p} \to [\Gamma_4]^{m,m}_{p,p}$, $\mathbf{P}^4_T : [\Gamma_4]^{m,m}_{p,p} \to [\Gamma_4]^{m,m}_{p,p}$ are as defined in [25, Lemmas 35-37], for any $S, T \in [\Gamma_4]^{m,m}_{p,p}$, we have

$$\mathcal{P}\left[\mathbf{P}^4_\times(S,T)\right] = \mathcal{P}\left[S\right]\mathcal{P}\left[T\right], \quad \mathcal{P}\left[\mathbf{P}^4_T(S)\right] = \mathcal{P}\left[S\right]^*,$$
$$\mathcal{P}\left[\mathbf{P}^4_+(S,T)\right] = \mathcal{P}\left[S\right] + \mathcal{P}\left[T\right].$$

B. Partial Integral Equations

A Partial Integral Equation (PIE) is an extension of the state-space representation of ODEs (vector-valued first-order differential equations on $\mathbb{R}^n$) to spatially-distributed states on the product space $\mathbb{R}L_2$. Mirroring the 9-matrix optimal control framework developed for state-space systems, a PIE system includes is parameterized by twelve 4-PI operators as

$$\begin{bmatrix} \mathcal{T}\dot{\underline{\mathbf{x}}}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{x}}(t) \\ w(t) \\ u(t) \end{bmatrix} - \begin{bmatrix} \mathcal{T}_w \dot{w}(t) + \mathcal{T}_u \dot{u}(t) \\ 0 \\ 0 \end{bmatrix},$$
$$\underline{\mathbf{x}}(0) = \underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{m,n}[a,b], \tag{4}$$

where $z(t) \in \mathbb{R}^{n_z}$ is the regulated output, $y(t) \in \mathbb{R}^{n_y}$ is the sensed output, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance, $u(t) \in \mathbb{R}^{n_u}$ is the control input, and $\underline{\mathbf{x}}(t) \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$ is the internal state.

No Spatial Derivatives or Boundary Conditions: A PIE system does permit spatial derivatives – only a first-order derivative with respect to time. The state of the PIE system, $\underline{\mathbf{x}} \in \mathbb{R}L_2[a,b]$ is not differentiable and consequently, no BCs are possible in the PIE framework.

Before formalizing the definition of solution for a PIE system, let us note two significant features of this definition. First, we observe that PIEs allow for the dynamics to depend on the time-derivative of the input signals: $\partial_t(\mathcal{T}_w w)$ and $\partial_t(\mathcal{T}_u u)$. Through some slight abuse of notation, in this paper we will use expressions such as $\mathcal{T}_w \dot{w}$ to represent $\partial_t(\mathcal{T}_w w)$. These terms are included in order to allow for PIEs to represent certain classes of PDEs wherein signals enter through the BCs.

Second, the internal state of the solution of a PIE system is required to be Frechét differentiable with respect to the $\mathcal{T}$-norm which is defined as $\|\underline{\mathbf{x}}\|_{\mathcal{T}} := \|\mathcal{T}\underline{\mathbf{x}}\|_{\mathbb{R}L_2}$, for $\underline{\mathbf{x}} \in \mathbb{R}L_2$.

Notation: Finally, and for brevity, we collect the 12 PI parameters which define a PIE system in Eq. (4) and introduce the shorthand notation $\mathbf{G}_{\text{PIE}}$ which represents the labelled tuple of such system parameters as

$$\mathbf{G}_{\text{PIE}} = \{\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u, \mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_{11}, \mathcal{D}_{12}, \mathcal{D}_{21}, \mathcal{D}_{22}\}.$$

When this shorthand notation is used, it is presumed that all parameters have appropriate dimensions.

We now define a notion of solution for a PIE system.

Definition 5 (Solution of a PIE system)**.** *For given inputs $u \in L_{2e}^{n_u}[\mathbb{R}_+]$, $w \in L_{2e}^{n_w}[\mathbb{R}_+]$ with $(\mathcal{T}_u u)(\cdot, s) \in W_{1e}^{n_x+n_{\hat{\mathbf{x}}}}[\mathbb{R}_+]$ and $(\mathcal{T}_w w)(\cdot, s) \in W_{1e}^{n_x+n_{\hat{\mathbf{x}}}}[\mathbb{R}_+]$ for all $s \in [a,b]$ and $\underline{\mathbf{x}}^0(t) \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$, we say that $\{\underline{\mathbf{x}}, z, y\}$ satisfies the PIE defined by $\mathbf{G}_{\text{PIE}} = \{\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u, \mathcal{A}, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_{ij}\}$ with initial condition $\underline{\mathbf{x}}^0$ and input $\{w, u\}$ if $z \in L_{2e}^{n_z}[\mathbb{R}_+]$, $y \in L_{2e}^{n_y}[\mathbb{R}_+]$, $\underline{\mathbf{x}}(t) \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}[a,b]$ for all $t \geq 0$, $\underline{\mathbf{x}}$ is Frechét differentiable with respect to the $\mathcal{T}$-norm almost everywhere on $\mathbb{R}_+$, $\underline{\mathbf{x}}(0) = \underline{\mathbf{x}}^0$, and Eq.* (4) *is satisfied for almost all $t \in \mathbb{R}_+$.*

III. GPDEs: A Generalized Class of Linear Models

Having introduced PIE systems and PI operators, we now parameterize the class of ODE-PDE models for which we may define associated PIE systems. To simplify the notation and analysis, we will represent these models as the interconnection of ODE and PDE subsystems – See Figure 3. This class of ODE-PDE models will be referred to as Generalized Partial Differential Equations (GPDEs). The parameterization of the ODE subsystem is defined in Section III-A, the parameterization of the PDE subsystem is defined in Section III-B, and the subsystems are combined in Section III-C.

A. ODE Subsystem

The ODE subsystem of the GPDE model, illustrated in Figure 1, is a typical state-space representation with real-valued inputs and outputs. These inputs and outputs are finite-dimensional and include both the interconnection with the PDE subsystem and the inputs and outputs of the GPDE model as a whole. Specifically, we partition both the input and output signals into 3 components, differentiating these channels by function. The input channels are: the control input to the GPDE ($u(t) \in \mathbb{R}^{n_u}$), the exogenous disturbance/source driving the GPDE ($w(t) \in \mathbb{R}^{n_w}$) and the internal feedback input ($r(t) \in \mathbb{R}^{n_r}$) which is the output of the PDE subsystem. The output channels of the ODE subsystem are: the regulated output of the GPDE ($z(t) \in \mathbb{R}^{n_z}$); the sensed outputs of the GPDE ($y(t) \in \mathbb{R}^{n_y}$); and the output from the ODE subsystem which becomes the input to the PDE subsystem ($v(t) \in \mathbb{R}^{n_v}$).

Definition 6 (Solution of an ODE Subsystem)**.** *Given matrices A, B_{xw}, B_{xu}, B_{xr}, C_z, D_{zw}, D_{zu}, D_{zr}, C_y, D_{yw}, D_{yu}, D_{yr}, C_v, D_{vw}, D_{vu} of appropriate dimension, we say $\{x, z, y, v\}$ with $\{x(t), z(t), y(t), v(t)\} \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_v}$ satisfies the ODE with initial condition $x^0 \in \mathbb{R}^{n_x}$ and input $\{w, u, r\}$ if x is differentiable, $x(0) = x^0$ and for $t \geq 0$*

$$\begin{bmatrix} \dot{x}(t) \\ \hline z(t) \\ y(t) \\ v(t) \end{bmatrix} = \left[\begin{array}{c|ccc} A & B_{xw} & B_{xu} & B_{xr} \\ \hline C_z & D_{zw} & D_{zu} & D_{zr} \\ C_y & D_{yw} & D_{yu} & D_{yr} \\ C_v & D_{vw} & D_{vu} & 0 \end{array}\right] \begin{bmatrix} x(t) \\ \hline w(t) \\ u(t) \\ r(t) \end{bmatrix}. \tag{5}$$

Notation: For brevity, we collect all matrix parameters from the ODE subsystem in (5) and introduce the shorthand notation $\mathbf{G}_{\text{o}}$ which represents the labelled tuple of such parameters as

$$\mathbf{G}_{\text{o}} = \{A, B_{xw}, B_{xu}, B_{xr}, C_z, D_{zw}, D_{zu}, D_{zr}, C_y, D_{yw}, D_{yu}, D_{yr}, C_v, D_{vw}, D_{vu}\}. \tag{6}$$

When this shorthand notation is used, it is presumed that all parameters have appropriate dimensions.

B. PDE Subsystem

Our parameterization of the PDE subsystem is divided into three parts: the continuity constraints, the in-domain dynamics, and the BCs. The continuity constraints specify the existence of partial derivatives and boundary values for each state as required by the in-domain dynamics and BCs. The BCs are represented as a real-valued algebraic constraint subsystem which maps the distributed state and inputs to a vector of boundary values. The in-domain dynamics (or generating equation) specify the time derivative of the state, $\hat{\mathbf{x}}(t, s)$, at every point in the interior of the domain, and are

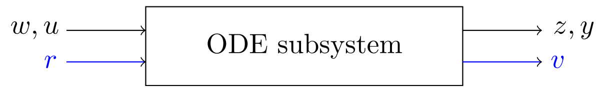

Fig. 1: Depiction of the ODE subsystem for use in defining a GPDE. All external input signals in the GPDE model pass through the ODE subsystem and are labelled as $u(t) \in \mathbb{R}^{n_u}$ and $w(t) \in \mathbb{R}^{n_w}$, corresponding to control input and disturbance/forcing input. Likewise all external outputs pass through the ODE subsystem and are labelled $y(t) \in \mathbb{R}^{n_y}$ and $z(t) \in \mathbb{R}^{n_z}$, corresponding to measured output and regulated output. All interaction with the PDE subsystem is routed through two vector-valued signals, where $r(t) \in \mathbb{R}^{n_r}$ is the sole output of the PDE subsystem and $v(t) \in \mathbb{R}^{n_v}$ is the sole input to the PDE subsystem.

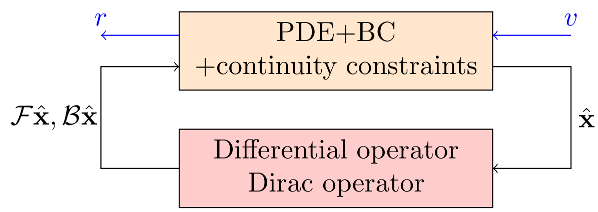

Fig. 2: Depiction of the PDE subsystem for use in defining a GPDE. All interaction of the PDE subsystem with the ODE subsystem is routed through the two vector-valued signals, r and v, where $r(t) \in \mathbb{R}^{n_r}$ is an output of the PDE subsystem (and input to the ODE subsystem) and $v(t) \in \mathbb{R}^{n_v}$ is an input to the PDE subsystem (and output from the ODE subsystem). Although there are no external inputs and outputs of the GPDE, such signals can be routed to and from the PDE subsystem through the ODE subsystem using r and v.

expressed using integral, dirac, and N^{th}-order spatial derivative operators. The PDE subsystem is illustrated in Figure 2. For simplicity, no external inputs or outputs are defined for the PDE subsystem, since these external signals may be included by routing the desired signal through the ODE subsystem using the internal signals, $v(t)$ and $r(t)$.

B.1 The continuity constraint

The 'continuity constraint' partitions the state vector of the PDE subsystem, $\hat{\mathbf{x}}(t,\cdot)$, and specifies the differentiability properties of each partition as required for existence of the partial derivatives in the generator and limit values in the boundary condition. This partition is defined by the parameter $n \in \mathbb{N}^{N+1} = \{n_0, \cdots n_N\}$, wherein n_i specifies the dimension of the ith partition vector so that $\hat{\mathbf{x}}_i(t,s) \in \mathbb{R}^{n_i}$. The partitions are ordered by increasing differentiability so that

$$\hat{\mathbf{x}}(t,\cdot) = \begin{bmatrix} \hat{\mathbf{x}}_0(t,\cdot) \\ \vdots \\ \hat{\mathbf{x}}_N(t,\cdot) \end{bmatrix} \in W^n := \begin{bmatrix} W_0^{n_0} \\ \vdots \\ W_N^{n_N} \end{bmatrix}.$$

Given the partition defined by $n \in \mathbb{N}^{N+1}$, and given $\hat{\mathbf{x}} \in W^n$, we would like to list all well-defined partial derivatives of $\hat{\mathbf{x}}$. To do this, we first define $n_{\hat{\mathbf{x}}} := |n|_1 = \sum_{i=0}^N n_i$ to be the number of states in $\hat{\mathbf{x}}$, $n_{S_i} := \sum_{j=i}^N n_j \le n_{\hat{\mathbf{x}}}$ to be the total number of i-times differentiable states, and $n_S = \sum_{i=1}^N n_{S_i}$ to be the total number of possible partial derivatives of $\hat{\mathbf{x}}$ as permitted by the continuity constraint.

Notation: For indexed vectors (such as n or $\hat{\mathbf{x}}$) we occasionally use the notation $\hat{\mathbf{x}}_{i:j}$ to denote the components i to j. Specifically, $\hat{\mathbf{x}}_{i:j} = \text{col}(\hat{\mathbf{x}}_i, \cdots, \hat{\mathbf{x}}_j)$, $n_{i:j} := \sum_{k=i}^j n_k$ and $n_{S_{i:j}} = \sum_{k=i}^j n_{S_k}$.

