Integral Quadratic Constraints with Infinite-Dimensional Channels

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Abstract— **Modern control theory provides us with a spectrum of methods for studying the interconnection of dynamic systems using input-output properties of the interconnected subsystems. Perhaps the most advanced framework for such input-output analysis is the use of Integral Quadratic Constraints (IQCs), which considers the interconnection of a nominal linear system with an unmodelled nonlinear or uncertain subsystem with known input-output properties. Although these methods are widely used for Ordinary Differential Equations (ODEs), there have been fewer attempts to extend IQCs to infinite-dimensional systems. In this paper, we present an IQC-based framework for Partial Differential Equations (PDEs) and Delay Differential Equations (DDEs). First, we introduce infinite-dimensional signal spaces, operators, and feedback interconnections. Next, in the main result, we propose a formulation of hard IQC-based input-output stability conditions, allowing for infinite-dimensional multipliers. We then show how to test hard IQC conditions with infinite-dimensional multipliers on a nominal linear PDE or DDE system via the Partial Integral Equation (PIE) state-space representation using a sufficient version of the Kalman-Yakubovich-Popov lemma (KYP). The results are then illustrated using four example problems with uncertainty and nonlinearity.**

I. INTRODUCTION

As developed in the 1970's and best exemplified by transfer-function-based properties of passivity and the small-gain condition, the Input-Output framework was a response to the increasing complexity of circuit-based subsystems. This framework obviated the need for a precise system model by characterizing the input-output behaviour of a system in terms of the input-output behaviour of its subsystems. However, by completely eliminating the model, and by only considering a subset of input-output properties, passivity and small-gain conditions resulted in substantially conservative results.

An attempt to improve the accuracy of the input-output framework was the use of multipliers proposed by Zames-Falb [1], Yakubovich [2] and others. However, verification of these multiplier-based conditions proved difficult. The Integral Quadratic Constraints (IQC) framework, introduced by Megretski and Rantzer [3], provided an attempt to simplify the multiplier-based input-output framework while also integrating modern model-based computational methods such as Linear Matrix Inequalities (LMIs) via generalizations of the Kalman-Yakubovich-Popov (KYP) lemma [4]. While this framework originally required homotopy in the unmodeled subsystem, recent works [5], [6] have attempted to remove the homotopy condition – thereby allowing for analysis of known nonlinear subsystems.

Despite the success of the IQC framework, its application to delayed and PDE systems has been limited. Specifically, most work on this topic treats the delayed or PDE dynamics as an unknown subsystem, with certain characterization of its input-output behaviour (e.g. [7], [8], [9], [10]). While this was a reasonable approach at a time when analysis of linear delayed and PDE systems was considered computationally challenging, recent work has shown that model-based computational evaluation of the input-output properties of linear delayed and PDE systems can be performed efficiently and accurately. As a result, the framework for IQC-based analysis has shifted, where now delayed and PDE components are contained in the nominal subsystem and nonlinearities and uncertainties are isolated in the unmodelled subsystem. This paradigm shift, however, means that the interconnecting signals between nominal and unknown subsystem may now be infinite-dimensional.

Recent attempts to consider delayed and PDE models in the known subsystem include the projection-based approach in [11] (wherein the interconnection signals are finite-dimensional) and the Sum-of-Squares-based dissipation inequalities in [12] (wherein the interconnection signals are infinite-dimensional). Neither of these results, however, directly consider the problem of extension of the IQC framework to subsystems interconnected by infinite-dimensional signal spaces.

The goal of this paper, therefore, is to propose a generalization of the hard IQC framework for a nominal infinite dimensional system interconnected with a nonlinear or uncertain subsystem by infinite-dimensional signals. We will accomplish this goal in three steps. First, we extend the IQC framework to infinite-dimensional systems, signals, interconnections, and multipliers, and generalize an IQC stability theorem to such interconnections.

Next, we assume both the nominal subsystem and multiplier can be represented as a Partial Integral Equation (PIE) as discussed in section V. PIE representations exist for most infinite-dimensional linear systems, including those with delays and those governed by PDEs. The existence of a PIE representation allows hard IQC conditions to be tested numerically using algorithms for the optimization of positive Partial Integral operators. Based on this PIE representation, we extend the KYP Lemma and use this extension to propose convex tests for conditions of the IQC theorem to be satisfied.

Finally, we examine several classes of nonlinearity and uncertainty with infinite-dimensional inputs and outputs and

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show that they satisfy a generalized version of the hard IQC constraints typically used for finite-dimensional systems.

Having completed these three steps, we then apply the results to several specific examples of delayed and PDE systems and show that the proposed approach is an improvement over alternatives such as quadratic stability.

II. Notation and Signal Spaces

a) Notation: We denote by $\mathbb{N}, \mathbb{R}, \mathbb{S}^n, I$ and $\mathbf{0}$ the natural numbers, the real numbers, the space of $n \times n$ symmetric matrices, the identity operator, and the null operator, respectively. For $\Omega \subset \mathbb{R}$, $L_2^n(\Omega)$ denotes the space of Lebesgue square integrable functions $f : \Omega \to \mathbb{R}^n$ with inner-product $\langle f, g\rangle_{L_2} = \int_\Omega f(s)^T g(s) ds$. For Hilbert space $\mathbf{H}$ we use $\mathbf{L_H}$ to denote the extension of L_2 to square-integrable functions $\mathbf{u} : [0, \infty) \to \mathbf{H}$. $\mathbf{L_H}$ is itself a Hilbert space [13] with associated inner product

$$\langle \mathbf{u}, \mathbf{v}\rangle_{\mathbf{L_H}} = \int_0^\infty \langle \mathbf{u}(t), \mathbf{v}(t)\rangle_{\mathbf{H}}\, dt.$$

Clearly, if $\mathbf{H} = \mathbb{R}^n$, then $\mathbf{L_H} = L_2^n[0, \infty)$. Associated with $\mathbf{L_H}$, we define an extended space, $\mathbf{L}_{e,\mathbf{H}}$, of functions, $\mathbf{u} : [0, \infty) \to \mathbf{H}$ such that for any $T \geq 0$, we have that

$$\int_0^T \|\mathbf{u}(t)\|_{\mathbf{H}}^2\, dt = \int_0^T \langle \mathbf{u}(t), \mathbf{u}(t)\rangle_{\mathbf{H}}\, dt$$

is finite.

Given $a, b \in \mathbb{R}$ and $\mathbf{n} \in \mathbb{N}^2$, we denote the Hilbert space $\mathbf{Z^n} := \mathbb{R}^{n_1} \times L_2^{n_2}[a, b]$ (with inner product $\langle (u, U), (v, V)\rangle = u^T v + \langle U, V\rangle_{L_2}$) and the extended signal space $\mathbf{L}^{\mathbf{n}}_{e,[a,b]} := \mathbf{L}_{e,\mathbf{Z^n}}$.

For notational convenience, given $\mathbf{u}(t) = (u(t), U(t)) \in \mathbf{Z^n}$ and $\mathbf{v}(t) = (v(t), V(t)) \in \mathbf{Z^m}$, we define the component-wise concatenation of $\mathbf{u}(t), \mathbf{v}(t)$ as

$$\begin{bmatrix}\mathbf{u}(t)\\ \mathbf{v}(t)\end{bmatrix} := \left(\begin{bmatrix}u(t)\\ v(t)\end{bmatrix}, \begin{bmatrix}U(t)\\ V(t)\end{bmatrix}\right) \in \mathbf{Z^{n+m}}.$$

Moreover, given $\mathbf{u} \in \mathbf{L}^{\mathbf{n}}_{e,[a,b]}$ and $\mathbf{v} \in \mathbf{L}^{\mathbf{m}}_{e,[a,b]}$, we use the notation

$$\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}(t) := \begin{bmatrix}\mathbf{u}(t)\\ \mathbf{v}(t)\end{bmatrix} \in \mathbf{Z^{n+m}}$$

b) Operators: For any Hilbert space, $\mathbf{H}$, we define the truncation operator $P_\tau : \mathbf{L}_{e,\mathbf{H}} \to \mathbf{L}_{e,\mathbf{H}}$ for any $\mathbf{y} \in \mathbf{L}_{e,\mathbf{H}}$ as

$$(P_\tau \mathbf{y})(t) = \begin{cases}\mathbf{y}(t), & 0 \leq t \leq \tau\\ 0, & \text{otherwise.}\end{cases}$$

