Computing Optimal Upper Bounds on the H_2-norm of ODE-PDE Systems using Linear Partial Inequalities

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Abstract: Recently, a broad class of linear delayed and ODE-PDEs systems was shown to have an equivalent representation using Partial Integral Equations (PIEs). In this paper, we use this PIE representation, combined with algorithms for convex optimization of Partial Integral (PI) operators to bound the H_2-norm for input-output systems of this class. Specifically, the methods proposed here apply to delayed and ODE-PDE systems (including delayed PDE systems) in one or two spatial variables where the disturbance does not enter through the boundary.
For such systems, we define a notion of H_2-norm using an initial state-to-output framework and show that this notion reduces to more traditional concepts under the assumption of existence of a strongly continuous semigroup. Next, we consider input-output systems for which there exists a PIE representation and for such systems show that computing a minimal upper bound on the H_2-norm of delayed and PDE systems can be equivalently formulated as a convex optimization problem subject to linear PI operator inequalities (LPIs). We convert, then, these optimization problems to Semi-Definite Programming (SDP) problems using the PIETOOLS toolbox. Finally, we apply the results to several numerical examples – focusing on time-delay systems (TDS) for which comparable H_2 approximation results are available in the literature. The numerical results demonstrate the accuracy of the computed upper bound on the H_2-norm.

Keywords: Infinite-dimensional systems; Polynomial methods; Convex optimization; Systems with time-delays; Linear systems; Time-invariant systems.

1. INTRODUCTION

The input-output properties of a linear system admit many characterizations, including small gain, passivity, Bounded Input Bounded Output Stability, Input-to-State Stability, H_∞ norm, and H_2-norm. Of these, however, the H_2-norm is arguably the most well-established metric for system performance – defining white noise amplification, the mean energy of the impulse response, and gain from initial condition to output. Furthermore, the H_2-optimal control is the generalization of the classical Linear Quadratic Regulator (LQR) and Linear Quadratic Gaussian (LQG). While in the context of robust control, the H_2-norm has been largely supplanted by the H_∞ norm, the H_2-norm still commonly appears in mixed-norm optimization problems where the goal is to improve performance while maintaining robustness with respect to model uncertainty – See, e.g. Iwasaki (1994) and Scherer et al. (1997). Unfortunately, despite the significance of the H_2-norm in analysis and control problems, and unlike in the finite-dimensional case, there are few (if any) results in the literature regarding the computation of *provable bounds* on the H_2-norm for delayed and Partial Differential Equation (PDE) input-output systems.

The most common alternative to finding provable bounds on the H_2-norm for infinite-dimensional systems is to approximate it numerically through some form of discretization – using so-called early or late lumping methods. For early lumping methods, the most common approach is to project the system state onto a finite-dimensional subspace using methods such as Galerkin (for PDEs) or to represent the transcendental delay term using a Padé approximation (for TDSs). These approaches result in an ODE which may be analyzed or controlled using standard computational approaches as in Balas (1978); Morris et al. (2015) (for PDEs) and Pekar and Kureckova (2011) (for TDSs).

Focusing on late-lumping methods, there exist operator-valued versions of both the Ricatti equation and Lyapunov equation characterization of the H_2-norm – primarily for delay systems. However, because the operators which define the Ricatti or Lyapunov equation are unbounded and do not form an algebra, the Ricatti equation is not easily solved without first projecting onto a finite-dimensional space. For example, by numerically solving the operator Lyapunov equation, numerical estimates of the H_2-norm of linear TDSs of both retarded and neutral type were found in Jarlebring et al. (2011), Mattenet et al. (2022), and in Michiels and Zhou (2019), which considered more efficient algorithms for larger scale delay systems. Furthermore, in the special case of commensurate delays, even for neutral-type delayed systems (NDSs), Sumacheva and Kharitonov (2014) give

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an analytical solution for the H_2 norm problem, and Jarlebring et al. (2011) also show that in this case the delay Lyapunov operator equation can be exactly reduced to a set of finite-dimensional linear equations.

Finally, still in the particular case of TDS systems, it is possible to obtain an analytical expression for the transfer function of the system, which is meromorphic. In many cases, it becomes possible to bind the resulting chains of poles – yielding accurate stability analysis results like in Fioravanti et al. (2012).

