STABILITY AND CONTROL OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract

This thesis addresses the question of stability of systems defined by differential equations which contain nonlinearity and delay. In particular, we analyze the stability of a well-known delayed nonlinear implementation of a certain Internet congestion control protocol. We also describe a generalized methodology for proving stability of time-delay systems through the use of semidefinite programming.

In Chapters 4 and 5, we consider an internet congestion control protocol based on the decentralized gradient projection algorithm. For a certain class of utility function, this algorithm was shown to be globally convergent for some sufficiently small value of a gain parameter. Later work gave an explicit bound on this gain for a linearized version of the system. This thesis proves that this bound also implies stability of the original system. The proof is constructed within a generalized passivity framework. The dynamics of the system are separated into a linear, delayed component and a system defined by an nonlinear differential equation with discontinuity in the dynamics. Frequency-domain analysis is performed on the linear component and time-domain analysis is performed on the nonlinear discontinuous system.

In Chapter 7, we describe a general methodology for proving stability of linear time-delay systems by computing solutions to an operator-theoretic version of the Lyapunov inequality via semidefinite programming. The result is stated in terms of a nested sequence of sufficient conditions which are of increasing accuracy. This approach is generalized to the case of parametric uncertainty by considering parameter-dependent Lyapunov functionals. Numerical examples are given to demonstrate convergence of the algorithm. In Chapter 8, this approach is generalized to nonlinear time-delay systems through the use of non-quadratic Lyapunov functionals.

Preface

This thesis covers my work done in the Networked Systems and Controls Lab at Stanford University under the direction of Professor Sanjay Lall during the period from August of 2002 through January of 2006. With the exception of Chapter 8, almost all of the work presented here has been previously published or submitted to peer-reviewed academic journals or conferences. The chapters of this thesis can be divided into those containing original research and those containing an overview of existing results. Specifically, Chapters 5, 7 and 8 contain original results, while most of the rest of the chapters can be considered as survey material.

Some attempt has been made in this thesis to keep the length of the chapters minimal in order to improve readability. The presentation style is mathematically oriented, with specifics of numerical implementation suppressed. Proofs, when especially long, have been moved to the appendices. To maintain focus on the theoretical contribution, in-depth treatment of various special cases has been omitted when such treatment can be inferred from previous exposition.

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Chapter 1

Introduction

1.1 Research Goals

The analysis of nonlinear systems is a subject of much recent interest. Although linearization is a well-established approach to analysis, results obtained in this manner are always, at best, local. When considering changes in critical systems such as the Internet, the guarantee of convergence associated with a global stability result carries significant weight. Furthermore, some dynamical systems, such as are found in biology or high performance aircraft, are dominated by nonlinear behavior. In such cases, the practical value of linear analysis is limited. Apart from nonlinearity, the study of decentralized systems and systems with delay is also the topic of much active research. Communications systems, an issue of significant current interest, are often decentralized and, by virtue of their geographic reach, inevitably contain delay. Earth based telescopes are currently being built which consist of thousands of mirror segments; each individually actuated but required to move in a coordinated manner. Such systems are difficult to model using a state space of reasonable dimension.

The recent development of efficient algorithms for optimization on the cone of positive semidefinite matrices has created a fundamental shift in how research is conducted. Specifically, semidefinite programming has been widely adopted in control theory and has led to a greater understanding of linear finite dimensional centralized systems. Unfortunately, this same understanding has not yet been extended to nonlinear, infinite dimensional or decentralized systems. In fact, many critical questions in analysis and control of these systems have been shown to be NP-hard. The goal of our research at Stanford has been to find new ways to address these types of problems. One approach is to expand the definition of solution. Instead of giving a single necessary and sufficient condition, expressible as a semidefinite program, one can construct a nested sequence of sufficient conditions, of increasing accuracy, which converge to a necessary and sufficient condition and are expressible as semidefinite programs. Examples of this technique as well as others can be found in the chapters of this thesis.

Internet Congestion Control The application which has motivated much of the research in this thesis has been stability analysis of proposed protocols for Internet congestion control. The analysis of congestion control protocols has received much attention recently. This work has been motivated by concern about the ability of current protocols to ensure stability and performance as the number of users and amount of bandwidth continues to increase. Although the protocols that have been used in the past have performed well as the Internet has increased in size, as capacities and delays increase instability will become a problem. In our work, we have considered global stability with delay of congestion control protocols which attempt to solve a distributed network optimization problem. These systems are described by differential equations with delay and contain non-static nonlinearity. Because of the nonlinearity and delays, proving convergence is difficult. By combining frequency and time domain techniques using a generalized passivity framework, in Chapters 4 and 5, we have shown for certain protocols that global stability with delay holds under the same conditions as local stability. Condensed versions of these results also appear in publication in [35] and [36]. These tight bounds, verified by experimental evidence, allow one to accurately predict when congestion control will fail.

An Approach to Analysis and Synthesis Our research at Stanford has produced a number of new results and tools concerning the analysis of systems with delay, nonlinearity and decentralized structure. Specifically, in Chapter 7, we have proposed two new types of refutation which can be used to construct the positive quadratic Lyapunov-Krasovskii functionals necessary for stability of linear time-delay systems. Furthermore, we have shown how these refutations can be parameterized using the space of positive semidefinite matrices. This has resulted in a nested sequence of sufficient conditions, of increasing accuracy and expressible as semidefinite programs, which prove stability of linear time-delay systems. Thus for any desired level of accuracy, these results give a condition, expressible as a semidefinite program, which will test stability to that level of accuracy. These results appear in publication in [37] and [38]. In addition, because of the structure of the refutations, in Chapter 8 we have also been able to generalize these results to time-delay systems with nonlinearities. Some of these results appear in publication in [34].