Definition 1: Let $\mathbf{H}, \mathbf{G}$ be Hilbert spaces, then an operator $G : \mathbf{L}_{e,\mathbf{H}} \to \mathbf{L}_{e,\mathbf{G}}$ is

1) **Causal** if $P_\tau G = P_\tau G P_\tau$. for any $\tau \geq 0$.
2) **Bounded on $\mathbf{L_H}$** if there exist $C \geq 0$ such that for all $\mathbf{v} \in \mathbf{L_H}$, we have that $\|G\mathbf{v}\|_{\mathbf{L_G}} \leq C\|\mathbf{v}\|_{\mathbf{L_H}}$.
3) **Bounded on $\mathbf{L}_{e,\mathbf{H}}$** if there exist $C \geq 0$ such that for all $\mathbf{v} \in \mathbf{L}_{e,\mathbf{H}}$, we have that $\|P_\tau G\mathbf{v}\|_{\mathbf{L_G}} \leq C\|P_\tau \mathbf{v}\|_{\mathbf{L_H}}$ for all $\tau \geq 0$.
4) **Incrementally $\mathbf{L}_{e,\mathbf{H}}$-bounded** if there exist $C \geq 0$ such that for all $\mathbf{v}, \mathbf{u} \in \mathbf{L}_{e,\mathbf{H}}$, we have that $\|P_\tau(G\mathbf{v} - G\mathbf{u})\|_{\mathbf{L_G}} \leq C\|P_\tau(\mathbf{v} - \mathbf{u})\|_{\mathbf{L_H}}$ for all $\tau \geq 0$.

For causal operators, bounded on $\mathbf{L_H}$ is equivalent to bounded on $\mathbf{L}_{e,\mathbf{H}}$. For a causal linear operator, bounded on $\mathbf{L_H}$ is equivalent to incrementally $\mathbf{L}_{e,\mathbf{H}}$-bounded.

The set of all causal, bounded linear operators between Hilbert spaces, $\mathbf{H}_1$ and $\mathbf{H}_2$, is denoted $\mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ and is a Banach space with induced norm [13]. We denote $\mathcal{L}(\mathbf{H}) := \mathcal{L}(\mathbf{H}, \mathbf{H})$.

Given a bounded linear operator $\mathcal{K} : \mathbf{H} \to \mathbf{H}$, we define the associated multiplication operator $M_\mathcal{K} : \mathbf{L}_{e,\mathbf{H}} \to \mathbf{L}_{e,\mathbf{H}}$ for $w \in \mathbf{L}_{e,\mathbf{H}}$ by

$$(M_\mathcal{K}\mathbf{w})(t) := \mathcal{K}\mathbf{w}(t) \in \mathbf{H}.$$

c) PI operators: We say $\mathcal{P}$ is a Partial Integral (PI) operator on $\mathbf{Z}^n$, if there exists matrix P, bounded functions Q_1, Q_2, R_0, and separable functions R_1, R_2 such that

$$\left(\mathcal{P}\begin{bmatrix}x\\ \mathbf{x}\end{bmatrix}\right)(s) := \begin{bmatrix}Px + \int_a^b Q_1(s)\mathbf{x}(s)ds\\ Q_2(s)x + (\mathcal{P}_{\{R_i\}}\mathbf{X})(s)\end{bmatrix},$$

where

$$\mathcal{P}_{\{R_i\}}\mathbf{x})(s) := R_0(s)\mathbf{x}(s) + \int_a^s R_1(s,\theta)\mathbf{x}(\theta)d\theta + \int_s^b R_2(s,\theta)\mathbf{x}(\theta)d\theta.$$

We denote the set of PI operators by Π_4

Given matrix P, bounded functions Q_1, Q_2, R_0, and separable functions R_1, R_2, the associated PI operator is denoted $\left(\mathcal{P}\begin{bmatrix}P & Q_1\\ Q_2 & \{R_i\}\end{bmatrix}\right) \in \Pi_4$. The set of PI operators form a $*$-algebra of bounded linear operators as discussed in, e.g. [14].

III. Feedback Interconnections on Hilbert Space

In this section, we consider basic definitions of the interconnection of systems G, Δ for the case when $\mathbf{H} = \mathbf{Z}$ where recall that $\mathbf{Z^n} := \mathbb{R}^{n_1} \times \mathbf{L}_2^{n_2}[a, b]$.

Definition 2 (Interconnection of G and Δ): Given operators $G : \mathbf{L}^{\mathbf{n}}_{e,[a,b]} \to \mathbf{L}^{\mathbf{m}}_{e,[a,b]}$ and $\Delta : \mathbf{L}^{\mathbf{m}}_{e,[a,b]} \to \mathbf{L}^{\mathbf{n}}_{e,[a,b]}$, and signals $\mathbf{e} \in \mathbf{L}^{\mathbf{n}}_{e,[a,b]}$ and $\mathbf{f} \in \mathbf{L}^{\mathbf{m}}_{e,[a,b]}$, we say that $\mathbf{u} \in \mathbf{L}^{\mathbf{n}}_{e,[a,b]}, \mathbf{v} \in \mathbf{L}^{\mathbf{m}}_{e,[a,b]}$ **satisfy the interconnection defined by** $[G, \Delta]$ if

$$\mathbf{v} = G\mathbf{u} + \mathbf{f} \qquad \text{and} \qquad \mathbf{u} = \Delta\mathbf{v} + \mathbf{e}. \tag{1}$$

Typically, G is a known causal bounded linear operator and Δ is either nonlinear or unknown, but lies in some set $\Delta \in \boldsymbol{\Delta}$ with known input-output properties. For a given G and Δ, we define the following notion of well-posedness of the feedback interconnection, guaranteeing existence and uniqueness of a causal mapping from inputs $\mathbf{e}, \mathbf{f}$ to outputs $\mathbf{u}, \mathbf{v}$.

Definition 3 (Well-posedness): Given operators $G : \mathbf{L}^{\mathbf{n}}_{e,[a,b]} \to \mathbf{L}^{\mathbf{m}}_{e,[a,b]}$ and $\Delta : \mathbf{L}^{\mathbf{m}}_{e,[a,b]} \to \mathbf{L}^{\mathbf{n}}_{e,[a,b]}$, we say the interconnection defined by $[G, \Delta]$ is well-posed if for any $\mathbf{e} \in \mathbf{L}^{\mathbf{n}}_{e,[a,b]}, \mathbf{f} \in \mathbf{L}^{\mathbf{n}}_{e,[a,b]}$, we have the following.

1) **Existence and Uniqueness:** There exist unique $\mathbf{u} \in \mathbf{L}^{\mathbf{n}}_{e,[a,b]}, \mathbf{v} \in \mathbf{L}^{\mathbf{m}}_{e,[a,b]}$ such that $\mathbf{u}, \mathbf{v}$ satisfy the interconnection defined by $[G, \Delta]$.
2) **Causality:** If $\mathbf{u}, \mathbf{v}$ satisfy the interconnection defined by $[G, \Delta]$ and $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ satisfy the interconnection defined

by $[G,\Delta]$ for $P_\tau \mathbf{e}, P_\tau \mathbf{f}$ for some $\tau \geq 0$, then $P_\tau(\mathbf{u} - \hat{\mathbf{u}}) = 0$ and $P_\tau(\mathbf{v} - \hat{\mathbf{v}}) = 0$.

Notation: Given $G, \Delta : \mathbf{L}_{e,[a,b]} \to \mathbf{L}_{e,[a,b]}$ if the interconnection defined by $[G,\Delta]$ is well-posed, then for $\mathbf{e}, \mathbf{f} \in \mathbf{L}_{e,[a,b]}$, we say that

$$\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} = \mathcal{F}_{G,\Delta}\left(\begin{bmatrix}\mathbf{e}\\ \mathbf{f}\end{bmatrix}\right)$$

if $\mathbf{u}, \mathbf{v}$ satisfies the interconnection defined by $[G,\Delta]$.