While early and late-lumping approaches are well-established, they inherently require the truncation of an infinite number of higher-order modes. This means that there may be little relationship between the solution of the original TDS or PDE and the reduced-order model. As a result, such methods may either over or under-approximate the H_2-norm with no easily-computed bound on accuracy. Furthermore, the accuracy of early and late lumping methods depends heavily on the dimension of the projected state space. As a result, such methods may become intractable – requiring hundreds of states – when accurate estimates of the H_2-norm are required.

Previous efforts at lumping-free analysis and control of TDS and PDE systems include the use of *Lyapunov* functions and backstepping. In the former case, an energy metric (*Lyapunov* function) is proposed or parameterized and if this function is uniformly decreasing, then the resulting system is stable. However, the accuracy of such methods depends on the *Lyapunov* functions used and how the negativity of the derivative of this function is verified. Successful examples of finding stability certificates for a class of PDE systems can be found in Gahlawat and Valmorbida (2019) for linear systems and in Valmorbida et al. (2015) for nonlinear PDEs. In particular, because TDS and PDE systems are defined in terms of unbounded operators, bounding the derivative of the *Lyapunov* function typically requires ad hoc steps such as integration by parts along with conservative inequalities such as the *Poincaré* inequality. In the case of backstepping, a strategy for boundary control and state estimation measuring at the boundaries has been developed by transforming the system to a target form with desired stable properties as in Krstic and Smyshlyaev (2008). Nevertheless, to date, the backstepping approach has not been used to compute input-output properties such as the H_2-norm.

Having now considered standard methods for computing the H_2-norm of TDS and PDE system, and having noted the lack of provable bounds associated with early and late lumping methods, and seeking to avoid the use of ad-hoc methods associated with typical Lyapunov functions, we consider next an alternative framework for representation, analysis, control, and simulation of TDS and PDE systems. Specifically, we consider Partial Integral Equations (PIEs) and the associated *-algebra of bounded linear Partial Integral (PI) operators.

Recently, in Peet (2021a), Shivakumar et al. (2022a), and Jagt and Peet (2022a), it has been shown that for a large class of linear TDS and ODE-PDE systems, there exists an associated PIE – a system of first-order differential equations parameterized using the algebra of PI operators and with no continuity or boundary condition restrictions on the state. Specifically, it has been shown that there exists an invertible map (defined by a PI operator) between solutions of the PDE and solutions of the PIE. This mapping allows one to study the properties of the solution of the PDE by studying its PIE representations. Furthermore, because the PIE representation is parameterized by a *-algebra of PI operators, and because there exist algorithms for optimization of positive PI operators (the LPIs), many Linear Matrix Inequality (LMI) methods designed for ODE systems have been generalized to PIEs and the PI algebra. For example, H_∞ norm computation for analysis and controller synthesis was extended to ODEs-PDEs using the PIE framework in Das et al. (2019) (analysis) and Shivakumar et al. (2020b) (optimal control). However, at present, the PIE representation has not been used to compute the H_2-norm of an input-output TDS or PDE system.

The main contribution of this work, then, is to use the PIE representation and algorithms for the optimization of PI operators to compute provable bounds on the H_2-norm for a broad class of infinite-dimensional systems. Naturally, the class of systems for which the proposed methods apply is determined by the class of systems for which there exists a suitable PIE representation. However, characterizing every system with a PIE representation is beyond the scope of this paper, and we will therefore rely heavily on the parameterization of TDS and ODE-PDE systems and construction of associated PIE operators given in Shivakumar et al. (2022a).

The first challenge with computing provable bounds on the H_2-norm is defining an appropriate characterization of this norm. While in finite dimensions, we can assume the existence of solutions and equivalence between frequency and time-domain characterizations, in infinite dimensions, the existence of a solution map cannot be assumed and there are no easy representations of the transfer function of a PDE system. As a result, in Section 3, for linear time-invariant systems described by ODE-PDEs in a standard form, we use a state-to-output characterization of the H_2-norm which reduces to the standard notion under the assumption of the existence of a strongly continuous semigroup.

Our metric, however, still depends on the general unbounded operators that parameterize this broad class of systems and have no special structure. Thus, numerically tractable methods cannot be applied directly to this formulation. We overcome this problem in Section 4 by relying on the PIE framework and the special class of PI operators. Assuming that the system is restricted to finite-dimensional inputs and outputs; inputs within the domain of PDEs, or on the finite-dimensional part of general ODE-PDEs, which excludes systems with boundary control; and has no algebraic delay terms – as exhibited by NDSs– we show the existence of a unitary map between the original system and the corresponding PIE. The resultant PIE representation is used in Section 5 to derive our main result, a LPI for the computation of an upper bound to the H_2-norm of the original system. By using polynomial methods, we finally show how the LPI translates into a SDP.