Next, we define the selection operator $S^i : \mathbb{R}^{n_{\hat{\mathbf{x}}}} \to \mathbb{R}^{n_{S_i}}$ which is used to select only those states in $\hat{\mathbf{x}}$ which are at least i-times differentiable. Specifically, for $\hat{\mathbf{x}} \in W^n$, we have $S^i = \begin{bmatrix} 0_{n_{S_i} \times n_{\hat{\mathbf{x}}} - n_{S_i}} & I_{n_{S_i}} \end{bmatrix}$, so that $(S^i\hat{\mathbf{x}})(s) = \begin{bmatrix} \hat{\mathbf{x}}_i(s) \\ \vdots \\ \hat{\mathbf{x}}_N(s) \end{bmatrix}$.
We may now conveniently represent all well-defined ith-order partial derivatives of $\hat{\mathbf{x}}$ as $\partial_s^i S^i \hat{\mathbf{x}}$ so that

$$(\partial_s^i S^i \hat{\mathbf{x}})(s) = \begin{bmatrix} \partial_s^i \hat{\mathbf{x}}_i(s) \\ \vdots \\ \partial_s^i \hat{\mathbf{x}}_N(s) \end{bmatrix} \text{ and } (\mathcal{F}\hat{\mathbf{x}})(s) := \begin{bmatrix} \hat{\mathbf{x}}(s) \\ (\partial_s S \hat{\mathbf{x}})(s) \\ \vdots \\ (\partial_s^N S^N \hat{\mathbf{x}})(s) \end{bmatrix}$$

where $\mathcal{F}$ concatenates all the $\partial_s^i S^i \hat{\mathbf{x}}$ for $i = 0, \cdots, N$ — creating an ordered list including both the PDE state, $\hat{\mathbf{x}}$, as well as all n_S possible partial derivatives of $\hat{\mathbf{x}}$ as permitted by the continuity constraint and the vector $(\mathcal{F}\hat{\mathbf{x}})(s) \in \mathbb{R}^{n_S + n_{\mathbf{x}}}$.

This notation also allows us to specify all well-defined boundary values of $\hat{\mathbf{x}} \in W^n$ and of its partial derivatives. Specifically, we may construct $(\mathcal{C}\hat{\mathbf{x}})(s) \in \mathbb{R}^{n_S}$, the vector of all absolutely continuous functions generated by $\hat{\mathbf{x}}$ and its partial derivatives. Using $\mathcal{C}\hat{\mathbf{x}}$, we may then construct $\mathcal{B}\hat{\mathbf{x}} \in \mathbb{R}^{2n_S}$, the list all possible boundary values of $\hat{\mathbf{x}} \in W^n$. Specifically, $\mathcal{C}\hat{\mathbf{x}}$ and $\mathcal{B}\hat{\mathbf{x}}$ are defined as

$$\mathcal{C}\hat{\mathbf{x}}(s) = \begin{bmatrix} (S\hat{\mathbf{x}})(s) \\ (\partial_s S^2 \hat{\mathbf{x}})(s) \\ \vdots \\ (\partial_s^{N-1} S^N \hat{\mathbf{x}})(s) \end{bmatrix} \text{ and } \mathcal{B}\hat{\mathbf{x}} = \begin{bmatrix} (\mathcal{C}\hat{\mathbf{x}})(a) \\ (\mathcal{C}\hat{\mathbf{x}})(b) \end{bmatrix}. \quad (7)$$

Combining $\mathcal{F}\hat{\mathbf{x}}$ and $\mathcal{B}\hat{\mathbf{x}}$, we obtain a complete list of all well-defined terms which may appear in either the in-domain dynamics or BCs.

B.2 Boundary Conditions (BCs)

Given the notational framework afforded by the continuity condition, and equipped with our list of well-defined terms ($\mathcal{F}\hat{\mathbf{x}}$ and $\mathcal{B}\hat{\mathbf{x}}$), we may now parameterize a generalized class of BCs consisting of a combination of boundary values, integrals of the PDE state, and the effect of the input signal, v. Specifically, the BCs are parameterized by the square integrable function $B_I : [a,b] \to \mathbb{R}^{n_{BC} \times (n_S + n_{\hat{\mathbf{x}}})}$ and matrices $B_v \in \mathbb{R}^{n_{BC} \times n_v}$ and $B \in \mathbb{R}^{n_{BC} \times 2n_S}$ as

$$\int_a^b B_I(s)(\mathcal{F}\hat{\mathbf{x}}(t))(s)ds + \begin{bmatrix} B_v & -B \end{bmatrix} \begin{bmatrix} v(t) \\ \mathcal{B}\hat{\mathbf{x}}(t) \end{bmatrix} = 0 \quad (8)$$

where n_{BC} is the number of user-specified BCs. For reasons of well-posedness, as discussed in Section IV, we typically require $n_{BC} = n_S$. If fewer BCs are available, it is likely that the continuity constraint is too strong – the user is advised to consider whether all the partial derivatives and boundary values are actually used in defining the PDE subsystem.

Now that we have parameterized a general set of BCs, we embed these BCs in what is typically referred to as the domain of the infinitesimal generator – which combines the BCs and continuity constraints into a set of acceptable states.

$$X_v := \left\{ \begin{matrix} \hat{\mathbf{x}} \in W^n[a,b] : \\ \int_a^b B_I(s)(\mathcal{F}\hat{\mathbf{x}})(s)ds + \begin{bmatrix} B_v & -B \end{bmatrix} \begin{bmatrix} v \\ \mathcal{B}\hat{\mathbf{x}} \end{bmatrix} = 0 \end{matrix} \right\} \quad (9)$$

The set X_v is used to restrict the state and initial conditions as $\hat{\mathbf{x}}(t) \in X_{v(t)}$ and $\hat{\mathbf{x}}(0) = \hat{\mathbf{x}}^0 \in X_{v(0)}$.

Notation: For convenience, we collect all the parameters which define the constraint in Eq. (8) and use $\mathbf{G}_\mathrm{b}$ to represent the labelled tuple of such parameters as

$$\mathbf{G}_\mathrm{b} = \{B,\ B_I,\ B_v\}. \quad (10)$$

When this shorthand notation is used, it is presumed that all parameters have appropriate dimensions.

B.3 In-Domain Dynamics of the PDE Subsystem

Having specified the continuity constraint and BCs using $\{n, \mathbf{G}_\mathrm{b}\}$, we once again use our list of well-defined terms ($\mathcal{F}\hat{\mathbf{x}}$ and $\mathcal{B}\hat{\mathbf{x}}$) to define the in-domain dynamics of the PDE subsystem and the output to the ODE subsystem. These dynamics are parameterized by the functions $A_0(s)$, $A_1(s,\theta)$, $A_2(s,\theta) \in \mathbb{R}^{n_{\hat{\mathbf{x}}}\times(n_S+n_{\hat{\mathbf{x}}})}$, $C_r(s) \in \mathbb{R}^{n_r\times(n_S+n_{\hat{\mathbf{x}}})}$, $B_{xv}(s) \in \mathbb{R}^{n_{\hat{\mathbf{x}}}\times n_v}$, $B_{xb}(s) \in \mathbb{R}^{n_{\hat{\mathbf{x}}}\times 2n_S}$, and matrices $D_{rv} \in \mathbb{R}^{n_r\times n_v}$ and $D_{rb}(s) \in \mathbb{R}^{n_r\times 2n_S}$ as follows.

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}}(t,s) \\ r(t) \end{bmatrix} = \begin{bmatrix} A_0(s)(\mathcal{F}\hat{\mathbf{x}}(t))(s) \\ 0 \end{bmatrix} + \begin{bmatrix} B_{xv}(s) & B_{xb}(s) \\ 0 & D_{rb} \end{bmatrix} \begin{bmatrix} v(t) \\ \mathcal{B}\hat{\mathbf{x}}(t) \end{bmatrix}$$
$$+ \begin{bmatrix} \int_a^s A_1(s,\theta)(\mathcal{F}\hat{\mathbf{x}}(t))(\theta)d\theta + \int_s^b A_2(s,\theta)(\mathcal{F}\hat{\mathbf{x}}(t))(\theta)d\theta \\ \int_a^b C_r(\theta)(\mathcal{F}\hat{\mathbf{x}}(t))(\theta)d\theta \end{bmatrix} \quad (11)$$

Note: Many commonly used PDE models are defined solely by A_0. For example, if we consider $u_t = \lambda u + u_{ss}$, then $A_0 = \begin{bmatrix} \lambda & 0 & 1 \end{bmatrix}$ and all other parameters are zero.

The motivation for the parameters in this representation (other than A_0) can be summarized as follows: The kernels A_1, A_2 model non-local effects of the distributed state; the function B_{xv} represents the distributed effect of the disturbance/forcing function v on the generating equation; and B_{xb} represents the distributed effect of the boundary values on the generating equation. In addition: C_r is used to model the influence of the PDE subsystem state on the dynamics and outputs of the ODE subsystem; and D_{rb} is used to model the effect of boundary values of the PDE subsystem on the dynamics and outputs of the ODE subsystem.

Notation: For convenience, we collect all parameters from the in-domain dynamics of the PDE subsystem (Eq. (11)) and use $\mathbf{G}_\mathrm{p}$ to represent the labelled tuple of such parameters as

$$\mathbf{G}_\mathrm{p} = \{A_0,\ A_1,\ A_2,\ B_{xv},\ B_{xb},\ C_r,\ D_{rb}\}. \quad (12)$$

When this shorthand notation is used, it is presumed that all parameters have appropriate dimensions. We may now define a notion of solution for a PDE subsystem.

Definition 7 (Solution of a PDE Subsystem)**.** *For given* $\hat{\mathbf{x}}^0 \in X_{v(0)}$ *and* $v \in L_{2e}^{n_v}[\mathbb{R}_+]$ *with* $B_v v \in W_{1e}^{2n_S}[\mathbb{R}_+]$, *we say that* $\{\hat{\mathbf{x}}, r\}$ *satisfies the PDE subsystem defined by* $n \in \mathbb{N}^{N+1}$ *and* $\{\mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ *(defined in Eqs.* (10) *and* (12)*) with initial condition* $\hat{\mathbf{x}}^0$ *and input* v *if* $r \in L_{2e}^{n_r}[\mathbb{R}_+]$, $\hat{\mathbf{x}}(t) \in X_{v(t)}$ *for all* $t \geq 0$, $\hat{\mathbf{x}}$ *is Fréchet differentiable with respect to the* L_2*-norm almost everywhere on* $\mathbb{R}_+$, $\hat{\mathbf{x}}(0) = \hat{\mathbf{x}}^0$, *and Eq.* (11) *is satisfied for almost all* $t \geq 0$.

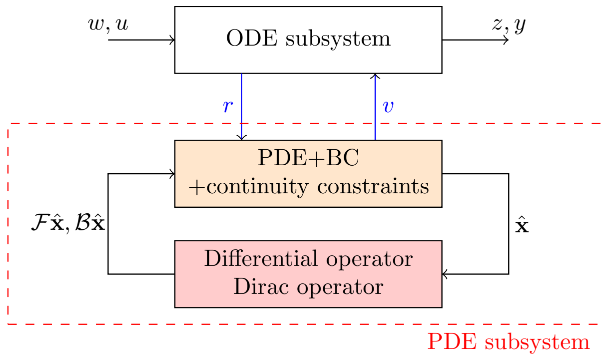

Fig. 3: A GPDE is the interconnection of an ODE subsystem (an ODE with finite-dimensional inputs w, u, v and outputs z, y, r) with a PDE subsystem (N^{th}-order PDEs and BCs with finite-dimensional input r and output v). The BCs and internal dynamics of the PDE subsystem are specified in terms of all well-defined spatially distributed terms as encoded in $\mathcal{F}\hat{\mathbf{x}}(t)$ and all well-defined limit values as encoded in $\mathcal{B}\hat{\mathbf{x}}(t)$.

C. GPDE: Interconnection of ODE and PDE Subsystems

Given the definition of ODE and PDE subsystems, a GPDE model is the mutual interconnection of these subsystems through the interconnection signals (r, v) and is collectively defined by Eqs. (5)-(11). This interconnection is illustrated in Figure 3.

Given suitable inputs w,u, for a GPDE model, parameterized by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$, we define the continuity constraint and time-varying BCs by $\{x(t), \hat{\mathbf{x}}(t)\} \in \mathcal{X}_{w(t),u(t)}$ where

$$\mathcal{X}_{w,u} := \left\{ \begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix} \in \mathbb{R}^{n_x} \times X_v \;\middle|\; v = C_v x + D_{vw} w + D_{vu} u \right\}. \quad (13)$$

We now define the solution of a GPDE model as follows.

Definition 8 (Solution of a GPDE model)**.** *For given* $\{x^0,\ \hat{\mathbf{x}}^0\} \in \mathcal{X}_{w(0),u(0)}$ *and* $w \in L_{2e}^{n_w}[\mathbb{R}_+]$, $u \in L_{2e}^{n_u}[\mathbb{R}_+]$ *with* $B_v D_{vw} w \in W_{1e}^{2n_S}[\mathbb{R}_+]$ *and* $B_v D_{vu} u \in W_{1e}^{2n_S}[\mathbb{R}_+]$, *we say that* $\{x,\ \hat{\mathbf{x}},\ z,\ y,\ v,\ r\}$ *satisfies the GPDE defined by* $\{n,\ \mathbf{G}_\mathrm{o},\ \mathbf{G}_\mathrm{b},\ \mathbf{G}_\mathrm{p}\}$ *(See Equations* (6), (10) *and* (12)*) with initial condition* $\{x^0,\ \hat{\mathbf{x}}^0\}$ *and input* $\{w,\ u\}$ *if* $z \in L_{2e}^{n_z}[\mathbb{R}_+]$, $y \in L_{2e}^{n_y}[\mathbb{R}_+]$, $v \in L_{2e}^{n_v}[\mathbb{R}_+]$, $r \in L_{2e}^{n_r}[\mathbb{R}_+]$, $\{x(t), \hat{\mathbf{x}}(t)\} \in \mathcal{X}_{w(t),u(t)}$ *for all* $t \geq 0$, x *is differentiable almost everywhere on* $\mathbb{R}_+$, $\hat{\mathbf{x}}$ *is Fréchet differentiable with respect to the* L_2*-norm almost everywhere on* $\mathbb{R}_+$, $x(0) = x^0$, $\hat{\mathbf{x}}(0) = \hat{\mathbf{x}}^0$, *and Eqs.* (5)-(11) *are satisfied for almost all* $t \geq 0$.