Definition 4: We say **the feedback system defined by** $[G,\Delta]$ **is stable** if the *interconnection defined by* $[G,\Delta]$ *is well-posed* and $\mathcal{F}_{G,\Delta}$ is bounded on $\mathbf{L}_{\mathbf{Z}}^{\mathbf{n+m}}$ where recall $\mathcal{F}_{G,\Delta}$ is bounded on $\mathbf{L}_{\mathbf{Z}}^{\mathbf{n+m}}$ if there exists a C such that for any $\mathbf{e} \in \mathbf{L}_{\mathbf{Z}}^{\mathbf{n}}, \mathbf{f} \in \mathbf{L}_{\mathbf{Z}}^{\mathbf{m}}$, if $\mathbf{u}, \mathbf{v}$ *satisfy the interconnection defined by* $[G,\Delta]$, then

$$\left\|\begin{bmatrix}\mathbf{v}\\ \mathbf{u}\end{bmatrix}\right\|_{\mathbf{L}} < C\left\|\begin{bmatrix}\mathbf{f}\\ \mathbf{e}\end{bmatrix}\right\|_{\mathbf{L}}$$

A. Integral Quadratic Constraints

Next, we extend the Hard IQC framework to infinite-dimensional systems.

Definition 5: We say the operator $\Delta : \mathbf{L}_{e,[a,b]}^{\mathbf{m}} \to \mathbf{L}_{e,[a,b]}^{\mathbf{n}}$ **satisfies the hard IQC** defined by operators $\Psi : \mathbf{L}_{e,[a,b]}^{\mathbf{n+m}} \to \mathbf{L}_{e,[a,b]}^{\mathbf{n+m}}$ and $\mathcal{K} : \mathbf{Z}^{\mathbf{n+m}} \to \mathbf{Z}^{\mathbf{n+m}}$, if for any $T > 0$ and for all $\mathbf{v} \in \mathbf{L}_{e,[a,b]}^{\mathbf{n+m}}$

$$\left\langle P_T\Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix}, P_T M_{\mathcal{K}}\Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix}\right\rangle_{\mathbf{L}} \geq 0, \tag{2}$$

where for all $\mathbf{w} \in \mathbf{L}_{e,[a,b]}^{\mathbf{n+m}}$

$$(M_{\mathcal{K}}\mathbf{w})(t) := \mathcal{K}\mathbf{w}(t) \in \mathbf{Z}^{\mathbf{n+m}}.$$

B. Problem formulation

Suppose we are given a known linear system/operator G and a set of nonlinear systems/operators $\mathbf{\Delta}$ where the "graph" of every $\Delta \in \mathbf{\Delta}$, defined as

$$\left\{\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix} \in \mathbf{L}_{e,[a,b]}^{\mathbf{n+m}} \;:\; \mathbf{v} \in \mathbf{L}_{e,[a,b]}^{\mathbf{n}}\right\},$$

is known to satisfy a set of certain Integral Quadratic Constraints (IQCs) as parameterized the set of operators $\mathbf{K}$ and $\mathbf{\Psi}$ – See Definition 5. Our goal is to show that if the inverse graph of G satisfies a similar IQC for some $(\mathcal{K},\Psi) \in \mathbf{K} \times \mathbf{\Psi}$, then the feedback interconnection of G and Δ is stable for all $\Delta \in \mathbf{\Delta}$.

IV. The Main Theorem

As discussed in the previous section, we would like to show that if the graph of Δ and the inverse graph of G are separated by some quadratic form defined by $\mathcal{K}$ and Ψ, then the feedback interconnection of G and Δ is stable. This result is given by Theorem 6, the proof of which is a generalization of the technique used in [15] and [16].

Theorem 6 (IQC theorem): Suppose $G : \mathbf{L}_{e,[a,b]}^{\mathbf{n}} \to \mathbf{L}_{e,[a,b]}^{\mathbf{m}}$ is bounded, $\Delta : \mathbf{L}_{e,[a,b]}^{\mathbf{m}} \to \mathbf{L}_{e,[a,b]}^{\mathbf{n}}$ and the interconnection defined by $[G,\Delta]$ is well-posed.

Further suppose there exists a causal, incrementally bounded on $\mathbf{L}_{e,[a,b]}^{\mathbf{n+m}}$ operator Ψ, and $\mathcal{K} \in \mathcal{L}(\mathbf{Z}^{\mathbf{n+m}})$ such that

1) Δ satisfies the hard IQC defined by $\Psi, \mathcal{K}$.
2) For any $\mathbf{u} \in \mathbf{L}_{e,[a,b]}^{\mathbf{n}}, \mathbf{v} \in \mathbf{L}_{e,[a,b]}^{\mathbf{m}}$ we have that

$$\left\langle P_T\Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}, P_T M_{\mathcal{K}}\Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}\right\rangle_{\mathbf{L}} \leq -\varepsilon\|P_T\mathbf{u}\|_{\mathbf{L}}^2, \tag{3}$$

for all $T > 0$.

Then the feedback system defined by $[G,\Delta]$ is stable.

Proof: Define the shorthand notation $\mathbf{u}_T := P_T\mathbf{u}$ and $\langle\mathbf{u},\mathbf{v}\rangle_T = \langle P_T\mathbf{u}, P_T\mathbf{v}\rangle_{\mathbf{L}}$. Clearly, by Cauchy Schwartz, $\langle\mathbf{u},\mathbf{v}\rangle_T \leq \|\mathbf{u}_T\|_{\mathbf{L}}\|\mathbf{v}_T\|_{\mathbf{L}}$.

Now, because the interconnection defined by $[G,\Delta]$ is well posed, the feedback system defined by $[G,\Delta]$ is stable if there exists a $C > 0$ such that for any $\mathbf{e}, \mathbf{f} \in \mathbf{L}_{\mathbf{Z}}$, if $\mathbf{u}, \mathbf{v}$ satisfy the interconnection defined by $[G,\Delta]$,

$$\left\|\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}\right\|_{\mathbf{L_Z}} \leq C\left\|\begin{bmatrix}\mathbf{e}\\ \mathbf{f}\end{bmatrix}\right\|_{\mathbf{L_Z}}.$$

Now, for $\mathbf{e}, \mathbf{f} \in \mathbf{L}_{[a,b]}$, suppose that $\mathbf{u}, \mathbf{v}$ satisfy the interconnection defined by G, Δ. Then $\mathbf{v} = G\mathbf{u} + \mathbf{f}$ and $\mathbf{u} = \Delta\mathbf{v} + \mathbf{e}$ and from equation (2),

$$\left\langle \Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix}, M_{\mathcal{K}}\Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix}\right\rangle_T + \left\langle \Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}, M_{\mathcal{K}}\Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}\right\rangle_T - \left\langle \Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}, M_{\mathcal{K}}\Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}\right\rangle_T \geq 0.$$

Subtracting Eqn. (3) from this expression, we find

$$\begin{aligned}\varepsilon\|\mathbf{u}_T\|_{\mathbf{L}} &\leq \left\langle \Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix}, M_{\mathcal{K}}\Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix}\right\rangle_T - \left\langle \Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}, M_{\mathcal{K}}\Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}\right\rangle_T\\ &= \left\langle \Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix} - \Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}, M_{\mathcal{K}}\Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix} - M_{\mathcal{K}}\Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}\right\rangle_T\\ &\quad + \left\langle \Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}, M_{\mathcal{K}}\Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix} - M_{\mathcal{K}}\Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}\right\rangle_T\\ &\quad + \left\langle \Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix} - \Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}, M_{\mathcal{K}}\Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}\right\rangle_T.\end{aligned}$$

Now, because Ψ and hence $M_{\mathcal{K}}\Psi$ are incrementally bounded with bounds C_Ψ and $C_{M_{\mathcal{K}}\Psi} = C_\Psi\|\mathcal{K}\|$, respectively, and also by Cauchy Schwartz inequality, we have

$$\begin{aligned}&\left\langle \Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix} - \Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}, M_{\mathcal{K}}\Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix} - M_{\mathcal{K}}\Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}\right\rangle_T\\ &\quad\leq \left\|\Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix} - \Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}\right\|\left\|M_{\mathcal{K}}\Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix} - M_{\mathcal{K}}\Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}\right\|_T\\ &\quad\leq C_\Psi C_{M_{\mathcal{K}}\Psi}\left(\left\|\begin{bmatrix}\mathbf{v} - G\mathbf{u}\\ \Delta\mathbf{v} - \mathbf{u}\end{bmatrix}_T\right\|_{\mathbf{L}}^2\right).\end{aligned}$$