Moreover, in Section 6, the derived SDP is applied to numerical examples of TDSs – taken from the literature – using the Matlab toolbox PIETOOLS, and the results are compared to other methods for validation. For details on PIETOOLS, the reader is referred to the most up-to-date user manual Shivakumar et al. (2022b), and to Shivakumar et al. (2020a). Finally, concluding remarks are presented in Section 7.

2. NOTATION

Greek letters are used to denote elements of an infinite-dimensional vector space over the field of scalars $\mathbb{R}$. For the sake of simplicity of notation, in analogy with the usual state-space representation of dynamical systems, when writing these vectors we denote $\phi(t) = \phi(\cdot,t)$ related to the function $\phi : \Omega \subset \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$, for natural d. On the other hand, elements of the vector space $\mathbb{R}^n$ over the field of scalars $\mathbb{R}$ are denoted by bold Latin letters when generally not scalars.

$L_2^k[\Omega]$ is the usual functional space of squared integrable functions evaluated at $\mathbb{R}^k$ and defined on the domain Ω. $W_n^k[\Omega]$ is the *Sobolev* subspace of $L_2^k[\Omega]$ for which the partial derivative up to n-order on the domain exist, such that $W_n^0[\Omega] = L_2^k[\Omega]$. These spaces are *Hilbert* spaces with their usual inner products. $\mathbb{1}$ denotes the indicator function, which equals 1 on the corresponding domain (clear from the context if not explicitly indicated) and zero otherwise.

Moreover: $\mathcal{L}(X,Y)$ denotes the space of linear operators from *Hilbert* space X to *Hilbert* space Y; $\mathcal{A}^*$ is the *Hilbert* adjoint of the operator $\mathcal{A}$ with respect to the appropriate inner-products; $\overline{S}$ denotes the closure of set S; the composition operation of maps with appropriate dimensions is omitted such that $\mathcal{AB}$ denote the resultant operator of composition between $\mathcal{A} : Y \to Z$ and $\mathcal{B} : X \to Y$ for arbitrary sets X, Y, and Z.

3. DEFINING THE H_2-NORM OF AN ODE-PDE

Consider the case of a system described by coupled linear ODE-PDEs of the form

$$\begin{aligned} \dot{\phi}(t) &= \mathscr{A}\phi(t) + \mathscr{B}u(t), & \phi(0) &= \phi_0 \\ \mathbf{y}(t) &= \mathscr{C}\phi(t), & \phi(t) &\in X \end{aligned} \qquad (1)$$

where $\mathscr{A} : X \to Z$, $\mathscr{B} : \mathbb{R}^{n_u} \to Z$, $\mathscr{C} : X \to \mathbb{R}^{n_y}$, and $\phi(t) \in X \subset Z$ is called the primal state at instant t. The subspace X enforces the boundary values and continuity constraints on ϕ, Z is a *Hilbert* space, and $\mathbf{u}_0 \in \mathbb{R}^{n_u}$. Hereafter, we assume that $Z = \mathbb{R}^m \times L_2^k[\Omega]$, and $X \subset \mathbb{R}^m \times \Pi_{i=0}^N W_i^{k_i}$, where $k = \sum_{i=0}^N k_i$, both endowed with the combined inner product of the finite and infinite-dimensional *Hilbert* spaces.

Definition 1. We say that $\{\phi, y\}$ is the solution of the system (1) defined by$\{\mathscr{A},\mathscr{B},\mathscr{C}\}$, given input $u \in L_2[\mathbb{R}_+]$ and initial condition $\phi_0 \in X$ if ϕ is suitably differentiable and Eqn. (1) is satisfied for all $t \geq 0$.

Next, we state the notion of the H_2-norm of an ODE-PDE that will be used henceforth. While our formulation may seem unfamiliar, it was chosen to not require: the existence of a strongly continuous semigroup; the existence of a transfer function; or the definition of the impulse response.