D. Illustrative Example of the GPDE Representation

In this subsection, we illustrate the process of identifying the GPDE parameters of a given system. We begin this process by introducing a conventional PDE representation. We then divide the system into ODE and PDE subsystems and focus on identifying the continuity constraint for the PDE subsystem – always the least restrictive constraint necessary for existence of the partial derivatives and boundary values. We then proceed to identify the remaining parameters.

Illustration 1 (Damped Wave equation with delay and motor dynamics) Let us consider a wave equation

$$\ddot{\eta}(t,s) = \partial_s^2 \eta(t,s), \text{ defined on the interval } s \in [0,1], \quad (14)$$

to which we apply the typical boundary feedback law

$\eta_s(t,1) = -\eta_t(t,1)$, but where there is an actuator disturbance and where the control is implemented using a DC motor and where the output from the DC motor experiences a distributed delay, so that $\eta_s(t,1) = w(t) + \int_{-\tau}^{0} \mu(s/\tau)T(t+s)$ where $T(t)$ is the output of the DC motor and $\mu(s)$ is a given multiplier. The delay is represented using a transport equation with distributed state $p(t,s)$ on the interval $[-1,0]$ so that

$$\dot{p}(t,s) = \frac{1}{\tau}p_s(t,s), p(t,0) = T(t), \eta(t,1) = \int_{-1}^{0} \mu(s)p(t,s)ds.$$

The DC motor dynamics relate the voltage input, $u(t)$ to the torque $T(t)$ through the current, $i(t)$ as

$$\dot{i}(t) = \frac{-R}{L}i(t) + u(t) \qquad T(t) = K_t i(t).$$

Finally, the sensed output is the typical feedback signal $\eta_t(1,t)$ and the regulated output is a combination of the integral of the displacement and controller effort so that

$$z(t) = \begin{bmatrix} \int_0^1 \eta(t,s)ds \\ u(t) \end{bmatrix}, \qquad y(t) = \eta_t(1,t).$$

Since we require all states to have first order derivatives in time and be defined on same spatial interval, we introduce the change of variables $\zeta_1 = \eta$, $\zeta_2 = \dot{\eta}$, $\zeta_3(t,s) = p(t,s-1)$. A complete list of equations is now $\dot{i}(t) = \frac{-R}{L}i(t) + u(t)$ and

$$\begin{aligned}
&\dot{\zeta}_1(t,s) = \zeta_2(t,s),\ \dot{\zeta}_2(t,s) = \partial_s^2\zeta_1(t,s),\\
&\dot{\zeta}_3(t,s) = \frac{1}{\tau}\partial_s\zeta_3(t,s),\ \zeta_1(t,0) = 0,\ \zeta_3(t,1) = K_t i(t),\\
&\partial_s\zeta_1(t,1) = w(t) + \int_0^1 \mu(s-1)\zeta_3(t,s)ds,\\
&z(t) = \begin{bmatrix} \int_0^1 \zeta_1(t,s)ds \\ u(t) \end{bmatrix},\ y(t) = \zeta_2(t,1),\ s\in[0,1], \quad t \geq 0.
\end{aligned}$$

ODE Subsystem: We start by identifying the parameters of the ODE subsystem. Since $i(t)$ is the only finite dimensional state we set $x(t) = i(t)$ to get $\dot{x}(t) = \frac{-R}{L}x(t) + u(t)$. The ODE subsystem influences the PDE subsystem via signals $w(t)$ and $T(t)$. The effect of the PDE subsystem on the regulated and observed outputs (z and y, respectively) is routed through $r(t)$. The outputs, z, y and internal signals, v, r, are now defined as

$$\begin{aligned}
&v(t) = \begin{bmatrix} T(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} K_t \\ 0 \end{bmatrix} i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t),\\
&r(t) = \begin{bmatrix} \int_0^1 \zeta_1(t,s)ds \\ \zeta_2(t,1) \end{bmatrix}, \quad \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ u(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} r(t).
\end{aligned}$$

Expressing these equations in the form of Eq. (5), we obtain

$$\begin{bmatrix} \dot{x}(t) \\ \hline z(t) \\ y(t) \\ v(t) \end{bmatrix} = \left[\begin{array}{c|cccc} -R/L & 0 & 1 & 0 \\ \hline \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ 0 & 0 & 0 & \begin{bmatrix} 0 & 1 \end{bmatrix} \\ \begin{bmatrix} K_t \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right] \begin{bmatrix} x(t) \\ \hline w(t) \\ u(t) \\ r(t) \end{bmatrix}.$$

Extracting the submatrices of this ODE subsystem, we obtain an expression for $\mathbf{G}_\mathrm{o}$ which has the following nonzero parameters: $A = \frac{-R}{L}$, $B_{xu} = 1$, $D_{yr} = \begin{bmatrix} 0 & 1 \end{bmatrix}$,

$$D_{zu} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_v = \begin{bmatrix} K_t \\ 0 \end{bmatrix},\ D_{vw} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ D_{zr} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

PDE subsystem: Next, we need to define: n, $\mathbf{G}_\mathrm{b}$, and $\mathbf{G}_\mathrm{p}$.
Continuity Constraint: To identify the continuity constraint, n, we consider the required partial derivatives and limit values for the three distributed states: ζ_1, ζ_2 and ζ_3. For ζ_1, $\partial_s^2\zeta_1$ appears in the in-domain dynamics and the BCs involve $\zeta_1(t,0)$ and $\partial_s\zeta_1(t,1)$. The least restrictive continuity constraint which guarantees existence of all three terms is $\zeta_1 \in \hat{\mathbf{x}}_2$. Next, no partial derivatives of ζ_2 are needed, but the limit value $\zeta(t,1)$ appears in the BCs – so we restrict $\zeta_2 \in \hat{\mathbf{x}}_1$. Finally, $\partial_s\zeta_3$ appears in the in-domain dynamics and $\zeta_3(t,1)$ appears in the BCs – implying $\zeta_3 \in \hat{\mathbf{x}}_1$. We conclude that $n = \{n_0, n_1, n_2\} = \{0,2,1\}$ and the GPDE state is

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}}_0 \\ \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{bmatrix} := \begin{bmatrix} \emptyset \\ \begin{bmatrix} \zeta_2(t,s) \\ \zeta_3(t,s) \end{bmatrix} \\ \zeta_1(t,s) \end{bmatrix}$$

Boundary Conditions: For this definition of the continuity constraint, n, we have $n_{\hat{\mathbf{x}}} = 3, n_{S_0} = 3, n_{S_1} = 3, n_{S_2} = 1$ and $n_S = 4$ – i.e., there are three 0^{th}-order, three 1^{st}-order and one 2^{nd}-order differentiable states. In addition, $n_{\hat{\mathbf{x}}} + n_S = 7$ indicates there are 7 possible distributed terms in $\mathcal{F}\hat{\mathbf{x}}$ and $2n_S = 8$ indicates there are 8 possible limit values in $\mathcal{B}\hat{\mathbf{x}}$. Specifically, recalling that $S^i\hat{\mathbf{x}}$ is the vector of all i^{th} order differentiable states, we have

$$S^0\hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}}_0 \\ \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \zeta_2 \\ \zeta_3 \\ \zeta_1 \end{bmatrix},\ S^1\hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \zeta_2 \\ \zeta_3 \\ \zeta_1 \end{bmatrix},\ S^2\hat{\mathbf{x}} = \hat{\mathbf{x}}_2 = \zeta_1,$$

$$\mathcal{F}\hat{\mathbf{x}} = \mathrm{col}(\zeta_2, \zeta_3, \zeta_1, \partial_s\zeta_2, \partial_s\zeta_3, \partial_s\zeta_1, \partial_s^2\zeta_1),$$

$$\mathcal{C}\hat{\mathbf{x}} = \mathrm{col}(\zeta_2, \zeta_3, \zeta_1, \partial_s\zeta_1) \quad \mathcal{B}\hat{\mathbf{x}} = \begin{bmatrix} \mathcal{C}\hat{\mathbf{x}}(0) \\ \mathcal{C}\hat{\mathbf{x}}(1) \end{bmatrix}.$$

We now define the BCs. Recall these appear in the form

$$\int_0^1 B_I(s)(\mathcal{F}\hat{\mathbf{x}}(t))(s)ds + \begin{bmatrix} B_v & -B \end{bmatrix} \begin{bmatrix} v(t) \\ \mathcal{B}\hat{\mathbf{x}}(t) \end{bmatrix} = 0.$$

Checking our BCs, we note that $\zeta_1(t,0) = 0$ can be differentiated in time to obtain $\zeta_2(t,0) = 0$. Collecting all the BCs, and placing these in the required form, we have

$$\int_0^1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mu(s-1)\zeta_3(s) \end{bmatrix} ds = \begin{bmatrix} \zeta_1(0) \\ \zeta_2(0) \\ \zeta_3(1) \\ \partial_s\zeta_1(1) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -v_1 \\ -v_2 \end{bmatrix}$$

Recalling the expansions of $\mathcal{F}\hat{\mathbf{x}}$ and $\mathcal{B}\hat{\mathbf{x}}$, we may identify the parameters in $\mathbf{G}_\mathrm{b}$ as

$$\begin{aligned}
&B = \begin{bmatrix} 0 & 0 & 1 & 0_{1,2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0_{1,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_{1,2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0_{1,2} & 0 & 0 & 1 \end{bmatrix},\\
&B_v = \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix},\ B_I(s) = \begin{bmatrix} 0_{3,1} & 0_{3,1} & 0_{3,5} \\ 0 & \mu(s-1) & 0_{1,5} \end{bmatrix}.
\end{aligned} \tag{15}$$

In-Domain Dynamics: To find the parameters $\mathbf{G}_\mathrm{p}$, first recall that PDE dynamics have the form

$$\begin{aligned}
&\begin{bmatrix} \dot{\hat{\mathbf{x}}}(t,s) \\ r(t) \end{bmatrix} = \begin{bmatrix} A_0(s)(\mathcal{F}\hat{\mathbf{x}}(t))(s) \\ 0 \end{bmatrix} + \begin{bmatrix} B_{xv}(s) & B_{xb}(s) \\ 0 & D_{rb} \end{bmatrix} \begin{bmatrix} v(t) \\ \mathcal{B}\hat{\mathbf{x}}(t) \end{bmatrix}\\
&+ \begin{bmatrix} \int_a^s A_1(s,\theta)(\mathcal{F}\hat{\mathbf{x}}(t))(\theta)d\theta + \int_s^b A_2(s,\theta)(\mathcal{F}\hat{\mathbf{x}}(t))(\theta)d\theta \\ \int_a^b C_r(\theta)(\mathcal{F}\hat{\mathbf{x}}(t))(\theta)d\theta \end{bmatrix}.
\end{aligned}$$

Recalling the expansion of $\mathcal{F}\hat{\mathbf{x}}$, we represent the dynamics as

$$\dot{\hat{\mathbf{x}}}(t,s) = \begin{bmatrix} \partial_s^2 \zeta_1(t,s) \\ 1/\tau \partial_s \zeta_3(t,s) \\ \zeta_2(t,s) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0_{1,3} & 0 & 0 & 1 \\ 0 & 0_{1,3} & \frac{1}{\tau} & 0 & 0 \\ 1 & 0_{1,3} & 0 & 0 & 0 \end{bmatrix}}_{A_0} (\mathcal{F}\hat{\mathbf{x}}(t))(s)$$

Likewise, from the definition of $r(t)$, we have

$$r(t) = \begin{bmatrix} \int_0^1 \zeta_1(t,s)ds \\ \zeta_2(t,1) \end{bmatrix} = \int_0^1 \overbrace{\begin{bmatrix} 0_{1,2} & 1 & 0_{1,4} \\ 0_{1,2} & 0 & 0_{1,4} \end{bmatrix}}^{C_r} (\mathcal{F}\hat{\mathbf{x}}(t))(\theta)d\theta, + \underbrace{\begin{bmatrix} 0_{1,4} & 0 & 0_{1,3} \\ 0_{1,4} & 1 & 0_{1,3} \end{bmatrix}}_{D_{rb}} \mathcal{B}\hat{\mathbf{x}}(t)$$

Thus we have A_0, C_r, D_{rb} – the only nonzero terms in $\mathbf{G}_\mathrm{p}$.