Similarly, because G is bounded with bound C_G,

$$\begin{aligned}&\left\langle \Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}, M_{\mathcal{K}}\Psi\begin{bmatrix}\mathbf{v}\\ \Delta\mathbf{v}\end{bmatrix} - M_{\mathcal{K}}\Psi\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}\right\rangle_T\\ &\quad\leq C_\Psi C_{M_{\mathcal{K}}\Psi}\left\|\begin{bmatrix}G\mathbf{u}\\ \mathbf{u}\end{bmatrix}_T\right\|_{\mathbf{L}}\left\|\begin{bmatrix}\mathbf{v} - G\mathbf{u}\\ \Delta\mathbf{v} - \mathbf{u}\end{bmatrix}_T\right\|_{\mathbf{L}}\end{aligned}$$

and

$$\left\langle \Psi \begin{bmatrix} \mathbf{v} \\ \Delta \mathbf{v} \end{bmatrix} - \Psi \begin{bmatrix} G\mathbf{u} \\ \mathbf{u} \end{bmatrix}, M_{\mathcal{K}} \Psi \begin{bmatrix} G\mathbf{u} \\ \mathbf{u} \end{bmatrix} \right\rangle_T \\ \le C_\Psi C_{M_{\mathcal{K}}\Psi} \left\| \begin{bmatrix} G\mathbf{u} \\ \mathbf{u} \end{bmatrix}_T \right\|_{\mathbf{L}} \left\| \begin{bmatrix} \mathbf{v} - G\mathbf{u} \\ \Delta\mathbf{v} - \mathbf{u} \end{bmatrix}_T \right\|_{\mathbf{L}}.$$

Since $\mathbf{v} = G\mathbf{u} + \mathbf{f}$ and $\mathbf{u} = \Delta\mathbf{v} + \mathbf{e}$, we conclude that

$$\begin{aligned} \varepsilon \|\mathbf{u}_T\|_{\mathbf{L}}^2 &\le C_\Psi C_{M_{\mathcal{K}}\Psi} \Bigg(\left\| \begin{bmatrix} \mathbf{v} - G\mathbf{u} \\ \Delta\mathbf{v} - \mathbf{u} \end{bmatrix}_T \right\|_{\mathbf{L}}^2 + \\ &\qquad + 2 \left\| \begin{bmatrix} G\mathbf{u} \\ \mathbf{u} \end{bmatrix}_T \right\|_{\mathbf{L}} \left\| \begin{bmatrix} \mathbf{v} - G\mathbf{u} \\ \Delta\mathbf{v} - \mathbf{u} \end{bmatrix}_T \right\|_{\mathbf{L}} \Bigg) \\ &= C_\Psi C_{M_{\mathcal{K}}\Psi} \left(\left\| \begin{bmatrix} \mathbf{f} \\ \mathbf{e} \end{bmatrix}_T \right\|_{\mathbf{L}}^2 + 2 \left\| \begin{bmatrix} G\mathbf{u} \\ \mathbf{u} \end{bmatrix}_T \right\|_{\mathbf{L}} \left\| \begin{bmatrix} \mathbf{f} \\ \mathbf{e} \end{bmatrix}_T \right\|_{\mathbf{L}} \right) \\ &\le C_\Psi C_{M_{\mathcal{K}}\Psi} \left(\left\| \begin{bmatrix} \mathbf{f} \\ \mathbf{e} \end{bmatrix}_T \right\|_{\mathbf{L}}^2 + 2(C_G + 1) \|\mathbf{u}_T\|_{\mathbf{L}} \left\| \begin{bmatrix} \mathbf{f} \\ \mathbf{e} \end{bmatrix}_T \right\|_{\mathbf{L}} \right). \end{aligned}$$

Next, if we complete the square by adding $C_\Psi C_{M_{\mathcal{K}}\Psi}(C_G + 1)^2 \|\mathbf{u}_T\|_{\mathbf{L}}^2$ to both sides, we get

$$\varepsilon \|\mathbf{u}_T\|_{\mathbf{L}}^2 + C_\Psi C_{M_{\mathcal{K}}\Psi}(C_G+1)^2 \|\mathbf{u}_T\|_{\mathbf{L}}^2 \\ = C_\Psi C_{M_{\mathcal{K}}\Psi} \left(\left\| \begin{bmatrix} \mathbf{f} \\ \mathbf{e} \end{bmatrix}_T \right\|_{\mathbf{L}} + (C_G+1) \|\mathbf{u}_T\|_{\mathbf{L}} \right)^2.$$

Hence

$$\sqrt{\frac{\varepsilon}{C_\Psi C_{M_{\mathcal{K}}\Psi}} + (C_G+1)^2} \|\mathbf{u}_T\|_{\mathbf{L}} \\ \le \left\| \begin{bmatrix} \mathbf{f} \\ \mathbf{e} \end{bmatrix}_T \right\|_{\mathbf{L}} + (C_G+1) \|\mathbf{u}_T\|_{\mathbf{L}}$$

or for $\hat{\varepsilon} = \frac{\varepsilon}{C_\Psi C_{M_{\mathcal{K}}\Psi}}$ and $K = (C_G + 1)$ we have

$$\left(\sqrt{\hat{\varepsilon} + K^2} - K \right) \|\mathbf{u}_T\|_{\mathbf{L}} \le \left\| \begin{bmatrix} \mathbf{f} \\ \mathbf{e} \end{bmatrix}_T \right\|_{\mathbf{L}}.$$

Then, defining $C = \left(\sqrt{\hat{\varepsilon} + K^2} - K\right)^{-1} > 0$ we have

$$\|\mathbf{u}_T\|_{\mathbf{L}} \le C \left\| \begin{bmatrix} \mathbf{f} \\ \mathbf{e} \end{bmatrix}_T \right\|_{\mathbf{L}}.$$

Finally, we define $C_{\mathbf{v}} := C_g C + 1$. Thus we have

$$\|\mathbf{v}_T\|_{\mathbf{L}} \le C_G \|\mathbf{u}_T\|_{\mathbf{L}} + \|f\|_{\mathbf{L}} \le C_{\mathbf{v}} \left\| \begin{bmatrix} \mathbf{f} \\ \mathbf{e} \end{bmatrix}_T \right\|_{\mathbf{L}}$$

for all $T > 0$.

We conclude that the interconnection is stable. ∎

Remark: For a given class of Δ, we are typically given a set of valid $\mathcal{K}$ and Ψ (e.g. Lemma 10). While it is easy to search over a convex set of $\mathcal{K}$ for a given Ψ, it is not as easy to search over all possible valid Ψ multipliers. A typical approach, therefore, is to choose a collection of IQCs $\{\mathcal{K}_i, \Psi_i\}_{i=1}^n$ and note that any conic combination of the following form is also a valid IQC:

$$\Psi = \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_n \end{bmatrix} \quad \mathcal{K}(\lambda) = \begin{bmatrix} \lambda_1 \mathcal{K}_1 & & \\ & \ddots & \\ & & \lambda_n \mathcal{K}_n \end{bmatrix},$$

where $\{\lambda_i\}_{i=1}^n$ are any non-negative constants.

The set of $\mathcal{K}(\lambda)$ is convex thus allowing convex optimization methods to search for feasible values of λ_i. Other convex parameterizations exists for certain classes of IQC multipliers [17].

Our goal, then, is to find some $\mathcal{K}(\lambda), \Psi$ for which Eqn. (3) is satisfied. To achieve this goal, we require some way of characterizing the input-output properties of the multiplier-mapped graph $\Psi \begin{bmatrix} G\mathbf{u} \\ \mathbf{u} \end{bmatrix}$. For this problem, we turn to state-space representations and the KYP lemma as extended to infinite-dimensional systems in the Partial Integral Equation (PIE) framework.