Definition 2. (Definition of H_2-norm). Suppose the ODE-PDE defined by $\{\mathscr{A},\mathscr{B},\mathscr{C}\}$ with $u = 0 \in L_2[\mathbb{R}_+]$, admits a solution for any initial condition of the form $\mathscr{B}\mathbf{u}_0$, where $\mathbf{u}_0 \in \mathbb{R}^{n_u}$. Then the H_2-norm of the ODE-PDE defined by $\{\mathscr{A},\mathscr{B},\mathscr{C}\}$ is

$$\mu := \sup_{\|\mathbf{u}_0\|_2 = 1} \|y\|_{L_2}, \quad s.t.$$

$\{\phi, y\}$ is the solution of the System (1) defined by$\{\mathscr{A},\mathscr{B},\mathscr{C}\}$ for zero input and initial condition $\mathscr{B}u_0$

This definition of the H_2-norm is simply a more convenient representation of what might be considered the standard H_2-norm of an input-output system where we presume there exists a strongly continuous semigroup Φ which, for any solution ϕ, satisfies

$$\phi(t) = \Phi(t)\phi_0 + \int_0^t \Phi(t-\tau)\mathscr{B}\mathbf{u}(\tau)d\tau. \qquad (2)$$

In this case, for $\mathbf{u} = 0 \in R^{n_u}$ and $\phi_0 = \mathscr{B}u_0$, we have

$$\|y\|_{L_2}^2 = \int_0^\infty \|\mathscr{C}\Phi(t)\phi_0\|_2^2 \, dt$$

$$\begin{aligned} &= \int_0^\infty \langle \mathscr{C}\Phi(t)\mathscr{B}u_0, \mathscr{C}\Phi(t)\mathscr{B}u_0 \rangle_2 \, dt \\ &= \int_0^\infty u_0^T \mathscr{B}^* \Phi(t)^* \mathscr{C}^* \mathscr{C} \Phi(t) \mathscr{B} dt u_0 \end{aligned}$$

so that the H_2-norm may then be equivalently expressed as

$$\mu = \sup_{u_0 \in \mathbb{R}^{n_u}} \frac{\sqrt{u_0^T \mathscr{B}^* \int_0^\infty \Phi(t)^* \mathscr{C}^* \mathscr{C} \Phi(t) \mathbf{d}t \mathscr{B} u_0}}{\|u_0\|}.$$

Note 1. Similarly, our notion of the H_2-norm may also be recast as the L_2 norm of the impulse response of the ODE-PDE system.

In Section 5, we will use this definition to derive a convex optimization problem for computing a minimal upper bound on the H_2-norm of an ODE-PDE. However, because the ODE-PDE is defined by unbounded operators and boundary constraints, we need first present an algebraic representation of this system which will facilitate the computation of the H_2-norm.

4. PARTIAL INTEGRAL EQUATIONS (PIES)

By necessity, the class of ODE-PDEs representable in the form (1) is very broad and not easily parameterized. Furthermore, the operators which define the ODE-PDE are unbounded and the state is restricted to lie in the subspace $\phi(t) \in X$. This makes the computation of the H_2-norm using this class of models either extremely difficult or requires us to consider a very limited subset of such problems.

Fortunately, however, there exists a very broad class of ODE-PDEs for which a Partial Integral Equation (PIE) representation can be found. Because this class is so broad, we do not repeat it here, but instead refer to Shivakumar et al. (2022a) for what is currently the largest class of such ODE-PDE systems, and to Jagt and Peet (2022a) for the extension to ODE-PDEs in two spatial dimensions.

For any ODE-PDE of form 1 which can be parameterized using the framework in Shivakumar et al. (2022a), there exists an associated system of the form

$$\begin{aligned} \mathcal{T}\dot{\varphi}(t) + \mathcal{T}_u \dot{u}(t) &= \mathcal{A}\varphi(t) + \mathcal{B}u(t), \quad \varphi(0) = \varphi_0 \in Z \\ \mathbf{y}(t) &= \mathcal{C}\varphi(t) \end{aligned} \qquad (3)$$

where $\varphi(t) \in Z$, $\{\mathcal{T},\mathcal{A},\mathcal{B},\mathcal{C}\}$ are so-called Partial Integral (PI) operators, and we require $\mathcal{T}_u = 0$. This latter restriction prohibits Dirac operators in $\mathscr{B}$, so that inputs cannot enter at the boundary. Note that, on the ODE-PDE representation of TDS systems, as parametrized in Peet (2021b), this implies that the input-to-boundary terms must be zero. Since this is not usually the case for NDSs, we exclude this class of systems from the TDSs covered by our results.

Definition 3. We say that $\{\varphi, y\}$ satisfies the PIE defined by $\{\mathcal{T},\mathcal{A},\mathcal{B},\mathcal{C}\}$ with initial condition $\varphi_0 \in Z$ and input, $u \in L_2$ if $\varphi(t)$ is differentiable in time, $\varphi(t) \in Z$ and (3) is satisfied for all $t \geq 0$.