IV. Representing a PDE Subsystem as a PIE

In Section III, we proposed a GPDE representation for a broad class of coupled ODE-PDEs Systems - See Eqs. (5)-(11). We now turn our attention to finding an alternative representation of such a GPDE model as a PIE. We begin this process by focusing on conversion of the PDE subsystem to a restricted class of PIE subsystem of the form

$$\begin{bmatrix} \hat{\mathcal{T}}\dot{\underline{\hat{\mathbf{x}}}}(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} \hat{\mathcal{A}} & \mathcal{B}_v \\ \mathcal{C}_r & \mathcal{D}_{rv} \end{bmatrix} \begin{bmatrix} \underline{\hat{\mathbf{x}}}(t) \\ v(t) \end{bmatrix} - \begin{bmatrix} \mathcal{T}_v \dot{v}(t) \\ 0 \end{bmatrix}, \tag{16}$$

with initial condition $\underline{\hat{\mathbf{x}}}(0) = \underline{\hat{\mathbf{x}}}^0 \in L_2^m$. Such PIE subsystems are a special case of Definition 5 with parameter set given by

$$\mathbf{G}_{\mathrm{PIE}_s} := \left\{ \hat{\mathcal{T}}, \mathcal{T}_v, \emptyset, \hat{\mathcal{A}}, \mathcal{B}_v, \emptyset, \mathcal{C}_{rv}, \emptyset, \mathcal{D}_{rv}, \emptyset, \emptyset, \emptyset \right\}.$$

In this section, we will show that for any admissible PDE subsystem defined by $\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$, there exists a corresponding PIE subsystem defined by $\{\hat{\mathcal{T}},\ \mathcal{T}_v,\ \hat{\mathcal{A}},\ \mathcal{B}_v,\ \mathcal{C}_{rv},\ \mathcal{D}_{rv}\}$ such that for any suitable signal v, $\{\underline{\hat{\mathbf{x}}}, r, \emptyset\}$ is a solution of the PIE subsystem with initial condition $\underline{\hat{\mathbf{x}}}^0$ and input v if and only if $\{\hat{\mathcal{T}}\underline{\hat{\mathbf{x}}}(t) + \mathcal{T}_v v(t), r\}$ is a solution of the PDE subsystem with initial condition $(\hat{\mathcal{T}}\underline{\hat{\mathbf{x}}}^0 + \mathcal{T}_v v(0))$ and input v.

A. Well-posedness of the BCs and Continuity Constraint

Before we map the PDE subsystem to an associated PIE subsystem, we first define a notion of admissibility. This definition imposes a notion of well-posedness on X_v, the domain of the PDE subsystem defined by the continuity constraints and the BCs. This condition ensures, e.g., that there are a correct number of independent BCs to establish a mapping between the distributed state and its partial derivatives. Without such a mapping, the solution to the PDE may not exist (too many BCs) or may not be unique (too few BCs).

Definition 9 (Admissible Boundary Conditions)**.** *Given an $n \in \mathbb{N}^{N+1}$ (with corresponding continuity constraint) and a parameter set, $\mathbf{G}_\mathrm{b} := \{B,\ B_I,\ B_v\}$, we say the pair $\{n, \mathbf{G}_\mathrm{b}\}$ is **admissible** if B_T is invertible where*

$$B_T := B \begin{bmatrix} T(0) \\ T(b-a) \end{bmatrix} - \int_a^b B_I(s) U_2 T(s-a) ds \in \mathbb{R}^{n_{BC} \times n_S},$$

and where T and U_2 are defined (See also Block 4) as

$$T_{i,j}(s) = \frac{s^{(j-i)}}{(j-i)!} \begin{bmatrix} 0_{n_{Si} - n_{Sj} \times n_{Sj}} \\ I_{n_{Sj}} \end{bmatrix} \in \mathbb{R}^{n_{Si} \times n_{Sj}}, \tag{17}$$

$$T(s) = \begin{bmatrix} T_{1,1}(s) & T_{1,2}(s) & \cdots & T_{1,N}(s) \\ 0 & T_{2,2}(s) & \cdots & T_{2,N}(s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{N,N}(s) \end{bmatrix} \in \mathbb{R}^{n_S \times n_S}, \tag{18}$$

$$U_{2i} = \begin{bmatrix} 0_{n_i \times n_{i+1:N}} \\ I_{n_{i+1:N}} \end{bmatrix} \in \mathbb{R}^{n_{S_i} \times n_{S_{i+1}}}, \tag{19}$$

$$U_2 = \begin{bmatrix} diag(U_{20}, \cdots, U_{2(N-1)}) \\ 0_{n_N \times n_S} \end{bmatrix} \in \mathbb{R}^{(n_{\hat{\mathbf{x}}} + n_S) \times n_S}. \tag{20}$$

Note that since B_T must be square to be invertible, admissibility requires $n_{BC} = n_S$. One way to interpret this condition is to note that whenever we differentiate a PDE state, we lose some of the information required to reconstruct that state. As a result, if we have n_S possible partial derivatives we need n_S BCs to relate all the partial derivatives back to the original state vector. However, while the constraint $n_{BC} = n_S$ is necessary for admissibility, it is not sufficient – the BCs must be both independent and provide enough information to allow us to reconstruct the PDE state. See Subsection 3.2.2 in [18] for an enumeration of pathological cases, including periodic BCs.

Finally, note that the test for admissibility depends only on the continuity condition, $n \in \mathbb{N}^{N+1}$ and the parameters which define the boundary condition – admissibility does not depend explicitly on the dynamics.

A.1 Illustration of the Admissibility Condition

Illustration 2 (Damped wave Equation with motor dynamics and delay) Let us revisit the coupled ODE-PDE from Section III-D. Recall that for this example, $n = \{0, 2, 1\}$, so $n_{S_0} = 3$, $n_{S_1} = 3$, $n_{S_2} = 1$, $n_S = 4$, and $n_{\hat{\mathbf{x}}} = 3$. In addition, $\mathbf{G}_\mathrm{b}$ has parameters as shown in Eq. (15). Then, using Eqs. (18) and (20), we compute T, U_2, and B_T as

$$T_{1,1} = \begin{bmatrix} 0_{3-3,3} \\ I_3 \end{bmatrix},\ T_{1,2} = s \begin{bmatrix} 0_{3-1,1} \\ I_1 \end{bmatrix},\ T_{2,2} = \begin{bmatrix} 0_{1-1,1} \\ I_1 \end{bmatrix},$$

$$U_{20} = \begin{bmatrix} 0_{0,3} \\ I_3 \end{bmatrix},\ U_{21} = \begin{bmatrix} 0_{2,1} \\ I_1 \end{bmatrix},\ T(s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & s \\ & & & 1 \end{bmatrix},$$

$$U_2 = \begin{bmatrix} I_3 & 0_{3,1} \\ 0_{2,3} & 0_{2,1} \\ 0_{1,3} & 1 \\ 0_{1,2} & 0_{1,2} \end{bmatrix},\ B_T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\int_a^b \mu(s-1)ds & 0 & 1 \end{bmatrix}.$$

Clearly, B_T is invertible for any μ which implies the pair $\{n, \mathbf{G}_\mathrm{b}\}$ is admissible.

B. A map between PIE and PDE states

Given an admissible pair $\{n, \mathbf{G}_\mathrm{b}\}$, we may construct a PIE subsystem which we will associate with the PDE subsystem defined by those parameters. The first step is to map $\hat{\mathbf{x}}(t) \in X_v$, the state of the PDE subsystem, to $\underline{\hat{\mathbf{x}}}(t) \in L_2$, the state of the PIE subsystem using

$$\underline{\hat{\mathbf{x}}} = \mathcal{D}\hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}}_0 \\ \partial_s \hat{\mathbf{x}}_1 \\ \vdots \\ \partial_s^N \hat{\mathbf{x}}_N \end{bmatrix} \in L_2^{n_{\hat{\mathbf{x}}}}.$$

where $\mathcal{D} := \mathrm{diag}(\partial_s^0 I_{n_0}, \cdots, \partial_s^N I_{n_N})$. The following theorem shows that this mapping is invertible and, moreover, the inverse is defined by PI operators.

Theorem 10. *Given an $n \in \mathbb{N}^{N+1}$, and $\mathbf{G}_\mathrm{b}$ with $\{n, \mathbf{G}_\mathrm{b}\}$ admissible, let $\{\hat{\mathcal{T}}, \mathcal{T}_v\}$ be as defined in Block 4, X_v as defined in Eq.* (9) *and $\mathcal{D} :=diag(\partial_s^0 I_{n_0}, \cdots, \partial_s^N I_{n_N})$. Then we have the following: (a) For any $v \in \mathbb{R}^{n_v}$, if $\hat{\mathbf{x}} \in X_v$, then $\mathcal{D}\hat{\mathbf{x}} \in L_2^{n_{\hat{\mathbf{x}}}}$ and $\hat{\mathbf{x}} = \hat{\mathcal{T}}\mathcal{D}\hat{\mathbf{x}} + \mathcal{T}_v v$; and (b) For any $v \in \mathbb{R}^{n_v}$ and $\underline{\hat{\mathbf{x}}} \in L_2^{n_{\hat{\mathbf{x}}}}$, $\hat{\mathcal{T}}\underline{\hat{\mathbf{x}}} + \mathcal{T}_v v \in X_v$ and $\underline{\hat{\mathbf{x}}} = \mathcal{D}(\hat{\mathcal{T}}\underline{\hat{\mathbf{x}}} + \mathcal{T}_v v)$.*

Proof. First, we generalize Cauchy's formula for repeated integration as

Lemma 11. *Suppose $\mathbf{x} \in C_N^n[a, b]$ for any $N \in \mathbb{N}$. Then*

$$\mathbf{x}(s) = \mathbf{x}(a) + \sum_{j=1}^{N-1} \frac{(s-a)^j}{j!}\partial_s^j \mathbf{x}(a) + \int_a^s \frac{(s-\theta)^{N-1}}{(N-1)!}\partial_s^N \mathbf{x}(\theta) d\theta.$$

This gives a map from $\partial_s^j \hat{\mathbf{x}}(a)$ and $\underline{\hat{\mathbf{x}}}$ to $\hat{\mathbf{x}}$. Next we express $\mathcal{B}\hat{\mathbf{x}}$ in terms of the $\partial_s^j \hat{\mathbf{x}}(a)$ and $\underline{\hat{\mathbf{x}}}$. Applying the boundary conditions in X_v, we may now invert this map (using B_T^{-1}) to obtain an expression for the $\partial_s^j \hat{\mathbf{x}}(a)$ in terms of $\underline{\hat{\mathbf{x}}}$ and v. Substituting this expression into Lemma 11, we obtain the theorem statement. For details, see [25, Appendix A]. □

For any given $v \in \mathbb{R}^{n_v}$, Theorem 10 provides an invertible map between the state of the PIE subsystem, $\underline{\hat{\mathbf{x}}}(t) \in L_2^{n_{\hat{\mathbf{x}}}}$ and the state of the PDE subsystem, $\hat{\mathbf{x}}(t) \in X_v$. Furthermore, as will be shown in Section VI, this transformation is unitary. In the following subsection, we apply this mapping to the internal dynamics of the PDE subsystem in order to obtain an equivalent PIE representation of this subsystem.

$$n_{\hat{\mathbf{x}}} = \sum_{i=0}^{N} n_i,\ n_{S_i} = \sum_{j=i}^{N} n_j,\ n_S = \sum_{i=1}^{N} n_{S_i}\ n_{i:j} = \sum_{k=i}^{j} n_k,$$

$$\tau_i(s) = \frac{s^i}{i!}, \quad T_{i,j}(s) = \tau_{(j-i)}(s)\begin{bmatrix} 0_{(n_{Si}-n_{Sj}),n_{Sj}} \\ I_{n_{Sj}} \end{bmatrix},$$

$$Q_i(s) = \begin{bmatrix} 0 & \tau_0(s) I_{n_i} & & & \\ 0 & & \tau_1(s) I_{n_{i+1}} & & \\ & & & \ddots & \\ 0 & & & & \tau_{N-i}(s) I_{n_N} \end{bmatrix}, Q(s) = \begin{bmatrix} Q_1(s) \\ \vdots \\ Q_N(s) \end{bmatrix}$$

$$T(s) = \begin{bmatrix} T_1(s) \\ \vdots \\ T_N(s) \end{bmatrix} = \begin{bmatrix} T_{1,1}(s) & \cdots & T_{1,N}(s) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T_{N,N}(s) \end{bmatrix},$$

$$U_{1i} = \begin{bmatrix} I_{n_i} \\ 0_{n_{i+1:N}, n_i} \end{bmatrix},\ U_1 = \mathrm{diag}(U_{10}, \cdots, U_{1N})$$

$$U_{2i} = \begin{bmatrix} 0_{n_i, n_{i+1:N}} \\ I_{n_{i+1:N}} \end{bmatrix},\ U_2 = \begin{bmatrix} \mathrm{diag}(U_{20}, \cdots, U_{2(N-1)}) \\ 0_{n_N, n_S} \end{bmatrix},$$

$$B_T = B\begin{bmatrix} T(0) \\ T(b-a) \end{bmatrix} - \int_a^b B_I(s) U_2 T(s-a) ds,$$

$$B_Q(s) = B_T^{-1}\left(B_I(s)U_1 + \int_s^b B_I(\theta)U_2 Q(\theta - s)d\theta - B\begin{bmatrix} 0 \\ Q(b-s) \end{bmatrix}\right),$$

$$G_0 = \begin{bmatrix} I_{n_0} & \\ & 0_{(n_{\hat{\mathbf{x}}}-n_0)} \end{bmatrix},\ G_2(s,\theta) = \begin{bmatrix} 0 \\ T_1(s-a)B_Q(\theta) \end{bmatrix},$$

$$G_1(s,\theta) = \begin{bmatrix} 0 \\ Q_1(s-\theta) \end{bmatrix} + G_2(s,\theta),\ G_v(s) = \begin{bmatrix} 0 \\ T_1(s-a)B_T^{-1}B_v \end{bmatrix},$$

$$\hat{\mathcal{T}} = \mathcal{P}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{G_i\} \end{array}\right], \qquad \mathcal{T}_v = \mathcal{P}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline G_v & \{\emptyset\} \end{array}\right].$$

Block 4: Definitions based on $n \in \mathbb{N}^{N+1}$ and the parameters of $\mathbf{G}_\mathrm{b} := \{B,\ B_I,\ B_v\}$ used in Theorem 10.