V. Input-Output Analysis using PIEs and the KYP Lemma

In this section, we propose a method of using convex optimization to test the conditions of Theorem 6. This is accomplished in three parts. First, we assume the nominal system and multipliers, Ψ, are represented as Partial Integral Equations (PIEs). Second, we generalize the KYP lemma to PIEs. Finally, we pose the conditions of Theorem 6 as a convex optimization problem over the cone of positive Partial Integral (PI) operators.

A. Partial Integral Equations

Our method for testing the conditions of Theorem 6 assumes there exists a state-space representation of the systems $G, \Psi : \mathbf{u} \mapsto \mathbf{y}$ of the form

$$\begin{aligned} \mathcal{T}\dot{\mathbf{x}}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathcal{C}\mathbf{x}(t) + \mathcal{D}\mathbf{u}(t), \end{aligned} \tag{4}$$

where $\mathbf{x}(t) \in \mathbf{Z}^{\mathbf{k}}$, $\mathbf{y}(t) \in \mathbf{Z}^{\mathbf{m}}$, $\mathbf{u}(t) \in \mathbf{Z}^{\mathbf{n}}$ and $\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are PI operators with appropriate dimensions.

Remark: Note that most linear delayed and well-posed PDE systems can be represented in this form — See [18] for PDE and [14] for delayed systems.

When such a representation exists, it is referred to as a Partial Integral Equation (PIE). We use the PIE representation (4) as it is possible to optimize over the cone of positive PI operators using, e.g. [19]. This allows us to test the conditions of the the following infinite-dimensional version of the KYP lemma.

B. KYP lemma

To test the conditions of Theorem 6, we presume that $\Delta \in \boldsymbol{\Delta}$ satisfies the IQC for some $K \in \Pi_4$ and Ψ, where Ψ admits a PIE representation with parameters $\mathcal{T}_\Psi, \mathcal{A}_\Psi, \mathcal{B}_\Psi, \mathcal{C}_\Psi \in \Psi_4$.

Assuming for now that $\Psi = I$, the following Lemma provides conditions under which G satisfies the conditions of Theorem 6.

Lemma 7 (sufficient version of KYP lemma): Suppose $G \in \mathcal{L}(\mathbf{L}_{e,[a,b]}^{\mathbf{n}}, \mathbf{L}_{e,[a,b]}^{\mathbf{m}})$ is a causal bounded linear operator and there exist $\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \Pi_4$ such that for any $\mathbf{u} \in \mathbf{L}_{e,[a,b]}^{\mathbf{n}}$, $\mathbf{y} = G\mathbf{u}$ implies that $\mathbf{y}$ satisfies (4) for some

$\mathbf{x} \in \mathbf{L}^{\mathbf{k}}_{e,[a,b]}$. Given $\mathcal{K} \in \Pi_4$, suppose there exists some $\varepsilon > 0$ and $\mathcal{P} \in \Pi_4$ such that $\mathcal{P} \geq 0$ and

$$\begin{bmatrix} \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} & \mathcal{P}\mathcal{B} \\ \mathcal{B}^*\mathcal{P} & \varepsilon I \end{bmatrix} + \begin{bmatrix} \mathcal{C}^* \\ \mathcal{D}^* \end{bmatrix} \mathcal{K} \begin{bmatrix} \mathcal{C} & \mathcal{D} \end{bmatrix} \leq 0. \tag{5}$$

Then for any $\mathbf{u} \in \mathbf{L}^{\mathbf{n}}_{e,[a,b]}$ we have that

$$\langle P_T G\mathbf{u}, P_T M_{\mathcal{K}} G\mathbf{u}\rangle_{\mathbf{L}} \leq -\varepsilon \|P_T \mathbf{u}\|^2_{\mathbf{L}},$$

for all $T > 0$.

Proof: Define $V(\mathbf{x}) = \langle \mathcal{T}\mathbf{x}, \mathcal{P}\mathcal{T}\mathbf{x}\rangle_{\mathbf{Z}}$. Suppose that $\mathbf{u} \in \mathbf{L}^{\mathbf{n}}_{e,[a,b]}$ and $\mathbf{y}(t) = (G\mathbf{u})(t)$ for some $\mathbf{x} \in \mathbf{L}^{\mathbf{k}}_{e,[a,b]}$, such that Eqns. (4) are satisfied. By inequality (5), we have that

$$\begin{aligned}
&\left\langle \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}, \left(\begin{bmatrix} \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} & \mathcal{P}\mathcal{B} \\ \mathcal{B}^*\mathcal{P} & \varepsilon I \end{bmatrix} \right.\right. \\
&\qquad\qquad \left.\left. + \begin{bmatrix} \mathcal{C}^* \\ \mathcal{D}^* \end{bmatrix} \mathcal{K} \begin{bmatrix} \mathcal{C} & \mathcal{D} \end{bmatrix} \right) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \right\rangle_{\mathbf{Z}} \\
&\quad = \langle \mathcal{T}\dot{\mathbf{x}}(t), \mathcal{P}\mathcal{T}\mathbf{x}(t)\rangle + \langle \mathcal{T}\mathbf{x}(t), \mathcal{P}\mathcal{T}\dot{\mathbf{x}}(t)\rangle \\
&\qquad\qquad + \varepsilon \|\mathbf{u}(t)\|^2_{\mathbf{Z}} + \langle \mathbf{y}(y), \mathcal{K}\mathbf{y}(t)\rangle_{\mathbf{Z}} \\
&\quad = \dot{V}(\mathbf{x}(t)) + \varepsilon \|\mathbf{u}(t)\|^2_{\mathbf{Z}} + \langle \mathbf{y}(t), \mathcal{K}\mathbf{y}(t)\rangle_{\mathbf{Z}} \leq 0.
\end{aligned}$$

Now, since $V(\mathbf{x}(0)) = V(0) = 0$ and $V(\mathbf{x}(T)) \geq 0$, and integrating in time, we obtain

$$\begin{aligned}
\varepsilon \|P_T \mathbf{u}\|^2_{\mathbf{L}} + \int_0^T \langle \mathbf{y}(t), \mathcal{K}\mathbf{y}(t)\rangle_{\mathbf{Z}}\, dt &\leq -V(\mathbf{x}(T)) + V(\mathbf{x}(0)) \\
&\leq 0.
\end{aligned}$$

We conclude that

$$\langle P_T G\mathbf{u}, P_T M_{\mathcal{K}} G\mathbf{u}\rangle_{\mathbf{L}} = \int_0^T \langle \mathbf{y}(t), \mathcal{K}\mathbf{y}(t)\rangle_{\mathbf{Z}}\, dt \leq -\varepsilon \|P_T \mathbf{u}\|^2_{\mathbf{L}}.$$

■

Given PI operator $\mathcal{K}$, the conditions of Lemma 7 may be verified using software for optimization of PI operators as in [19].

C. Augmentation of the Dynamics

Now, we suppose that the multiplier $\Psi \in \mathcal{L}(\mathbf{L}^{\mathbf{n}+\mathbf{m}}_{e,[a,b]})$ also admits a PIE representation of the form

$$\begin{aligned}
\mathcal{T}_\Psi \dot{\mathbf{z}}(t) &= \mathcal{A}_\Psi \mathbf{z}(t) + \mathcal{B}_\Psi \mathbf{v}(t) \\
\mathbf{w}(t) &= \mathcal{C}_\Psi \mathbf{z}(t) + \mathcal{D}_\Psi \mathbf{v}(t).
\end{aligned} \tag{6}$$

Corollary 8 (Augmented KYP lemma): Suppose $G \in \mathcal{L}(\mathbf{L}^{\mathbf{n}}_{e,[a,b]}, \mathbf{L}^{\mathbf{m}}_{e,[a,b]})$ is a causal bounded linear operator and there exist $\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \Pi_4$ such that for any $\mathbf{u} \in \mathbf{L}^{\mathbf{n}}_{e,[a,b]}$, $\mathbf{y} = G\mathbf{u}$ implies that $\mathbf{y}$ satisfies (4) for some $\mathbf{x} \in \mathbf{L}^{\mathbf{k}_G}_{e,[a,b]}$.