The advantage of the PIE representation is that the set of PI operators, denoted $\Pi \subset \mathcal{L}(Z)$, form a *-algebra, being bounded linear operators closed under composition, addition, concatenation and, in certain cases, inversion – meaning they can be manipulated similarly as matrices. Furthermore, any PI operator is parameterized by square integral functions and the set of PI operators with polynomial parameters forms a *-subalgebra. For brevity, however, we will not present the parameterized structure of the PI algebra, as it becomes rather complicated (especially in 2D). Nevertheless, the parameterization can be found in Shivakumar et al. (2022a), where these operators are extended to the ODE-PDE systems in the 1D case and in Jagt and Peet (2022a) for the 2D case. In both cases, however, the software package PIETOOLS automates the expression and manipulation of PI operators by overloading on an associate class most commands associated with matrices.

The following critical result shows that for any admissible ODE-PDE system in the class defined in Shivakumar et al. (2019) or Jagt and Peet (2022a), there exists an invertible map from the solution of the ODE-PDE to the solution of the associated PIE.

Theorem 4. Given admissible $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, and X, as parameterized in Shivakumar et al. (2019); Jagt and Peet (2022a). Let $\mathcal{T},\mathcal{A},\mathcal{B},\mathcal{C}$ be as defined in these references. Then $\mathcal{T}: Z \to X$, $\mathcal{A} = \mathscr{A}\mathcal{T}$, $\mathcal{C} = \mathscr{C}\mathcal{T}$, $\mathscr{B} = \mathcal{B}$ and there is a differential operator, $\mathcal{D}$ such that $\mathcal{T}\mathcal{D}\phi = \phi$ and $\mathcal{D}\mathcal{T}\varphi = \varphi$ for any $\phi \in X$ and $\varphi \in Z$. Furthermore, the following are equivalent

(1) $\{\phi, y\}$ satisfies the ODE-PDE defined by $\{\mathscr{A},\mathscr{B},\mathscr{C},X\}$ for input u and initial condition ϕ_0.
(2) $\{\mathcal{D}\phi, y\}$ satisfies the PIE defined by $\{\mathcal{T},\mathcal{A},\mathcal{B},\mathcal{C}\}$ for input u and initial condition $\mathcal{D}\phi_0$.

Alternatively, we may say the following are equivalent

(1) $\{\varphi, y\}$ satisfies the PIE defined by $\{\mathcal{T},\mathcal{A},\mathcal{B},\mathcal{C}\}$ for input u initial condition φ_0.
(2) $\{\mathcal{T}\varphi, y\}$ satisfies the ODE PDE defined by $\{\mathscr{A},\mathscr{B},\mathscr{C},X\}$ for input u and initial condition $\mathcal{T}\varphi_0$.

Proof. The proof may be found in Shivakumar et al. (2022a). □

Note 2. We refer to the solution of the ODE-PDE, $\phi(t)$, as the primal state, and the associated solution of the PIE, $\varphi(t)$ as the fundamental state.

5. COMPUTING OPTIMAL BOUNDS ON THE H_2-NORM

Now that we have defined the H_2-norm and established the existence of a PIE representation of the ODE-PDE, we provide a method for computing the H_2-norm of the ODE-PDE system.

Theorem 5. Suppose (1) defined by $\{\mathscr{A},\mathscr{B},\mathscr{C}\}$ is observable. Let $\mathcal{T},\mathcal{A},\mathcal{C} \in \Pi$ be PI operators as defined in Theorem (4), and μ the H_2-norm of System (1). Suppose there is a PI operator $W_0 \succ 0$ such that:

$$\begin{aligned} trace(\mathcal{B}^* W_o \mathcal{B}) &< \gamma^2, \\ \mathcal{A}^* W_o \mathcal{T} + \mathcal{T}^* W_o \mathcal{A} + \mathcal{C}^*\mathcal{C} &\prec 0. \end{aligned} \tag{4}$$

Then the system (1) is internally stable and, if $\{\phi, y\}$ satisfies (1) for some initial $\mathbf{u}_0$, we have that $\|y\|_{L2} \le \gamma \|\mathbf{u}_0\|_2$ – i.e. $\mu \le \gamma$.

Proof.