C. PIE representation of a PDE Subsystem

For finite-dimensional state-space systems, similarity transforms are used to construct equivalent representations of the input-output map. Specifically, for any invertible T, the system $G := \{A, B, C, D\}$ with internal state x may be equivalently represented as $G := \{T^{-1}AT, T^{-1}B, CT, D\}$ with internal state $\hat{x} = T^{-1}x$. In this subsection, we apply this approach to PDE subsystems. Specifically, now that we have obtained an invertible transformation from $L_2^{n_{\hat{\mathbf{x}}}}$ to X_v, we apply the logic of the similarity transform to the internal dynamics of the PDE subsystem in order to obtain an equivalent PIE subsystem representation. Specifically, in Theorem 12, we substitute $\hat{\mathbf{x}} = \hat{\mathcal{T}}\underline{\hat{\mathbf{x}}} + \mathcal{T}_v v$ in the internal dynamics of the PDE subsystem. The result is a set of equations parameterized entirely using PI operators. These PI operators, as defined in Block 5, specify a PIE subsystem whose input-output behavior mirrors that of the PDE subsystem and whose solution can be constructed using the solution of the PDE subsystem. Conversely, any solution of the associated PIE subsystem can be used to construct a solution for the PDE subsystem.

Theorem 12. *Given an $n \in \mathbb{N}^{N+1}$ and a set of PDE parameters $\{\mathbf{G}_\mathrm{b},\ \mathbf{G}_\mathrm{p}\}$ as defined in Equations* (10) *and* (12) *with $\{n, \mathbf{G}_\mathrm{b}\}$ admissible, suppose $v \in L_{2e}^{n_v}[\mathbb{R}_+]$ with $B_v v \in W_{1e}^{2n_S}[\mathbb{R}_+]$, $\{\hat{\mathcal{T}},\ \mathcal{T}_v\}$ are as defined in Block 4 and $\{\hat{\mathcal{A}}, \mathcal{B}_v, \mathcal{C}_r, \mathcal{D}_{rv}\}$ are as defined in Block 5. Define*

$$\mathbf{G}_{\mathrm{PIE}} = \left\{\hat{\mathcal{T}}, \mathcal{T}_v, \emptyset, \hat{\mathcal{A}}, \mathcal{B}_v, \emptyset, \mathcal{C}_r, \emptyset, \mathcal{D}_{rv}, \emptyset, \emptyset, \emptyset\right\}$$

Then we have the following.

1) *For any $\hat{\mathbf{x}}^0 \in X_{v(0)}$ (X_v is as defined in Equation* (9)*), if $\{\hat{\mathbf{x}}, r\}$ satisfies the PDE defined by $\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\hat{\mathbf{x}}^0$ and input v, then $\{\mathcal{D}\hat{\mathbf{x}}, r, \emptyset\}$ satisfies the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ with initial condition $\mathcal{D}\hat{\mathbf{x}}^0 \in L_2^{n_{\hat{\mathbf{x}}}}$ and input $\{v, \emptyset\}$ where $\mathcal{D}\hat{\mathbf{x}} = col(\partial_s^0 \hat{\mathbf{x}}_0, \cdots, \partial_s^N \hat{\mathbf{x}}_N)$.*
2) *For any $\underline{\hat{\mathbf{x}}}^0 \in L_2^{n_{\hat{\mathbf{x}}}}$, if $\{\underline{\hat{\mathbf{x}}}, r, \emptyset\}$ satisfies the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ for initial condition $\underline{\hat{\mathbf{x}}}^0$ and input $\{v, \emptyset\}$, then $\{\hat{\mathcal{T}}\underline{\hat{\mathbf{x}}} + \mathcal{T}_v v,\ r\}$ satisfies the PDE defined by $\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\hat{\mathbf{x}}^0 = \hat{\mathcal{T}}\underline{\hat{\mathbf{x}}}^0 + \mathcal{T}_v v(0)$ and input v.*

Proof. The proof is based on a partial similarity transform induced by $\hat{\mathbf{x}} = \hat{\mathcal{T}}\underline{\hat{\mathbf{x}}} + \mathcal{T}_v v$ and details may be found in [25, Appendix B]. □

The first part of Theorem 12 shows that well-posedness of the PDE subsystem guarantees well-posedness of the associated PIE subsystem and, furthermore, shows that the input-output behavior of the PIE subsystem mirrors that of the PDE subsystem. The second, converse, result, shows that well-posedness of the PIE subsystem guarantees well-posedness of the PDE subsystem and, furthermore, shows that the input-output behavior of the PDE subsystem mirrors that of the PIE subsystem. Because PIEs are potentially easier to numerically analyze, control and simulate, this converse result suggests that the tasks of analysis, control and simulation of a PDE subsystem may be more readily accomplished by performing the desired task on the PIE subsystem and then applying the result to the original PDE subsystem.

C.1 Illustration of the Construction of the PIE Subsystem

In this subsection, we detail the application of the formulae in Blocks 4 and 5 to a given GPDE model. Additional, less detailed examples are given in Section VIII.

$$R_{D,2}(s,\theta) = U_2 T(s-a) B_Q(\theta)$$
$$R_{D,1}(s,\theta) = R_{D,2}(s,\theta) + U_2 Q(s-\theta),$$
$$\Upsilon = \left[\begin{array}{c|c} \begin{bmatrix} I_{n_v} \\ B_T^{-1}B_v \\ T(b-a)B_T^{-1}B_v \end{bmatrix} & \begin{bmatrix} 0_{n_r\times n_x} \\ B_Q(s) \\ T(b-a)B_Q(s)+Q(b-s) \end{bmatrix} \\ \hline U_2T(s-a)B_T^{-1}B_v & \{U_1, R_{D,1}, R_{D,2}\} \end{array}\right],$$
$$\Xi = \left[\begin{array}{c|c} \begin{bmatrix}0 & D_{rb}\end{bmatrix} & C_r \\ \hline \begin{bmatrix}B_{xv} & B_{xb}\end{bmatrix} & \{A_i\} \end{array}\right], \quad \left[\begin{array}{c|c} D_{rv} & C_{rx} \\ \hline B_{xv} & \{\hat{A}_i\} \end{array}\right] = \mathbf{P}^4_\times(\Xi, \Upsilon)$$
$$\hat{\mathcal{A}} = \mathcal{P}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{\hat{A}_i\} \end{array}\right], \; \mathcal{B}_v = \mathcal{P}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline B_{xv} & \{\emptyset\} \end{array}\right], \; \mathcal{C}_r = \mathcal{P}\left[\begin{array}{c|c} \emptyset & C_{rx} \\ \hline \emptyset & \{\emptyset\} \end{array}\right]$$
$$\mathcal{D}_{rv} = \mathcal{P}\left[\begin{array}{c|c} D_{rv} & \emptyset \\ \hline \emptyset & \{\emptyset\} \end{array}\right], \; \mathcal{T} = \begin{bmatrix} I_{n_x} & 0 \\ G_vC_v & \hat{\mathcal{T}} \end{bmatrix}, \; \mathcal{T}_w = \begin{bmatrix} 0 & 0 \\ G_vD_{vw} & 0 \end{bmatrix},$$
$$\mathcal{T}_u = \begin{bmatrix} 0 & 0 \\ G_vD_{vu} & 0 \end{bmatrix}, , \; \mathcal{A} = \begin{bmatrix} A + B_{xr}\mathcal{D}_{rv}C_v & B_{xr}\mathcal{C}_r \\ \mathcal{B}_vC_v & \hat{\mathcal{A}} \end{bmatrix},$$
$$\mathcal{B}_1 = \begin{bmatrix} B_{xw} + B_{xr}\mathcal{D}_{rv}D_{vw} \\ \mathcal{B}_vD_{vw} \end{bmatrix}, \; \mathcal{B}_2 = \begin{bmatrix} B_{xu} + B_{xr}\mathcal{D}_{rv}D_{vu} \\ \mathcal{B}_vD_{vu} \end{bmatrix},$$
$$\mathcal{C}_1 = \begin{bmatrix} C_z + D_{zr}\mathcal{D}_{rv}C_v & D_{zr}\mathcal{C}_r \end{bmatrix}, \; \mathcal{C}_2 = \begin{bmatrix} C_y + D_{yr}\mathcal{D}_{rv}C_v & D_{yr}\mathcal{C}_r \end{bmatrix}$$
$$\mathcal{D}_{11} = D_{zw} + D_{zr}\mathcal{D}_{rv}D_{vw}, \quad \mathcal{D}_{12} = D_{zu} + D_{zr}\mathcal{D}_{rv}D_{vu},$$
$$\mathcal{D}_{21} = D_{yw} + D_{yr}\mathcal{D}_{rv}D_{vw}, \quad \mathcal{D}_{22} = D_{yu} + D_{yr}\mathcal{D}_{rv}D_{vu}.$$

Block 5: Definitions based on the PDE and GPDE parameters in $\mathbf{G}_\mathrm{p} = \{A_0,\, A_1,\, A_2,\, B_{xv},\, B_{xb},\, C_r,\, D_{rb}\}$ and $\mathbf{G}_\mathrm{o} = \{A,\, B_{xw},\, B_{xu},\, B_{xr},\, C_z,\, D_{zw},\, D_{zu},\, D_{zr},\, C_y,\, D_{yw},\, D_{yu},\, D_{yr},\, C_v,\, D_{vw},\, D_{vu}\}$, the Definitions from $\mathbf{G}_\mathrm{b}$ as listed in Block 4 and the map $\mathbf{P}^4_\times$ as defined in [25, Lemma 36].

Illustration 3 (A simple PIE: The Entropy PDE) A PDE model for entropy change in a 1D linear thermoelastic rod clamped at both ends is given by [9]

$$\dot{\eta}(t,s) = \partial_s^2 \eta(t,s),$$

subject to the BCs

$$\eta(t,0) + \int_0^1 \eta(t,s)ds = 0, \quad \eta(t,1) + \int_0^1 \eta(t,s)ds = 0.$$

The GPDE representation of this model is defined by $n = \{0,0,2\}$, $\mathbf{G}_\mathrm{p} = \{A_0 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}\}$, and

$$\mathbf{G}_\mathrm{b} = \left\{ B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; B_I = -\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}.$$

Using the formulae in Blocks 4 and 5, we find the PIE subsystem as follows (we neglect interconnection to the ODE subsystem as there are no ODEs, inputs, or outputs).

$$U_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \; U_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; T(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \; Q(s) = \begin{bmatrix} s \\ 1 \end{bmatrix},$$
$$B_T = \begin{bmatrix} 2 & 1/2 \\ 2 & 3/2 \end{bmatrix}, \quad B_Q(s) = (1-s)\begin{bmatrix} \frac{s}{4} \\ -1 \end{bmatrix}, \quad G_0(s) = 0,$$
$$G_1(s,\theta) = G_2(s,\theta) + (s-\theta), \quad G_2(s,\theta) = 3s\frac{(s-1)}{4}.$$

The PIE form ($\underline{\eta} = \partial_2^2 \eta$) of the entropy PDE is then given by

$$\int_0^s \left(s^2 + \frac{s}{4} - \theta\right) \dot{\underline{\eta}}(t,\theta)d\theta + \int_s^1 \frac{3}{4}(s^2 - s)\dot{\underline{\eta}}(t,\theta)d\theta = \underline{\eta}(t,s).$$

V. PIE Representation of a GPDE

Having converted the PDE subsystem to a PIE, integration of the ODE dynamics is a simple matter of augmenting the PIE subsystem (Equation (16)) with the differential equations which define the ODE (Equation (5)), followed by elimination of the interconnection signals v and r. The result is an augmented PIE system, as defined in Equation (4) whose parameters are 4-PI operators, as defined in Blocks 4 and 5.

A. A map between PIE and GPDE states

Our first step in constructing the augmented PIE system which will be associated to a given GPDE model is to construct the augmented map from GPDE state (defined on $\mathcal{X}_{w,u}$) to the associated PIE state (defined on $\mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$). Specifically, given a GPDE model $\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{p}\}$ with $\{n, \mathbf{G}_\mathrm{b}\}$ admissible and state $\mathbf{x} = \begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix} \in \mathcal{X}_{w,u}$, the associated PIE system state is $\underline{\mathbf{x}} = \begin{bmatrix} x \\ \mathcal{D}\hat{\mathbf{x}} \end{bmatrix} \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$ where $\mathcal{D} := \mathrm{diag}(\partial_s^0 I_{n_0}, \cdots, \partial_s^N I_{n_N})$. Using this definition, Corollary 13 shows that if $\{\mathcal{T},\, \mathcal{T}_w,\, \mathcal{T}_u\}$ are as defined in Block 5, then the map $\mathbf{x} \to \underline{\mathbf{x}}$ can be inverted as $\mathbf{x} = \mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w + \mathcal{T}_u u$.