Furthermore, suppose that there exist $\mathcal{T}_\Psi, \mathcal{A}_\Psi, \mathcal{B}_\Psi, \mathcal{C}_\Psi, \mathcal{D}_\Psi \in \Pi_4$ such that for any $\mathbf{v} \in \mathbf{L}^{\mathbf{n}+\mathbf{m}}_{e,[a,b]}$, $\mathbf{w} = \Psi\mathbf{v}$ implies that $\mathbf{w}$ satisfies (6) for some $\mathbf{z} \in \mathbf{L}^{\mathbf{k}_\Psi}_{e,[a,b]}$.

Given $\mathcal{K} \in \Pi_4$, suppose there exists some $\mathcal{P} \in \Pi_4$ such that $\mathcal{P} \geq 0$ and

$$\begin{bmatrix} \hat{\mathcal{T}}^*\mathcal{P}\hat{\mathcal{A}} + \hat{\mathcal{A}}^*\mathcal{P}\hat{\mathcal{T}} & \mathcal{P}\hat{\mathcal{B}} \\ \hat{\mathcal{B}}^*\mathcal{P} & \varepsilon I \end{bmatrix} + \begin{bmatrix} \hat{\mathcal{C}}^* \\ \hat{\mathcal{D}}^* \end{bmatrix} \mathcal{K} \begin{bmatrix} \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{bmatrix} \leq 0. \tag{7}$$

where

$$\begin{aligned}
&\hat{\mathcal{T}} = \begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T}_\psi \end{bmatrix}, \hat{\mathcal{A}} = \begin{bmatrix} \mathcal{A} & 0 \\ \mathcal{B}_\Psi \begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix} & \mathcal{A}_\Psi \end{bmatrix}, \hat{\mathcal{B}} = \begin{bmatrix} \mathcal{B} \\ \mathcal{B}_\Psi \begin{bmatrix} D \\ I \end{bmatrix} \end{bmatrix}, \\
&\hat{\mathcal{C}} = \begin{bmatrix} \mathcal{D}_\Psi \begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix} & \mathcal{C}_\Psi \end{bmatrix}, \quad \hat{\mathcal{D}} = \mathcal{D}_\Psi \begin{bmatrix} \mathcal{D} \\ I \end{bmatrix}.
\end{aligned} \tag{8}$$

Then for any $\mathbf{u} \in \mathbf{L}^{\mathbf{n}}_{e,[a,b]}$, we have that

$$\left\langle P_T \Psi \begin{bmatrix} G\mathbf{u} \\ \mathbf{u} \end{bmatrix}, P_T M_{\mathcal{K}} \Psi \begin{bmatrix} G\mathbf{u} \\ \mathbf{u} \end{bmatrix} \right\rangle_{\mathbf{L}} \leq -\varepsilon \|P_T \mathbf{u}\|_{\mathbf{L}},$$

for all $T > 0$.

Proof: The proof follows immediately from Lemma 7 since if G has PIE representation $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ and Ψ has PIE representation $\{\mathcal{T}_\Psi, \mathcal{A}_\Psi, \mathcal{B}_\Psi, \mathcal{C}_\Psi, \mathcal{D}_\Psi\}$, then $\Psi \begin{bmatrix} G \\ I \end{bmatrix}$ has PIE representation $\{\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{D}}\}$ – i.e. if $\mathbf{w} = \Psi \begin{bmatrix} G \\ I \end{bmatrix} \mathbf{u}$, then for some $\mathbf{x}, \mathbf{z}$,

$$\begin{aligned}
\begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T}_\psi \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix} &= \begin{bmatrix} \mathcal{A} & 0 \\ \mathcal{B}_\Psi \begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix} & \mathcal{A}_\Psi \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t) \end{bmatrix} + \begin{bmatrix} \mathcal{B} \\ \mathcal{B}_\Psi \begin{bmatrix} D \\ I \end{bmatrix} \end{bmatrix} \mathbf{u}(t) \\
\mathbf{w}(t) &= \begin{bmatrix} \mathcal{D}_\Psi \begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix} & \mathcal{C}_\Psi \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t) \end{bmatrix} + \mathcal{D}_\Psi \begin{bmatrix} \mathcal{D} \\ I \end{bmatrix} \mathbf{u}(t).
\end{aligned}$$

■

D. Testing the Conditions of Thm. 6

We now suppose the existence of a PIE representation of G and Ψ and propose a convex optimization problem whose feasibility verifies the conditions of Thm. 6.

Theorem 9: Suppose $G \in \mathcal{L}(\mathbf{L}^{\mathbf{n}}_{e,[a,b]}, \mathbf{L}^{\mathbf{m}}_{e,[a,b]})$ is a causal bounded linear operator and for $\mathbf{u} \in \mathbf{L}^{\mathbf{n}}_{e,[a,b]}$, $\mathbf{y} = G\mathbf{u}$ implies Eqns. (4) are satisfied for $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ and some $\mathbf{x} \in \mathbf{L}^{\mathbf{k}_G}_{e,[a,b]}$.

Further suppose that for any $\Delta \in \mathbf{\Delta}$, the interconnection defined by $[G, \Delta]$ is well-posed and Δ satisfies the Hard IQC defined by $\mathcal{K} \in \Pi_4$ and Ψ where $\mathbf{w} = \Psi\mathbf{v}$ implies Eqns. (6) are satisfied for $\{\mathcal{T}_\Psi, \mathcal{A}_\Psi, \mathcal{B}_\Psi, \mathcal{C}_\Psi, \mathcal{D}_\Psi\}$ and some $\mathbf{z} \in \mathbf{L}^{\mathbf{k}_\Psi}_{e,[a,b]}$.

Then we have the following.

1) If there exists $\mathcal{P} \in \Pi_4$ such that $\mathcal{P} \geq 0$ and Inequality (7) holds for $\{\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{D}}\}$ as defined in (8), we have that the feedback interconnection defined by $[G, \Delta]$ is stable for all $\Delta \in \mathbf{\Delta}$.

Proof: Suppose there exists $\mathcal{P} \in \Pi_4$ such that $\mathcal{P} \geq 0$ and Inequality (7) is satisfied. As per Corollary 8, we have that Inequality (3) holds for all $\mathbf{u} \in \mathbf{L}^{\mathbf{n}}_{e,[a,b]}$ – i.e.

$$\left\langle P_T \Psi \begin{bmatrix} G\mathbf{u} \\ \mathbf{u} \end{bmatrix}, P_T M_{\mathcal{K}} \Psi \begin{bmatrix} G\mathbf{u} \\ \mathbf{u} \end{bmatrix} \right\rangle_{\mathbf{L}} \leq -\varepsilon \|P_T \mathbf{u}\|_{\mathbf{L}}.$$

Since any $\Delta \in \mathbf{\Delta}$ satisfies the Hard IQC (3), Theorem 6 implies that feedback system defined by $[G, \Delta]$ is stable for any $\Delta \in \mathbf{\Delta}$. ■

VI. Types of IQC

In Section V, we have assumed that the causal uncertain or nonlinear subsystem $\Delta : \mathbf{L}^{\mathbf{m}}_{e,[a,b]} \to \mathbf{L}^{\mathbf{n}}_{e,[a,b]}$ is known a priori to satisfy a hard IQC defined by some $\mathcal{K}$ and Ψ. As is typical in the finite-dimensional case, the set of $\mathcal{K}$ and Ψ for which the hard IQC hold are determined by the input-output properties of Δ. In this section, we review the infinite-dimensional equivalent of several well-studied classes of uncertainty/nonlinearity and provide corresponding infinite-dimensional extensions of the relevant finite-dimensional IQCs.