Consider the storage function $V(\phi) := \langle \phi, W_0 \phi \rangle_Z$. Suppose that $\{\phi, y\}$ is a solution to the ODE-PDE defined by $\{\mathscr{A},\mathscr{B},\mathscr{C}\}$ with state-space X, zero input, and initial condition $\mathscr{B}\mathbf{u}_0 \in X$. Then $\{\varphi(t) := \mathcal{D}\phi(t), \mathbf{y}(t)\}$ is a solution to the PIE with zero input and initial condition $\varphi_0 = \mathscr{B}\mathbf{u}_0$, by Theorem 4. Then $\phi(t) = \mathcal{T}\varphi(t)$, $\phi(0) = \mathcal{T}\varphi(0)$, and we have that

$$\begin{aligned} \dot V(\phi(t)) &= \langle \dot\phi(t), W_0\phi(t)\rangle_Z + \langle \phi(t), W_0\dot\phi(t)\rangle_Z \\ &= \langle \mathcal{A}\varphi(t), W_o\mathcal{T}\varphi(t)\rangle_Z + \langle \mathcal{T}\varphi(t), W_o\mathcal{A}\varphi(t)\rangle_Z \\ &= \langle \varphi(t), (\mathcal{A}^*W_o\mathcal{T} + \mathcal{T}^*W_o\mathcal{A})\varphi(t)\rangle_Z \\ &\le -\langle \varphi(t), \mathcal{C}^*\mathcal{C}\varphi(t)\rangle_Z \\ &= -\langle \mathcal{C}\varphi(t), \mathcal{C}\varphi(t)\rangle_2 \\ &= -\|\mathbf{y}(t)\|_2^2. \end{aligned}$$

Since the ODE-PDE is observable, this implies internal stability and, furthermore

$$\int_0^\infty \dot V(\phi(t))dt \le -\|y\|_{L_2}^2,$$

which (since $\phi(0) = \mathcal{T}\varphi(0) = \mathscr{B}\mathbf{u}_0 = \mathcal{B}\mathbf{u}_0$) implies that

$$\begin{aligned} \|y\|_{L_2}^2 &\le V(\phi(0)) - \lim_{t\to\infty} V(\phi(t)) \\ &\le V(\phi(0)) \\ &= \langle \mathcal{T}\varphi(0), W_o\mathcal{T}\varphi(0)\rangle_Z \\ &= \langle \mathcal{B}u_0, W_o\mathcal{B}u_0\rangle_Z \\ &= u_0^T\mathcal{B}^*W_o\mathcal{B}u_0 \\ &= \text{trace}(u_0^T\mathcal{B}^*W_o\mathcal{B}u_0) \\ &= \text{trace}(\mathcal{B}^*W_o\mathcal{B}u_0u_0^T) \\ &\le \text{trace}(\mathcal{B}^*W_o\mathcal{B})\,\text{trace}(u_0u_0^T) \\ &= \gamma^2\, trace(u_0u_0^T) \\ &= \gamma^2 \|u_0\|_2^2, \end{aligned}$$

where we used the circularity property of trace operation and the inequality $\text{trace}(AB) \le \sum_{i=0}^{n_u} \lambda_i(A)\lambda_i(B)$ proved by Richter (1958), with λ_i representing the ith eigenvalue of the matrix. Note that, for positive semidefinite matrices, it implies that $\text{trace}(AB) \le \sum_{i=0}^{n_u} \lambda_i(A) \sum_{i=0}^{n_u} \lambda_i(B) = \text{trace}(A)\,\text{trace}(B)$.

Thus we have that for any solution $\{\phi, y\}$ of the ODE-PDE with initial condition $\mathscr{B}u_0$,

$$\|y\|_{L2} \le \gamma \|\mathbf{u}_0\|_2$$

which implies, by Definition 2, that the ODE-PDE H_2-norm is upper bounded by γ. □

Having proposed a convex optimization problem for calculating the metric using operator inequalities, we show next how to enforce these conditions using semidefinite programming.

Definition 6. We say that $W \in \Pi^+$ if there exists some matrix $P \succcurlyeq 0$ and $Z \in \Pi$ such that

$$W = \mathcal{Z}^*\mathcal{M}_P\mathcal{Z}$$

Where $(\mathcal{M}_P\mathbf{u})(s) := P\mathbf{u}(s)$ is the multiplier operator defined by P.

Given $W \in \Pi$, constraints of the form $W \in \Pi^+$ can be represented as an LMI. The PIETOOLS software suite automates the conversion of such constraints to LMI problems – See Shivakumar et al. (2020a).