Corollary 13 (Corollary of Theorem 10). *Given an $n \in \mathbb{N}^{N+1}$, and $\mathbf{G}_\mathrm{b}$ with $\{n, \mathbf{G}_\mathrm{b}\}$ admissible, let $\{\mathcal{T},\, \mathcal{T}_w,\, \mathcal{T}_u\}$ be as defined in Block 5, $\mathcal{X}_{w,u}$ as defined in Eq. (13) and $\mathcal{D}$:=diag$(\partial_s^0 I_{n_0}, \cdots, \partial_s^N I_{n_N})$. Then for any $w \in \mathbb{R}^{n_w}$ and $u \in \mathbb{R}^{n_u}$ we have:*

(a) If $\mathbf{x} := \{x, \hat{\mathbf{x}}\} \in \mathcal{X}_{w,u}$, then $\underline{\mathbf{x}} := \{x, \mathcal{D}\hat{\mathbf{x}}\} \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$ and $\mathbf{x} = \mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w + \mathcal{T}_u u$.

(b) If $\underline{\mathbf{x}} \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$, then $\mathbf{x} := \mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w + \mathcal{T}_u u \in \mathcal{X}_{w,u}$ and $\underline{\mathbf{x}} = \begin{bmatrix} I_{n_x} & 0 \\ 0 & \mathcal{D} \end{bmatrix}\mathbf{x}$.

Proof. The proof simply applies the definitions of $\mathbf{x}$, $\underline{\mathbf{x}}$, and v – See [25, Appendix C]. □

Thus, for any given w, u, we have an invertible transformation from $\mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$ to $\mathcal{X}_{w,u}$.

B. Representation of a GPDE model as a PIE system

In this subsection, we define the PIE system associated with a given admissible GPDE model. This associated PIE system is defined by 4-PI parameters as defined in Blocks 4 and 5. For convenience, we use $\mathbf{M} : \{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{p}\} \mapsto \{\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u, \mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_{11}, \mathcal{D}_{12}, \mathcal{D}_{21}, \mathcal{D}_{22}\}$ to represent the several formulae used to map GPDE parameters to PIE parameters.

Definition 14. *Given $\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{p}\}$ where*

$$\mathbf{G}_\mathrm{b} = \{B, B_I, B_v\}, \quad \mathbf{G}_\mathrm{p} = \{A_0, A_1, A_2, B_{xv}, B_{xb}, C_r, D_{rb}\}$$
$$\mathbf{G}_\mathrm{o} = \{A, B_{xw}, B_{xu}, B_{xr}, C_z, D_{zw}, D_{zu}, D_{zr}, C_y, D_{yw}, D_{yu}, D_{yr}, C_v, D_{vw}, D_{vu}\}$$

we say that $\mathbf{G}_\mathrm{PIE} = \mathbf{M}(\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{p}\})$ if $\mathbf{G}_\mathrm{PIE} = \{\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u, \mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_{11}, \mathcal{D}_{12}, \mathcal{D}_{21}, \mathcal{D}_{22}\}$ where $\{\mathcal{T},\, \mathcal{T}_w,\, \mathcal{T}_u,\, \mathcal{A},\, \mathcal{B}_1,\, \mathcal{B}_2,\, \mathcal{C}_1,\, \mathcal{C}_2,\, \mathcal{D}_{11},\, \mathcal{D}_{12},\, \mathcal{D}_{21},\, \mathcal{D}_{22}\}$ are as defined in Blocks 4 and 5.

Having specified the PIE system associated with a given GPDE model, we now extend the results of Theorem 12 to show that the map $\mathbf{x} \mapsto \begin{bmatrix} I & 0 \\ 0 & \mathcal{D} \end{bmatrix}\mathbf{x}$ proposed in Corollary 13 maps a solution of a given GPDE model to a solution of the associated PIE system and that the inverse map $\underline{\mathbf{x}} \mapsto \mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w + \mathcal{T}_u u$ maps a solution of the associated PIE to a solution of the given GPDE model.

Corollary 15 (Corollary of Theorem 12)**.** *Given an $n \in \mathbb{N}^{N+1}$ and parameters $\{\mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ as defined in Equations* (6), (10) *and* (12) *with $\{n, \mathbf{G}_\mathrm{b}\}$ admissible, let $w \in L_{2e}^{n_w}[\mathbb{R}_+]$ with $B_v D_{vw} w \in W_{1e}^{2n_S}[\mathbb{R}_+]$, $u \in L_{2e}^{n_u}[\mathbb{R}_+]$ with $B_v D_{vu} u \in W_{1e}^{2n_S}[\mathbb{R}_+]$. Define*

$$\begin{aligned}\mathbf{G}_\mathrm{PIE} &= \{\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u, \mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_{11}, \mathcal{D}_{12}, \mathcal{D}_{21}, \mathcal{D}_{22}\} \\ &= \mathbf{M}(\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{p}\}.\end{aligned}$$

Then we have the following:

1) *For any $\{x^0, \hat{\mathbf{x}}^0\} \in \mathcal{X}_{w(0),u(0)}$ (where $\mathcal{X}_{w,u}$ is as defined in Equation* (13)*), if $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfies the GPDE defined by $\{n,\ \mathbf{G}_\mathrm{o},\ \mathbf{G}_\mathrm{b},\ \mathbf{G}_\mathrm{p}\}$ with initial condition $\{x^0, \hat{\mathbf{x}}^0\}$ and input $\{w, u\}$, then $\left\{\begin{bmatrix} x \\ \mathcal{D}\hat{\mathbf{x}}\end{bmatrix}, z, y\right\}$ satisfies the PIE defined by $\mathbf{G}_\mathrm{PIE}$ with initial condition $\begin{bmatrix} x^0 \\ \mathcal{D}\hat{\mathbf{x}}^0\end{bmatrix}$ and input $\{w, u\}$ where $\mathcal{D}\hat{\mathbf{x}} = col(\partial_s^0 \hat{\mathbf{x}}_0, \cdots, \partial_s^N \hat{\mathbf{x}}_N)$.*
2) *For any $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$, if $\{\underline{\mathbf{x}},\ z,\ y\}$ satisfies the PIE defined by $\mathbf{G}_\mathrm{PIE}$ with initial condition $\underline{\mathbf{x}}^0$ and input $\{w, u\}$, then $\{x,\ \hat{\mathbf{x}},\ z,\ y,\ v,\ r\}$ satisfies the GPDE defined by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0\end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}^0 + \mathcal{T}_w w(0) + \mathcal{T}_u u(0)$ and input $\{w, u\}$ where*

$$\begin{aligned}\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t)\end{bmatrix} &:= \mathcal{T}\underline{\mathbf{x}}(t) + \mathcal{T}_w w(t) + \mathcal{T}_u u(t), \\ v(t) &:= C_v x(t) + D_{vw} w(t) + D_{vu} u(t), \\ r(t) &:= \begin{bmatrix} 0_{n_{\hat{\mathbf{x}}} \times n_x} & \mathcal{C}_r\end{bmatrix} \underline{\mathbf{x}}(t) + \mathcal{D}_{rv} v(t),\end{aligned}$$

and where $\mathcal{C}_r$ and $\mathcal{D}_{rv}$ are as defined in Block 5.

Proof. The proof is simply a matter of applying Theorem 12 to the augmented states and verifying the definition of solution is satisfied for both the GPDE and PIE. A detailed proof can be found in [25, Appendix D]. □

Several examples of the conversion of GPDE models to PIE systems can be found in Section VIII.

VI. Equivalence of Properties of GPDE and PIE

We have motivated the construction of PIE representations of GPDE models by stating that many analysis, control, and simulation tasks may be more readily accomplished in the PIE framework. However, this motivation is predicated on the assumption that the results of analysis, control and simulation of a PIE system somehow translate to analysis, control and simulation of the original GPDE model. For simulation, the conversion of a numerical solution of a PIE system to the numerical solution of the GPDE is trivial, as per Corollary 15 through the mapping $\underline{\mathbf{x}}(t) \mapsto \mathcal{T}\underline{\mathbf{x}}(t) + \mathcal{T}_w w(t) + \mathcal{T}_u u(t)$. In this section, we show that analysis and control of the PIE system may also be translated to the GPDE model. For input-output properties, this translation is trivial. For internal stability and control, additional mathematical structure is required.

A. Equivalence of Input-Output Properties

Because the translation of PIE solution to GPDE solution is limited to the internal state of the PIE (inputs and outputs are unchanged), Corollary 15 implies that all input-output (I/O) properties of the GPDE model are inherited by the PIE system and vice versa. As a result, we have the following.

Corollary 16 (Input-Output Properties)**.** *Given an $n \in \mathbb{N}^{N+1}$ and parameters $\{\mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ as defined in Equations* (6), (10) *and* (12) *with $\{n, \mathbf{G}_\mathrm{b}\}$ admissible, let $w \in L_{2e}^{n_w}[\mathbb{R}_+]$ with $B_v D_{vw} w \in W_{1e}^{2n_S}[\mathbb{R}_+]$. Let $\mathbf{G}_\mathrm{PIE} = \mathbf{M}(\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{p}\}$. Suppose $\{x^0, \hat{\mathbf{x}}^0\} = \{0, 0\}$. Then the following are equivalent.*

1) *If $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfies the GPDE defined by $\{n,\ \mathbf{G}_\mathrm{o},\ \mathbf{G}_\mathrm{b},\ \mathbf{G}_\mathrm{p}\}$ with initial condition $\{0, 0\}$ and input $\{w, 0\}$, then $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$.*
2) *If $\{\underline{\mathbf{x}},\ z,\ y\}$ satisfies the PIE defined by $\mathbf{G}_\mathrm{PIE}$ with initial condition 0 and input $\{w, 0\}$, then $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$.*

Suppose $\mathcal{K} :\in L_{2e}^{n_y} \to L_{2e}^{n_u}$. Then the following are equivalent.

1) *If $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfies the GPDE defined by $\{n,\ \mathbf{G}_\mathrm{o},\ \mathbf{G}_\mathrm{b},\ \mathbf{G}_\mathrm{p}\}$ with initial condition $\{0, 0\}$ and input $\{w, \mathcal{K}y\}$, then $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$.*
2) *If $\{\underline{\mathbf{x}},\ z,\ y\}$ satisfies the PIE defined by $\mathbf{G}_\mathrm{PIE}$ with initial condition 0 and input $\{w, \mathcal{K}y\}$, then $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$.*

Proof. Corollary 16 follows directly from Corollary 15. □

B. Equivalence of Internal Stability

Unlike I/O properties, the question of internal stability of a GPDE model is complicated by the fact that there is no universally accepted definition of stability for such models. Specifically, if the state-space of a GPDE model is defined to be $\mathcal{X}_{u,w}$ (a subspace of the Sobolev space W^n), then the obvious norm is the Sobolev norm – implying that exponential stability requires exponential decay with respect to the Sobolev norm. However, many results on stability of PDE models use the L_2 norm as a simpler notion of size of the state.

In this section, we show that while both notions of stability are reasonable, the use of the Sobolev norm and associated inner product confers significant advantages in terms of mathematical structure on the GPDE model and offers a clear equivalence between internal stability of the GPDE model and associated PIE system. In particular, we first show that $\mathcal{X}_{0,0}$ is a Hilbert space when equipped with the Sobolev inner product and furthermore, exponential stability of the GPDE model with respect to the Sobolev norm is equivalent to exponential stability of the PIE system with respect to the L_2 norm.

B.1 Topology of $\mathcal{X}_{0,0}$ (state space of a GPDE with no inputs)

Before we begin, for $n \in \mathbb{N}^N$, let us recall the standard inner product on $\mathbb{R}^{n_x} \times W^n$

$$\left\langle \begin{bmatrix} u \\ \mathbf{u}\end{bmatrix}, \begin{bmatrix} v \\ \mathbf{v}\end{bmatrix} \right\rangle_{\mathbb{R}^{n_x} \times W^n} = u^T v + \sum\nolimits_{i=0}^N \langle \mathbf{u}_i, \mathbf{v}_i \rangle_{W_i^{n_i}},$$

$$\langle \mathbf{u}_i, \mathbf{v}_i \rangle_{W_i^{n_i}} := \sum\nolimits_{j=0}^i \left\langle \partial_s^j \mathbf{u}_i, \partial_s^j \mathbf{u}_i \right\rangle_{L_2}$$

with associated norms $\|\mathbf{u}_i\|_{W_i^{n_i}} := \sum_{j=0}^i \left\|\partial_s^j \mathbf{x}_i\right\|_{L_2^{n_i}}$ and

$$\left\| \begin{bmatrix} u \\ \mathbf{u}\end{bmatrix} \right\|_{\mathbb{R}^{n_x} \times W^n} = \|u\|_{\mathbb{R}} + \sum\nolimits_{i=0}^N \|\mathbf{u}_i\|_{W_i^{n_i}}.$$

As we will see, however, the standard inner product $\mathbb{R}^{n_x} \times W^n$ is not quite the right inner product for $\mathcal{X}_{0,0}$. For this reason, we propose a slightly modified inner product which we will denote $\langle \cdot, \cdot \rangle_{X^n}$, and show that this new inner product is equivalent to the standard inner product on W^n. Specifically, we have

$$\langle \mathbf{u}, \mathbf{v} \rangle_{X^n} := \sum\nolimits_{i=0}^N \left\langle \partial_s^i \mathbf{u}_i, \partial_s^i \mathbf{v}_i \right\rangle_{L_2^{n_i}} = \langle \mathcal{D}\mathbf{u}, \mathcal{D}\mathbf{v} \rangle_{L_2^{n_\mathrm{x}}} \quad (21)$$

and define the obvious extension

$$\left\langle \begin{bmatrix} u \\ \mathbf{u} \end{bmatrix}, \begin{bmatrix} v \\ \mathbf{v} \end{bmatrix} \right\rangle_{\mathbb{R}^{n_x} \times X^n} := u^T v + \langle \mathbf{u}, \mathbf{v} \rangle_{X^n}.$$

We now show that the norms $\|\cdot\|_{\mathbb{R}^{n_x}\times W^n}$ and $\|\cdot\|_{\mathbb{R}^{n_x}\times X^n}$ are equivalent on the subspace $\mathcal{X}_{0,0}$.