A. Real Constant Multiplication

Lemma 10: Suppose that $(\Delta \mathbf{v})(t) = \delta \mathbf{v}(t)$ for some $\delta \in \mathbb{R}$ such that $|\delta| \le 1$. Then for any $\mathcal{P}, \mathcal{R} \in \Pi_4$ such that $\mathcal{R}^* = -\mathcal{R}$, $\mathcal{P}^* = \mathcal{P} \ge 0$ and for any causal bounded linear $H \in \mathcal{L}(\mathbf{L}^{\mathbf{n}+\mathbf{m}}_{e,[a,b]})$ we have that Δ satisfies the Hard IQC defined by

$$\mathcal{K} = \begin{bmatrix} \mathcal{P} & \mathcal{R} \\ \mathcal{R}^* & -\mathcal{P} \end{bmatrix} \quad \text{and} \quad \Psi = \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}. \tag{9}$$

Proof: For any $\mathcal{P}, \mathcal{R} \in \Pi_4$ such that $\mathcal{R}^* = -\mathcal{R}$, $\mathcal{P}^* = \mathcal{P}$ and for any causal bounded linear $H \in \mathcal{L}(\mathbf{L}^{\mathbf{n}+\mathbf{m}}_{e,[a,b]})$ we have that

$$\begin{aligned} &\left\langle P_T \Psi \begin{bmatrix} \mathbf{v} \\ \Delta \mathbf{v} \end{bmatrix}, P_T M_{\mathcal{K}} \Psi \begin{bmatrix} \mathbf{v} \\ \Delta \mathbf{v} \end{bmatrix} \right\rangle \\ &= \int_0^T \left\langle \begin{bmatrix} H\mathbf{v} \\ H\Delta \mathbf{v} \end{bmatrix}(t), M_{\mathcal{K}} \begin{bmatrix} H\mathbf{v} \\ H\Delta \mathbf{v} \end{bmatrix}(t) \right\rangle_{\mathbf{Z}} dt \\ &= \int_0^T (1-\delta^2) \left\langle (H\mathbf{v})(t), M_{\mathcal{P}} (H\mathbf{v})(t) \right\rangle_{\mathbf{Z}} dt \\ &\ge 0. \end{aligned}$$

Therefore Δ satisfies the IQC defined by

$$\mathcal{K} = \begin{bmatrix} \mathcal{P} & \mathcal{R} \\ \mathcal{R}^* & -\mathcal{P} \end{bmatrix} \quad \text{and} \quad \Psi = \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}.$$

■

Corollary 11: Suppose that $(\Delta \mathbf{v})(t) = \delta(t)\mathbf{v}(t)$ for some $\delta : \mathbb{R} \to \mathbb{R}$ such that $\sup_{t>0} |\delta(t)| \le 1$.

Then for any $\mathcal{P}, \mathcal{R} \in \Pi_4$ such that $\mathcal{R}^* = -\mathcal{R}$ and $\mathcal{P}^* = \mathcal{P} \ge 0$ we have that Δ satisfies the Hard IQC defined by

$$\mathcal{K} = \begin{bmatrix} \mathcal{P} & \mathcal{R} \\ \mathcal{R}^* & -\mathcal{P} \end{bmatrix} \quad \text{and} \quad \Psi = I,$$

Proof: The proof is similar to that of Lemma 10 ■

B. Polytopic uncertainty

Lemma 12: Let $\boldsymbol{\Delta} := \{\sum_i \mu_i \Delta_i \ : \ \sum_i \mu_i = 1\}$ where $\Delta_i \in \Pi_4$. Then Δ satisfies the Hard IQC defined by

$$\mathcal{K} = \begin{bmatrix} \mathcal{P} & \mathcal{R} \\ \mathcal{R}^* & \mathcal{Q} \end{bmatrix} \quad \text{and} \quad \Psi = I$$

where $\mathcal{P}, \mathcal{R}, \mathcal{Q} \in \Pi_4$ are such that $\mathcal{Q} < 0$ and

$$P + \Delta_i^* R^* + R\Delta_i + \Delta_i^* Q \Delta_i \ge 0 \quad \text{for all } i. \tag{10}$$

Proof: Suppose $\mathcal{P}, \mathcal{R}, \mathcal{Q} \in \Pi_4$ are such that $\mathcal{Q} < 0$ and Inequality (10) is satisfied. Then for any $\Delta \in \boldsymbol{\Delta}$ and $\mathbf{v} \in \mathbf{L}^{\mathbf{m}}_{e,[a,b]}$ we have that

$$\begin{aligned} &\left\langle \begin{bmatrix} \mathbf{v}(t) \\ (\Delta \mathbf{v})(t) \end{bmatrix}, \begin{bmatrix} \mathcal{P} & \mathcal{R} \\ \mathcal{R}^* & \mathcal{Q} \end{bmatrix} \begin{bmatrix} \mathbf{v}(t) \\ (\Delta \mathbf{v})(t) \end{bmatrix} \right\rangle_{\mathbf{Z}} \\ &= \langle \mathbf{v}(t), P + \Delta^* R^* + R\Delta + \Delta^* Q \Delta \mathbf{v}(t) \rangle_{\mathbf{Z}}. \end{aligned}$$

Thus, by the convexity of the set $\boldsymbol{\Delta}$ and $\mathcal{Q} < 0$ we have that any $\Delta \in \boldsymbol{\Delta}$ satisfies the IQC defined by

$$\mathcal{K} = \begin{bmatrix} \mathcal{P} & \mathcal{R} \\ \mathcal{R}^* & \mathcal{Q} \end{bmatrix} \quad \text{and} \quad \Psi = I.$$

■

C. Sector-bounded uncertainty

Lemma 13: Suppose that for $\mathbf{v} \in \mathbf{L}_{e,[a,b]}$, $(\Delta \mathbf{v})(s,t) = \phi(\mathbf{v}(s,t))$ for all $s \in [a,b]$ and $t > 0$, where ϕ satisfies

$$\alpha v^2 \le v\phi(v) \le \beta v^2 \quad \text{for all } v \in \mathbb{R}. \tag{11}$$

Then Δ satisfies Hard IQC defined by $\Psi = I$ and

$$\begin{aligned} \mathcal{K} &= \begin{bmatrix} \beta I & -I \\ -\alpha I & I \end{bmatrix}^* \begin{bmatrix} 0 & \mathcal{R} \\ \mathcal{R} & 0 \end{bmatrix} \begin{bmatrix} \beta I & -I \\ -\alpha I & I \end{bmatrix} \\ &= \begin{bmatrix} -\beta^* \mathcal{R} \alpha - \alpha^* \mathcal{R} \beta & \beta^* \mathcal{R} + \alpha^* \mathcal{R} \\ \mathcal{R}\beta + \mathcal{R}\alpha & -2\mathcal{R} \end{bmatrix}, \end{aligned} \tag{12}$$

for any $\mathcal{R} \in \Pi_4$ where

$$(\mathcal{R}\mathbf{x})(s,t) := R_0(s)\mathbf{x}(s,t)$$

for some $R_0(s) \ge 0$.

Proof: Suppose $\mathcal{R} \in \Pi_4$ is such that $(\mathcal{R}\mathbf{x})(s,t) = R_0(s)\mathbf{x}(s,t) + \int_a^s R_1(s,\theta)\mathbf{x}(\theta,t)d\theta + \int_s^b R_2(s,\theta)\mathbf{x}(\theta,t)d\theta$ with $R_0(s), R_1(s,\theta), R_2(s,\theta) \ge 0$. Then for all $s \in [a,b]$ and $t > 0$ we have that

$$(\beta \mathbf{v}(s,t) - \phi(\mathbf{v}(s,t)))(\phi(\mathbf{v}(s,t)) - \alpha \mathbf{v}(s,t)) \ge 0,$$

and hence we have that

$$(\beta \mathbf{v}(s,t) - \phi(\mathbf{v}(s,t)))R_0(s)(\phi(\mathbf{v}(s,t)) - \alpha \mathbf{v}(s,t)) \ge 0.$$

Therefore Δ satisfies the IQC defined by $\Psi = I$ and $\mathcal{K}$. ■

VII. Numerical examples

In the following examples, we consider the problem of robust input-output stability of several systems by separating the system into nominal and uncertain subsystems and then testing the conditions of Theorem 6 using Lemma 7 and the software package PIETOOLS. Unless otherwise stated, the conversion of the nominal system to a PIE is performed using the conversion utilities in PIETOOLS. For simplicity, we do not include the external disturbances in the original model, although the effect of these disturbances can be inferred by the definition of the interconnection.