Lemma 7. Given $\{\mathscr{A},\mathscr{B},\mathscr{C}\}$, $\varepsilon \ge 0$. Let $\{\mathcal{T},\mathcal{A},\mathcal{B},\mathcal{C}\}$ be the associated PI operators as stated on Theorem (4). Then, if

$$\gamma^* := \min_{\gamma\in\mathbb{R}, Wo\in\Pi} \gamma$$

$$\begin{aligned} W - \varepsilon I &\in \Pi^+, \\ trace(\mathcal{B}^*W\mathcal{B}) &\le \gamma, \\ -(\mathcal{A}^*W\mathcal{T} + \mathcal{T}^*W\mathcal{A} + \mathcal{C}^*\mathcal{C}) &\in \Pi^+, \end{aligned} \tag{5}$$

and $\{\phi, y\}$ satisfies the PDE for some initial ϕ_0, we have that $\|y\|_{L_2}^2 \le \gamma^2 \|\mathbf{u}_0\|_2^2$ – i.e. $\mu \le \gamma$.

In the next section, we solve this problem to estimate the H_2-norm of TDS systems for which the literature provides alternative methods.

6. SOLVING NUMERICAL EXAMPLES

In this section, we apply Lemma 7 to several delayed systems and a PDE systems. In each case, the PIE representation is constructed using the formulae in Peet (2021a) and Peet (2021b). Although there are very few analytic expressions for the H_2-norm of delayed and PDE systems in the literature, the accuracy of the proposed algorithm can be estimated by comparing it to approximation schemes such as that proposed in Jarlebring et al. (2011) based on a projection of the delay-Lyapunov Equation.

For each example, we list the computed bound on the H_2-norm, along with an analytical value (if available), the estimated H_2-norm provided by Jarlebring et al. (2011) (if available), and the estimated norm provided by replacing the delay term with a low-order Padé approximation.

The computed bound on the H_2-norm from Lemma 7 is obtained by first using PIETOOLS 2021b to construct the PIE representation. Then, the conditions of Lemma 7 are enforced using `poslpivar` to create an operator variable W_0 and the PIETOOLS `lpi_ineq` command to enforce the inequality constraints. For both of these steps, the PIETOOLS 'light' settings were used – possibly resulting in some loss of accuracy.

The first numerical comparison used is the method proposed in Jarlebring et al. (2011). This result is based on a numerical discretization of the operator Lyapunov equation, expanding and truncating the desired solution as a series of *Chebyshev* polynomials. This method can be classified as a late-lumping on the infinite-dimensional part of the problem (the delays).

The second numerical comparison is the early-lumping approach of replacing the delay term in the frequency domain with a *Padé* approximation. In particular, we use the Matlab `pade` function to convert a delay system structure to an ODE using a 10th-order approximation.

Example 1. First, consider the scalar, single delay (τ) system

$$\dot{x}(t) = -ax(t-\tau) + bu(t),$$
$$y(t) = cx(t),$$

where $a, b, c, > 0$. This system has an analytic expression for the H_2-norm, which is listed in the first column of Table 1.

For computation, we take values $a = 1$, $b = 2$, $c = 2$, and $\tau = 0.5$, for which the system is open-loop stable. The results of the numerical test are listed in Table 1. All tests agree to 4 decimal places in this case.

Example 2. Next, consider the two-delay system

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau_1) + A_2x(t-\tau_2) + Bu(t), y(t) = Cx(t)$$

For numerical testing, we take $\tau_1 = \pi/10$, $\tau_2 = 1$, so that the delays are incommensurate and

$$A_0 = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -3 & 2 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_1 = \frac{1}{5}\begin{bmatrix} -3 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 2 & -2 \end{bmatrix},$$

$$A_2 = \frac{1}{5}\begin{bmatrix} -4 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C = [1\ 1\ 1].$$

In this case, the method in Jarlebring et al. (2011) and the bound on the H_2-norm from Lemma 7 agree to four significant figures. However, the 10th-order Padé approximation yields a slightly higher value.