Lemma 17. *Suppose pair $\{n, \mathbf{G}_\mathrm{b}\}$ is admissible. Then $\|\mathbf{u}\|_{\mathbb{R}^{n_x}\times X^n} \le \|\mathbf{u}\|_{\mathbb{R}^{n_x}\times W^n}$ and there exists $c_0 > 0$ such that for any $\mathbf{u} \in \mathcal{X}_{0,0}$, we have $\|\mathbf{u}\|_{\mathbb{R}^{n_x}\times W^n} \le c_0 \|\mathbf{u}\|_{\mathbb{R}^{n_x}\times X^n}$.*

Proof. Because the map $\underline{\mathbf{x}} \to \mathbf{x}$ is a PI operator, it is bounded, which allows a bound on all terms in the Sobolev norm. See [25, Appendix F] for a complete proof. □

Trivially, using $n_x = 0$, this result also extends to equivalence of $\|\cdot\|_{W^n}$ and $\|\cdot\|_{X^n}$ on X_0.

Next, we will show that $\hat{\mathcal{T}}$ and $\mathcal{T}$ are isometric when X_0 and $\mathcal{X}_{0,0}$ are endowed with the inner products $\langle\cdot,\cdot\rangle_{\mathbb{R}^{n_x}\times W^n}$ and $\langle\cdot,\cdot\rangle_{\mathbb{R}^{n_x}\times X^n}$, respectively. This implies that these spaces are complete with respect to both $\|\cdot\|_{\mathbb{R}^{n_x}\times X^n}$ ($\|\cdot\|_{X^n}$) and $\|\cdot\|_{\mathbb{R}^{n_x}\times W^n}$ ($\|\cdot\|_{W^n}$).

B.2 $\mathcal{X}_{0,0}$ is Hilbert and $\mathcal{T}$ is unitary

First, note X_0 and $\mathcal{X}_{0,0}$ are defined by $\{n, \mathbf{G}_\mathrm{b}\}$ as

$$X_0 := \left\{ \hat{\mathbf{x}} \in W^n[a,b] : B\mathcal{B}\hat{\mathbf{x}} = \textstyle\int_a^b B_I(s)(\mathcal{F}\hat{\mathbf{x}})(s)ds \right\},$$

$$\mathcal{X}_{0,0} := \left\{ \begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix} \in \mathbb{R} \times X_v \;:\; v = C_v x \right\}.$$

The sets X_0 and $\mathcal{X}_{0,0}$ are the subspaces of valid PDE subsystem and GPDE model states when $v = 0$ and when $u = 0, w = 0$, respectively. Previously, in Theorem 10, we have shown that $\hat{\mathcal{T}}$ is a bijective map. In Theorem 18 we extend this result to show that $\hat{\mathcal{T}} : L_2^{n_{\hat{\mathbf{x}}}} \to X^n$ and $\mathcal{T} : \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}} \to \mathbb{R}^{n_x} \times X^n$ are unitary in that the respective inner products are preserved under these transformations.

Theorem 18. *Suppose $\{n, \mathbf{G}_\mathrm{b}\}$ is admissible, $\{\hat{\mathcal{T}}, \mathcal{T}_v\}$ are as defined in Block 4, and $\{\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u\}$ are as defined in Block 5 for some matrices C_v, D_{vw} and D_{vu}. If $\langle\cdot,\cdot\rangle_{X^n}$ is as defined in Equation (21), then we have the following:*

a) for any $v_1, v_2 \in \mathbb{R}^{n_v}$ and $\hat{\underline{\mathbf{x}}}$, $\hat{\underline{\mathbf{y}}} \in L_2^{n_{\hat{\mathbf{x}}}}$

$$\left\langle \left(\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v_1\right), \left(\hat{\mathcal{T}}\hat{\underline{\mathbf{y}}} + \mathcal{T}_v v_2\right) \right\rangle_{X^n} = \left\langle \hat{\underline{\mathbf{x}}}, \hat{\underline{\mathbf{y}}} \right\rangle_{L_2^{n_{\hat{\mathbf{x}}}}}. \tag{22}$$

b) for any $w_1, w_2 \in \mathbb{R}^{n_w}$, $u_1, u_2 \in \mathbb{R}^{n_u}$, $\underline{\mathbf{x}}, \underline{\mathbf{y}} \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$,

$$\begin{aligned} &\left\langle (\mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w_1 + \mathcal{T}_u u_1), (\mathcal{T}\underline{\mathbf{y}} + \mathcal{T}_w w_2 + \mathcal{T}_u u_2) \right\rangle_{\mathbb{R}^{n_x}\times X^n} \\ &\quad = \left\langle \underline{\mathbf{x}}, \underline{\mathbf{y}} \right\rangle_{\mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}}. \end{aligned} \tag{23}$$

Proof. The proof follows directly from the definition of the X^n inner product and the map $\mathbf{x} \mapsto \underline{\mathbf{x}}$. See [25, Appendix E] for more details. □

Corollary 19. *Suppose $\{n, \mathbf{G}_\mathrm{b}\}$ is admissible, $\hat{\mathcal{T}}$ is as defined in Block 4, $\mathcal{T}$ is as defined in Block 5, X_v is as defined in Eq. (9) and, for any matrices C_v, D_{vw} and D_{vu}, $\mathcal{X}_{w,u}$ is as defined in Eq. (13). Then X_0 is complete with respect to $\|\cdot\|_{X^n}$ and $\mathcal{X}_{0,0}$ is complete with respect to $\|\cdot\|_{\mathbb{R}^{n_x}\times X^n}$. Furthermore, $\hat{\mathcal{T}} : L_2^{n_{\hat{\mathbf{x}}}} \to X_0$ and $\mathcal{T} : \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}} \to \mathcal{X}_{0,0}$ are unitary (isometric surjective mappings between Hilbert spaces).*

Proof. From Theorem 10 and Corollary 13, we have that $\mathcal{T}$ is a bijective mapping from $\mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$ to $\mathcal{X}_{0,0}$. From Theorem 18, we have that $\mathcal{T}$ is isometric with respect to the $\mathbb{R}^{n_x}\times X^n$ inner product. Since $\mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$ is complete, we conclude that $\mathcal{X}_{0,0}$ is complete with respect to the $\mathbb{R}^{n_x} \times X^n$ norm. Completeness of X_0 follows trivially from the special case $n_x = 0$. □

As a direct consequence of Corollary 19 and Lemma 17, X_0 and $\mathcal{X}_{0,0}$ are also complete with respect to $\|\cdot\|_{W^n}$ and $\|\cdot\|_{\mathbb{R}^{n_x}\times W^n}$, respectively.

B.3 Equivalence of Internal Stability Properties

As shown in Theorem 18, the natural definition of exponential stability of a GPDE model is with respect to the $\mathbb{R}^{n_x}\times X^n$ norm. However, as shown in Lemma 17, exponential stability with respect to the $\mathbb{R}^{n_x}\times X^n$ norm is equivalent to exponential stability with respect to the $\mathbb{R}^{n_x} \times W^n$ norm. Hence, we formally define stability with respect to the $\mathbb{R}^{n_x} \times W^n$ norm.

Definition 20 (Exponential Stability of a GPDE model)**.** *We say a GPDE model defined by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ is exponentially stable if there exist constants M, $\alpha > 0$ such that for any $\{x^0, \hat{\mathbf{x}}^0\} \in \mathcal{X}_{0,0}$, if $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfies the GPDE defined by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\{x^0, \hat{\mathbf{x}}^0\}$ and input $\{0, 0\}$, then*

$$\left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} \le M \left\| \begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} e^{-\alpha t} \quad \text{for all } t \ge 0.$$

Clearly, internal stability of a PIE system is with respect to the $\mathbb{R}L_2$ norm.

Definition 21 (Exponential Stability of a PIE system)**.** *We say a PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ is exponentially stable if there exist M, $\alpha > 0$ such that for any $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$, if $\{\underline{\mathbf{x}}, z, y\}$ satisfies the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ with initial condition $\underline{\mathbf{x}}^0$ and input $\{0, 0\}$, then $\|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2} \le M \|\underline{\mathbf{x}}^0\|_{\mathbb{R}L_2} e^{-\alpha t}$ for all $t \ge 0$.*

Exponential stability of a GPDE model is equivalent to exponential stability of the associated PIE system.

Theorem 22. *Given $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with $\{n, \mathbf{G}_\mathrm{b}\}$ admissible, the GPDE model defined by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ is exponentially stable if and only if the PIE defined by $\mathbf{G}_{\mathrm{PIE}} := \mathbf{M}(\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{p}\})$ is exponentially stable.*

Proof. The proof is a direct application of the stability definitions, Theorem 18, and Lemma 17 ([25, Appendix G]). □

The results of Theorem 22 also imply that Lyapunov and asymptotic stability of the GPDE model in the $\mathbb{R}^{n_x} \times W^n$ norm are equivalent to Lyapunov and asymptotic stability of the associated PIE system in the $\mathbb{R}L_2$ norm. Lyapunov and asymptotic stability of GPDEs and PIEs are defined as follows.

Definition 23 (Lyapunov Stability)**.**

1) We say a GPDE model defined by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ is Lyapunov stable, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for any $\{x^0, \hat{\mathbf{x}}^0\} \in \mathcal{X}_{0,0}$ with $\left\| \begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} < \delta$, if $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfies the GPDE defined by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\{x^0, \hat{\mathbf{x}}^0\}$ and input $\{0, 0\}$, then

$$\left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} < \epsilon \qquad \text{for all } t \ge 0.$$

2) We say a PIE model defined by $\mathbf{G}_{\mathrm{PIE}}$ is Lyapunov stable if for every $\epsilon > 0$ there exists a constant $\delta > 0$ such that for any $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{m,n}$ with $\|\underline{\mathbf{x}}^0\|_{\mathbb{R}L_2^{m,n}} < \delta$, if $\{\underline{\mathbf{x}}, z, y\}$ satisfies the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ with initial condition $\underline{\mathbf{x}}^0$ and input $\{0, 0\}$, then $\|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2^{m,n}} < \epsilon$ for all $t \geq 0$.

Definition 24 (Asymptotic Stability)**.**

1) We say a GPDE defined by $\{n, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ is asymptotically stable, if for every $\{x^0, \hat{\mathbf{x}}^0\} \in \mathcal{X}_{0,0}$ and $\epsilon > 0$, there exists a $T_\epsilon > 0$ such that if $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfies the GPDE defined by $\{n, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ with initial condition $\{x^0, \hat{\mathbf{x}}^0\}$ and input $\{0, 0\}$, then $\left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x} \times W^n} < \epsilon$ for all $t > T_\epsilon$.

2) We say a PIE model defined by $\mathbf{G}_{\mathrm{PIE}}$ is asymptotically stable, if for every $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{m,n}$ and $\epsilon > 0$, there exists a $T_\epsilon > 0$ such that if $\{\underline{\mathbf{x}}, z, y\}$ satisfies the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ with initial condition $\underline{\mathbf{x}}^0$ and input $\{0, 0\}$, then there exists $T_\epsilon > 0$ such that $\|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2^{m,n}} < \epsilon$ for all $t > T_\epsilon$.

Corollary 25. *Given $\{n, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ with $\{n, \mathbf{G}_{\mathrm{b}}\}$ admissible, let $\mathbf{G}_{\mathrm{PIE}} := \mathbf{M}(\{n, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{p}}\})$. Then*

1) The GPDE model defined by $\{n, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ is Lyapunov stable if and only if the PIE system defined by $\mathbf{G}_{\mathrm{PIE}}$ is Lyapunov stable.

2) The GPDE model defined by $\{n, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ is asymptotically stable if and only if the PIE system defined by $\mathbf{G}_{\mathrm{PIE}}$ is asymptotically stable.

Proof. Based on the stability definitions, this result is a direct corollary of Theorem 22 (See [Appendix G] [25]). ☐

C. Convex Conditions for Internal Stability of a GPDE model

In this subsection, we show how the PIE system representation may be used to determine internal stability of the corresponding GPDE model in the $\mathbb{R}L_2$ norm (as opposed to the $\mathbb{R} \times W^n$ norm). In this case, however, we do not establish stability of the PIE system itself in any sense – bounds on the $\mathbb{R}L_2$ norm cannot be used to bound the $\mathbb{R} \times W^n$ norm. Thus the stability test is defined in terms of the PIE system representation but is not actually a test for stability of the PIE system. In addition, the stability test is defined in terms of the existence of positive semidefinite PI operators subject to affine equality constraints. Such forms of convex optimization are labelled Linear PI Inequalities (LPIs) and LMI-based methods for the feasibility of LPIs have been discussed in, e.g. [18]. The following is a direct extension of Theorem 6 in [18].