Example 1: We begin with a system modeled by a diffusion equation. We would like to find the largest $\lambda_{\max}$ such that

$$\mathbf{x}_t(s,t) = \lambda\mathbf{x}(s,t) + \mathbf{x}_{ss}(s,t)$$

is stable for all $\lambda \in [0, \lambda_{\max}]$. For this problem, we split the dynamics into nominal and uncertain subsystems, defining the nominal G by

$$\begin{aligned}\mathbf{x}_t(s,t) &= \frac{\lambda_{\max}}{2}\mathbf{x}(s,t) + \mathbf{x}_{ss}(s,t) + \mathbf{u}(s,t)\\ (G\mathbf{u})(t,s) &= \mathbf{x}(t,s),\end{aligned}$$

where $s \in [0,1]$ and boundary conditions are $\mathbf{x}(0,t) = \mathbf{x}(1,t) = 0$. We consider the uncertain subsystem as $(\Delta\mathbf{v})(s,t) := \lambda\mathbf{v}(s,t)$ where $\lambda \in [-\frac{\lambda_{max}}{2}, \frac{\lambda_{max}}{2}]$.

By Lemma 10, Δ satisfies the hard IQC defined as in Eqn. (9) for any suitable $\mathcal{P}, \mathcal{R}$ and Ψ. For this test, we choose Ψ to be defined as $\Psi := \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}$, where H is defined as $(H\mathbf{y})(s,t) := \begin{bmatrix} \mathbf{z}(s,t) \\ \mathbf{y}(s,t) \end{bmatrix}$ where

$$\mathbf{z}_t(s,t) = \mathbf{z}_{ss}(s,t) + 0.5\pi^2\mathbf{z}(s,t) + \mathbf{y}(s,t)$$

By testing the conditions of Theorem 9 using PIETOOLS, it can be shown that the conditions of Eqn. (7) are feasible for $\lambda_{\max} = .99\pi^2$, implying that the diffusion equation is stable for any $\lambda \in [0, .99\pi^2]$. Note that non-robust approaches to stability analysis [20] confirm the stability interval as approximately $\lambda \in [0, \pi^2]$.

Example 2

Consider the time-delay system

$$\dot{x}(t) = Ax(t) + A_1x(t-\tau), \tag{13}$$

where $\tau > 0$ is an uncertain delay parameter. Given τ_0, the goal is to maximize λ such that for $\tau_{\max} = \tau_0 + \lambda$ and $\tau_{\min} = \tau_0 - \lambda$, System (13) is stable for all $\tau \in [\tau_{min}, \tau_{max}]$ where $\tau_{\min} > 0$.

For this problem, we use the nominal DDE system G defined using a PIE as

$$\begin{aligned}\mathcal{T}\dot{\mathbf{x}}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}\mathbf{u}(t)\\ (G\mathbf{u})(t) &= \mathbf{x}(t),\end{aligned}$$

where $\mathbf{x} \in \mathbf{Z}$ and

$$\begin{aligned}\mathcal{T} &= \mathcal{P}\begin{bmatrix} I & 0 \\ I & \{0,0,I\} \end{bmatrix},\\ \mathcal{A} &= \mathcal{P}\begin{bmatrix} A + \sum_{i=1}^n A_i & -\begin{bmatrix} A_1 & \dots & A_n \end{bmatrix} \\ 0 & \{\frac{1}{2}(\tau_{min}^{-1} + \tau_{max}^{-1}), 0, 0\} \end{bmatrix},\\ \mathcal{B} &= \mathcal{P}\begin{bmatrix} 0 & 0 \\ 0 & \{\frac{1}{2}(\tau_{min}^{-1} - \tau_{max}^{-1}), 0, 0\} \end{bmatrix}.\end{aligned}$$

The uncertain system Δ is defined as $(\Delta\mathbf{v})(t) = \lambda\mathbf{v}(t)$ where $|\lambda| < 1$.

Note that Δ satisfies the Hard IQC defined in Lemma 10 as in Eqn. (9) for any suitable $\mathcal{P}, \mathcal{R}$ and $\Psi := \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}$.

We now consider the system defined in [21], where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{14}$$

In details, We use $(H\mathbf{y})(s,t) := \begin{bmatrix} G\mathbf{y}(s,t) \\ \mathbf{y}(s,t) \end{bmatrix}$.

Using Theorem 9 with the multiplier Ψ, as defined above, we find a robust stability region of $\tau \in [0.1008, 1.66]$ using a single storage function. Note that if we do not include the multiplier Ψ (the quadratic stability case), we obtain the smaller interval $\tau \in [0.11, 0.63]$. For comparison, this problem has a known an stability range of $\tau \in [0.1, 1.717]$, that was shown in [22].

Example 3: The next example is a modification of a PDE in studied in [23].

$$\mathbf{x}_t(s,t) = a(s)\mathbf{x}_{ss}(s,t) + b(s)\mathbf{x}_s(s,t) + c(s)\mathbf{x}(s,t) + \lambda\mathbf{x}_s(s,t), \tag{15}$$

where $a(s) = s^3 - s^2 + 2$, $b(s) = 3s^2 - 2s$, $c(s) = -0.5s^3 + 1.3s^2 - 1.5s + 3.03$ and $\mathbf{x}(0,t) = \mathbf{x}_s(1,t) = 0$. We would like to find the maximal λ_{max} such that the system (15) is stable for all $\lambda \in [-\lambda_{max}, \lambda_{max}]$.

For this task, we consider the feedback interconnection defined by G

$$\begin{aligned}\mathbf{x}_t(s,t) &= a(s)\mathbf{x}_{ss}(s,t) + b(s)\mathbf{x}_s(s,t) + c(s)\mathbf{x}(s,t) + \mathbf{u}(s,t)\\ (G\mathbf{u})(s,t) &= \mathbf{x}_s(s,t).\end{aligned}$$

And the uncertainty is defined as $(\Delta\mathbf{v})(s,t) = \lambda\mathbf{v}(s,t)$. Thus, Δ satisfies the Hard IQC defined in Lemma 10, where we used $(H\mathbf{y})(s,t) = \begin{bmatrix} \mathbf{z}(s,t) \\ \mathbf{y}(s,t) \end{bmatrix}$ where

$$\mathbf{z}_t(s,t) = \mathbf{z}_{ss}(s,t) + 4.9\mathbf{z}(s,t) + \mathbf{y}(s,t).$$

Using Theorem 9 and this multiplier, we may show the stability region for any $\lambda \in [-2.8, 2.8]$.

Example 4: The nonlinear example is adapted version [24] of the diffusion PDE in Examples 1 and 3.

$$\mathbf{x}_t(s,t) = \mathbf{x}_{ss}(s,t) + \lambda\mathbf{x}(s,t) + \phi(\mathbf{x}(s,t)), \tag{16}$$

where $\mathbf{x}(0,t) = \mathbf{x}(1,t) = 0$ and the nonlinear feedback part is defined by the sector bounded function $\phi : \mathbb{R} \to \mathbb{R}$ where $-u^2 \leq \phi(u)u \leq u^2$ for $u \in \mathbb{R}$. The goal, then, is to find the largest λ such that the system (16) is stable for any ϕ which satisfies the given sector bound.

First, we represent this system as the interconnection defined by $[G, \Delta]$, where G is

$$\begin{aligned}\mathbf{x}_t(s,t) &= \mathbf{x}_{ss}(s,t) + \lambda\mathbf{x}(s,t) + \mathbf{u}(s,t)\\ (G\mathbf{u})(s,t) &= \mathbf{x}(s,t).\end{aligned}$$

Second, we define the uncertain system as $(\Delta\mathbf{y})(s,t) = \phi(\mathbf{v}(s,t))$.

By Lemma 13, we have that Δ satisfies the Hard IQC defined by $\Psi = I$ and $\mathcal{K}$ as in Eq. 12. Using Theorem 6, we are able to prove stability when $\lambda \leq 1.7$ – mirroring the results in [24].

VIII. Conclusion

In this paper, we proposed a framework for using convex optimization to study the interconnection of infinite-dimensional subsystems. First, we extended the IQC framework to infinite-dimensional systems, signals, interconnections, and multipliers, and generalized an IQC stability theorem to such interconnections. Second, we assumed both the nominal subsystem and multiplier were represented as PIEs and extended the KYP Lemma to such systems, proposing convex tests for conditions of the IQC theorem to be satisfied. Third, we examined several classes of nonlinearity and uncertainty with infinite-dimensional inputs and outputs and showed that they satisfy a generalized version of the hard IQC constraints typically used for finite-dimensional systems. Finally, we applied the results to several example problems and showed that the proposed approach offers an improvement over alternatives such as quadratic stability.

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