Example 3. For a third example, we now consider a system modeled using PDEs, but with two incommensurate delays, $\tau_1 = 0.25$, $\tau_2 = 0.3$ and $h = \frac{\pi}{10}$. The spatial domain is $x \in [0, \pi]$. The dynamics are as follows.

$$\begin{aligned} \frac{\partial \xi}{\partial t} &= \frac{\partial^2 \xi}{\partial x^2} - 20\xi(x,t) - 4\xi(x,t-\tau_1)... \\ &\qquad ... - 0.1\xi(x,t-\tau_2) + \mathbb{1}(x)u(t), \\ \xi(0,t) &= 0, \\ \xi(\pi,t) &= 0, \\ y(t) &= \frac{1}{h}\int_0^\pi \xi(x,t)dx. \end{aligned} \tag{6}$$

For this system, of course, there is neither an analytical expression nor a numerical method available to compute the H_2-norm without performing a spatial discretization. However, this system admits a 2D PIE representation using the result in Jagt and Peet (2022b). The bound on the H_2-norm for this system is listed in Table 1. Note that for this problem, to improve numerical reliability, a bisection approach was taken for minimization of γ in Lemma 7.

Example 4. (Example (3), discretized) For the last example, we consider the question of whether discretization of the PDE in Example (3) yields accurate estimates of the H_2-norm of the delayed PDE. Specifically, we use a finite-difference discretization of System (6), taking the spatial domain $\bar{\Omega} = [0, \pi]$ and dividing it in n_e equally spaced of $\frac{\pi}{n_e}$disjoint subdomains such that $\Omega_1 \cup \Omega_2 \cup ... \cup \Omega_{n_e}$, each with associated lumped state, v_i. The resulting system of delayed ODEs is given by

$$\begin{aligned} \dot{v}(t) &= (T + D_0)v(t) + D_1v(t-\tau_1) + D_2v(t-\tau_2) + B_2u(t) \\ y(t) &= C_2v(t) \end{aligned}$$

where

$$T = \frac{(n_e+1)^2}{\pi^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & & 1 & -2 \end{bmatrix} \in \mathbb{R}^{n_e \times n_e},$$

and

$$D_0 = \begin{bmatrix} -20 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & -20 \end{bmatrix}, \quad D_1 = \begin{bmatrix} -4 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & -4 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} -.1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & -.1 \end{bmatrix}, \quad \begin{aligned} C_2 &= [1\ 1\ \cdots\ 1] \in \mathbb{R}^{1\times n_e}, \\ B_2 &= [1\ 1\ \cdots\ 1]^T \in \mathbb{R}^{n_e \times 1}. \end{aligned}$$

For this delayed ODE discretization of the delayed PDE, tests were performed at both 10 and 15 discretization points. At 10 discretization points, all methods produced a significantly lower number than the bound from the original PDE. At 15 discretization points, both the method in Jarlebring et al. (2011) and the Padé approximation yielded higher values but were still significantly lower than the bound on from the original delayed PDE. None of the numerical methods tested were able to obtain results at 20 discretization points. For the method in Jarlebring et al. (2011), matrix sizes exceeded preset tolerances. For Lemma 7, computation time exceeded 4 hours. For the

Padé approximation, the computed bound diverged, yielding an estimate of 2.8339 at 20 discretization points and 14.3521 at 100 discretization points. It is unclear if this divergence is due to the instability of the discretization method or numerical problems with computing the norm of the Padé approximation.

Table 1. Computed H_2-norm of numerical examples 1,2, and 3, along with a spatially discretized version of Example 3 using both 10 (i.e., 3d(10)) and 15 (i.e., 3d(15)) discretization points. The first column is the analytic value, when available. The second column is the numerical estimate from Jarlebring et al. (2011). The third column is the bound from Lemma 7. The fourth column is the numerical estimate using a Padé approximation to generate an ODE representation of the delay.

Ex. #	analytic	[Jarl. (2011)]	Lem. 7	Padé
1	3.6724	3.6724	3.6727	3.6724
2	–	3.8299	3.8305	3.9275
3	–	–	2.3236	–
3d (10)	–	1.3696	1.4122	1.3696
3d (15)	–	2.1052	–	2.1053

7. CONCLUSION

The numerical examples presented here validate our main result and illustrate how to formulate a computational method to measure the H_2-norm of a broad class of infinite-dimensional systems, including coupled linear ODE-PDE systems up to two spatial dimensions, with in-domain inputs and outputs, boundary effects in the dynamics, boundary conditions that combine boundary values with inputs and integrals of the state; delay systems described by delay-differential equations with concentrated and distributed delays on the states, on the inputs, or even on both. This was achieved by using the PIE framework, which allows a convex optimization problem that can be numerically treated. This work contributes to expanding the range of problems covered by the PIE framework as an alternative to conventional lumping methods.

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