Theorem 26. *Given $\{n, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ with $\{n, \mathbf{G}_{\mathrm{b}}\}$ admissible, let $\mathbf{G}_{\mathrm{PIE}} := \mathbf{M}(\{n, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{p}}\})$. Suppose there exist $\epsilon, \delta > 0$, and $\mathcal{P} \in \Pi_{n_x, n_{\hat{\mathbf{x}}}}^{n_{\hat{\mathbf{x}}}, n_{\hat{\mathbf{x}}}}$ such that $\mathcal{P} = \mathcal{P}^* \geq \epsilon I$ and*

$$\mathcal{A}^* \mathcal{P} \mathcal{T} + \mathcal{T}^* \mathcal{P} \mathcal{A} \leq -\delta \mathcal{T}^* \mathcal{T}.$$

Then the GPDE model defined by $\{n, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ is exponentially stable in the $\mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$ norm.

While a complete discussion of LPI tests for analysis and control of PIEs and GPDEs is beyond the scope of this paper, we note that other LPI tests for properties of the GPDE in terms of the associated PIE include L_2-gain [23], H_∞-optimal estimator design [7], and H_∞-optimal full-state feedback controller synthesis [24].

VII. PIETOOLS: A Software Package for GPDE models and PIE Representation

Because the GPDE class of model is meant to be "universal", construction of a GPDE requires the identification of a large number of system parameters — most of which are typically zero or sparse. Furthermore, construction of the associated PIE system using the formulae in Blocks 4 and 5 can be cumbersome, requiring one to parse a rather complicated notational system. This complicated process of identification of parameters and application of formulae may thus be an impediment to practical application of the results in this paper. For this reason, PIETOOLS versions 2021a and later include software interfaces for construction of GPDE models which do not require the user to understand of the notational system defined in this paper. For example, PIETOOLS 2021b (Available from [26]) includes a Graphical User Interface (GUI) which allows the user to define a GPDE data structure one term at a time. Because many GPDE models only consist of a few terms, this GUI dramatically reduces the time required to declare a GPDE model. Furthermore, this GUI automates the application of the formulae in Blocks 4 and 5 – allowing the user to construct an associated PIE system data structure which is compatible with the PIETOOLS utilities for analysis, control and simulation of PIEs. Additional details can be found in the PIETOOLS user manual [22]. In addition to the GUI, PIETOOLS includes many tools for the analysis, control, estimation and simulation of PIE systems in the context of: simple PDE models, advanced GPDE models and Delay Differential Equations. In the following section, we apply this GUI to several GPDE models. In some cases, we will also include results generated by the analysis, control and simulation tools in PIETOOLS as applied to the PIE systems associated with these GPDE models.

VIII. Examples of the PIE representation

In this section, we illustrate the PIE representation of three GPDE models. In most cases, we use the PIETOOLS GUI, as described in Section VII, to construct the associated PIE system. More examples can be found in [25, Appendix J].

A. Damped wave equation with delay and motor dynamics

First, we revisit the GPDE model studied in Section III-D and Section IV-A1. Since we have already identified the parameters of the GPDE model and applied the formulae in Block 4, we do not use the GUI defined in Section VII and instead directly construct the associated PIE system using the formulae in Block 5. This allows us to retain the parameter dependencies of the original GPDE model which now appear in the 4-PI parameters of $\mathbf{G}_{\mathrm{PIE}}$ which defines the associated PIE system. However, for simplicity, we choose $\mu(s) = 1$ which yields the following nonzero PIE system parameters.

$$\mathcal{T} = \mathcal{P} \left[\begin{array}{c|c} 1 & \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 \\ -K_t \\ -K_t s \end{bmatrix} & \left\{ 0_3, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -s\theta & -\theta \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -s\theta & -s \end{bmatrix} \right\} \end{array} \right],$$

$$\mathcal{B}_2=\mathcal{P}\left[\begin{array}{c|c}1 & \emptyset\\ \hline 0_{3,1} & \{\emptyset\}\end{array}\right],\ \mathcal{C}_2=\mathcal{P}\left[\begin{array}{c|c}0 & \begin{bmatrix}1&0&0\end{bmatrix}\\ \hline \emptyset & \{\emptyset\}\end{array}\right]$$

$$\mathcal{A}=\mathcal{P}\left[\begin{array}{c|c}-\frac{R}{L} & 0_{1,3}\\ \hline \begin{bmatrix}0\\0\\0\end{bmatrix} & \left\{\begin{bmatrix}0&0&1\\0&\frac{1}{\tau}&0\\0&0&0\end{bmatrix},0_3,0_3\right\}\end{array}\right],\ \mathcal{T}_w=\mathcal{P}\left[\begin{array}{c|c}\emptyset & \emptyset\\ \hline \begin{bmatrix}0\\0\\s\end{bmatrix} & \{\emptyset\}\end{array}\right]$$

$$\mathcal{C}_1=\mathcal{P}\left[\begin{array}{c|c}\begin{bmatrix}-0.5K_t\\0\end{bmatrix} & \begin{bmatrix}0&-0.5s&-0.5s^2-s\\0&0&0\end{bmatrix}\\ \hline \emptyset & \{\emptyset\}\end{array}\right],$$

$$\mathcal{D}_{12}=\mathcal{P}\left[\begin{array}{c|c}\begin{bmatrix}0\\1\end{bmatrix} & \emptyset\\ \hline \emptyset & \{\emptyset\}\end{array}\right],\ \mathcal{D}_{11}=\mathcal{P}\left[\begin{array}{c|c}\begin{bmatrix}0.5\\0\end{bmatrix} & \emptyset\\ \hline \emptyset & \{\emptyset\}\end{array}\right].$$

B. A 4^{th} order PDE: Timoshenko Beam Equation

In this example, we find the PIE system associated with a GPDE model with a 4^{th} order spatial derivative. While the dynamics of the Timoshenko beam [28] are often modeled as two coupled 2^{nd} order PDEs, if the beam is elastic, isotropic and homogeneous with constant cross-section then these equations can be combined to obtain a 4^{th} order GPDE representation.

$$\rho A\ddot{w}(t,s)-\left(\rho I+\frac{EI\rho}{\kappa G}\right)\partial_s^2\ddot{w}(t,s)+\frac{\rho^2 I}{\kappa G}\ddddot{w}(t,s)$$
$$=-EI\partial_s^4 w(t,s),$$

where ρ is the density of the beam material, A is the cross section area, I is the second moment of area, κ is the Timoshenko beam constant, E is the elastic modulus and G is the shear modulus. For simplicity, we take $\rho=A=I=\kappa=G=E=1$. The BCs are given by $w(t,0)=0$, $\partial_s w(t,0)=0$, $\partial_s^2 w(t,1)=w(t,1)$, and $\partial_s^3 w(t,1)=\partial_s w(t,1)$.

To eliminate the higher-order time derivates, we define the state variables as $w,\dot{w},\ddot{w}$ and $\dddot{w}$ and based on the generator and BCs, we partition the state variables as $\hat{\mathbf{x}}_0=\text{col}(\dot{w},\dddot{w})$, $\hat{\mathbf{x}}_1=\ddot{w}$, and $\hat{\mathbf{x}}_2=w$. The full state is then $\hat{\mathbf{x}}=col(\hat{\mathbf{x}}_0,\hat{\mathbf{x}}_2,\hat{\mathbf{x}}_4)$ so that the continuity condition is $n=\{2,0,1,0,1\}$, implying $n_{S_1}=2$, $n_{S_2}=2$, $n_{S_3}=1$, $n_{S_4}=1$, and hence $n_S=6$. Because we require $n_{BC}=n_S$, we need two additional BCs. To get these new BCs, we differentiate $w(t,0)=0$ and $\partial_s w(t,0)=0$ twice in time to obtain $\ddot{w}(t,0)=0$ and $\partial_s\ddot{w}(t,0)=0$. We now use the PIETOOLS GUI to calculate the PIE representation as

$$\underbrace{\mathcal{P}\left[\begin{array}{c|c}\emptyset & \emptyset\\ \hline \emptyset & \{G_i\}\end{array}\right]}_{\mathcal{T}}\dot{\underline{\hat{\mathbf{x}}}}(t,s)=\underbrace{\mathcal{P}\left[\begin{array}{c|c}\emptyset & \emptyset\\ \hline \emptyset & \{A_i\}\end{array}\right]}_{\mathcal{A}}\underline{\hat{\mathbf{x}}}(t,s)$$

where $\underline{\hat{\mathbf{x}}}=\text{col}(\dot{w},\dddot{w},\partial_s^2\ddot{w},\partial_s^4 w)$ and

$$A_0(s)=\begin{bmatrix}0&0&0&0\\0&0&1&-1\\0&1&0&0\\1&0&0&0\end{bmatrix},\ A_1(s,\theta)=\begin{bmatrix}0&0&s-\theta&0\\0&0&\theta-s&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

$$A_2(s,\theta)=0_4,\ f_0(s,\theta)=-\frac{1}{39}s^3\theta^3+\frac{s^2\theta^2}{26}(3s-\theta-2),$$

$$G_0(s)=\begin{bmatrix}I_2&\\&0_2\end{bmatrix},\ G_2(s,\theta)=\begin{bmatrix}0_3&\\&f_0(s,\theta)-\frac{1}{6}s^2(s+3\theta)\end{bmatrix},$$

$$G_1(s,\theta)=\begin{bmatrix}0_2&\\&\begin{bmatrix}s-\theta&0\\0&f_0(s,\theta)+\frac{1}{6}\theta^2(3s-\theta)\end{bmatrix}\end{bmatrix}.$$

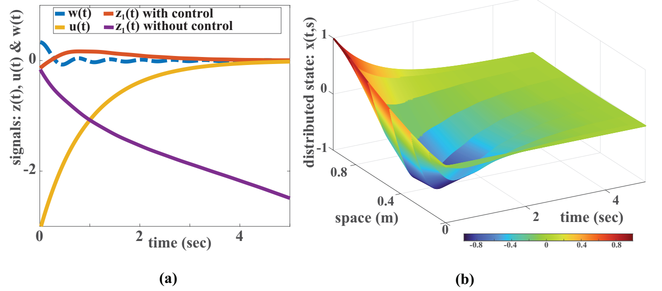

Fig. 6: A simulation of the closed-loop GPDE model in Section VIII-C using the stabilizing state-feedback controller generated using PIETOOLS. The initial conditions and decaying sinusoidal disturbance, w, are as defined in section VIII-C.

C. Controller Simulation of Reaction-Diffusion Equations

Consider the a reaction-diffusion PDE model with an ODE-based controller acting at the boundary.

$$\dot{x}(t)=-x(t)+u(t),\ \dot{\mathbf{x}}(t,s)=\lambda\mathbf{x}(t,s)+\partial_s^2\mathbf{x}(t,s)+w(t),$$
$$z(t)=\begin{bmatrix}\int_0^1\mathbf{x}(t,s)ds\\u(t)\end{bmatrix},\quad s\in(0,1),\quad t\geq 0$$
$$\mathbf{x}(t,0)=0,\ \mathbf{x}(t,1)=x(t),\ x(0)=0,\ \mathbf{x}(0,s)=\sin(\pi s),$$

where x is the state of the dynamic boundary controller, $\mathbf{x}$ is the distributed state, z is the regulated output and w is a disturbance. The control input, $u(t)$, enters the system through an ODE (typical for RC motor implementation) which is then coupled with the PDE state x at the boundary. Using the PIETOOLS GUI to define the GPDE, we construct the associated PIE system. For $\lambda=10$, the open loop GPDE model is unstable. However, the PIETOOLS tool for stabilizing state-feedback controller synthesis (based on [24]) provides the following state-feedback controller.

$$u(t)=-13.45x(t)+\int_0^1 k(s)\partial_s^2\mathbf{x}(s,t)ds,\qquad\text{where}$$
$$k(s)=-9.39s^{10}+19.7s^9+34.7s^8-124s^7+83.5s^6$$
$$+48.2s^5-78.9s^4+25.4s^3+3.98s^2-8.73s+6.61.$$

We now use the PIESIM package in PIETOOLS to simulate the closed-loop PIE system and reconstruct the GPDE solution where the disturbance is $w(t)=\frac{sin(10t)}{10t+10^{-5}}$ (the 10^{-5} term is added to avoid ill-conditioning at $t=0$). Both the output and control input are shown in Figure 6(a) – verifying that the proposed controller stabilizes the system.

IX. Conclusion

We have considered a generalized class of coupled ODE-PDE models (GPDEs) which can be used to define analysis, simulation and optimal control/estimation problems. This generalized class allows for inputs and outputs which enter through the limit values of the GPDE model, through the in-domain dynamics of the PDE subsystem and through a coupled ODE. The GPDE class allows for integral constraints on the PDE state. Additionally, we may model integrals of the PDE state acting: on the PDE dynamics; on a coupled ODE; or on the outputs of the system. Finally, this class includes PDE models with n^{th}-order spatial derivatives. The GPDE model unifies several existing classes of PDE models in a single parameterized framework.

Having parameterized a broad class of coupled ODE-PDE models, we proposed a test for admissibility of a given GPDE model and shown that admissibility implies the existence of an associated Partial Integral Equation (PIE) representation of the GPDE model with a unitary map from state of the PIE system to state of the GPDE model. The class of PIE systems is parameterized by the $*$-algebra of PI operators and we have furthermore shown that the unitary map from PIE to GPDE state is itself a PI operator. Finally, we have shown that many properties of the GPDE model and associated PIE system are equivalent – including existence of solutions, input-output properties, internal stability, and controllability.

To aid in practical application of the proposed GPDE models and PIE conversion formulae, we have described efficient open-source software (PIETOOLS) for the construction of the GPDE model, conversion to PIE system, simulation of the GPDE/PIE, and analysis/control of the GPDE/PIE. This software includes a GUI for construction of GPDE models and conversion to an associated PIE system – a feature demonstrated on several example problems.

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