1.2 Prior Work

The contents of this thesis utilize results from a number of different areas of research. A complete survey of the results in any one of these fields would be beyond the scope of this document. In this section we briefly list some landmark contributions which have directly influenced the direction of our research.

Stability Theory Early results include the work of the Russian mathematician A. M. Lyapunov [25], who, in 1892, standardized the definition of stability and generalized the potential energy work of Lagrange [21] to systems of ordinary differential equations of the form $\dot{x}(t) = f(x(t))$. In the century that has followed, the use of Lyapunov functions to prove stability has become commonplace and is known alternatively as the "Direct method of Lyapunov" or "Lyapunov's second method".

In the 1960's, the input-output approach to stability analysis emerged as an alternative to Lyapunov's method. This approach was motivated by the development of complicated electronic systems for which a detailed analysis of the stability of the internal states was impractical. The work was pioneered by Sandberg [42, 42] and Zames [51, 52] and can be found in such works as [48] and [9]. The input-output framework was of practical importance for a number of primary reasons. First, it could be used with "black-box" frequency sweeping techniques to quickly determine properties of complicated linear systems even when those systems contained internal delay. In addition, these frequency-domain properties could then be used to predict the stability of the interconnection of systems through the use of concepts such as passivity and small-gain. Development of the input-output framework has had some relatively recent advances with the introduction of the theory of "Integral Quadratic Constraints"(IQCs) by Rantzer and Megretskii[40]. This work on IQCs generalizes the passivity framework by formulating stability conditions which allow for the use of a broad class of multipliers. This work will be discussed in more depth in Chapter 3.

Semidefinite Programming The first stability problem to be posed as a linear matrix inequality(LMI) can be attributed to Lyapunov himself, who stated that the system

$$\dot{x}(t) = Ax(t)$$

is stable if and only if there exists a $P \geq 0$ such that

$$A^T P + PA < 0.$$

Of course, in the absence of algorithms for this LMI, Lyapunov was forced to express this conditional analytically as the unique solution, P, for arbitrary $Q > 0$, of the Lyapunov equation $A^T P + PA = -Q$. The analysis of LMIs arising in control theory through the use of analytical solutions was continued in the 1940's and 1950's by the work of Lur'e, Postnikov, and others in the Soviet Union.

The first general method for the solution of LMIs was developed in the 1960's by Kalman, Yakubovich, and others. The famous positive-real or KYP lemma showed that a certain class of LMI problems could be solved graphically by considering a frequency domain inequality. Today, of course, the KYP lemma is currently used more often in the opposite sense, i.e. to convert a frequency-domain inequality to a semidefinite program. The LMI associated with the KYP lemma was later shown to admit an analytic solution through use of the algebraic Riccati equation(ARE).

The most significant advance in the solution of LMI problems was the recognition in the 1970's that many LMIs could be expressed as convex optimization problems which could be solved using recently developed efficient numerical algorithms for linear programming. Thus the use of analytic solutions to LMI problems was replaced by *semidefinite programming* which is defined as optimization over the convex cone of positive semidefinite matrices.

The algorithms for numerical optimization which were first used to solve LMI problems were originally developed to solve linear programming problems of the following form where the inequality is defined by the positive orthant.

$$\begin{aligned} &\max c^T y : \\ &Ay \leq b \end{aligned}$$

The solution of linear programming problems was first addressed by the development of the Simplex algorithm by G. Dantzig in the mid-40's. This algorithm performed well when applied to most problems but was shown to fail in certain special cases. Concern about this "worst-case" complexity prompted the search for what we now refer to as polynomial-time algorithms, which have provable bounds on worst-case performance. The first such polynomial-time algorithm was introduced by Khachiyan in 1979 and is referred to as the ellipsoid algorithm. This algorithm, however, proved inferior to the Simplex algorithm in most practical cases. The first major improvement over the Simplex algorithm was developed in 1984, when N. Karmakar proposed a new type of algorithm which used what we now refer to as an "interior-point" method. This algorithm was also of polynomial-time complexity but performed far better than the ellipsoid algorithm in practice. The final hurdle towards numerical solution of LMI problems was overcome in 1988, when Nesterov and Nemirovskii developed interior point algorithms which applied directly to semidefinite programming problems. Today, semidefinite programming programs can be solved simply and efficiently using interior-point solvers such as SeDuMi [45] combined with user-friendly interfaces such as Yalmip. A summary of standard LMI problems in control theory can be found in the seminal work by Boyd et al. [4].

Recently, the success of semidefinite programming in addressing problems in control has led to research into whether these algorithms can be applied to convex problems in polynomial optimization. Polynomial optimization problems arise in many areas of nonlinear systems analysis and control and this topic will be covered in significant depth in Chapter 6.

Internet Congestion Control The origins of research interest in Internet congestion control from a mathematical perspective can be traced back to the the paper by Kelly et al. [18], wherein the the congestion control problem was first cast as a decentralized optimization problem. This paper showed that certain congestion control protocols would converge to the global optimum in the absence of delay through the use of a Lyapunov argument. Subsequently, in Low and Lapsley [23], it was shown that the dynamics of the delayed Internet model with a certain class of control algorithms could be interpreted as a decentralized implementation of the asynchronous gradient projection algorithm to solve the dual to the network optimization problem thus showing global convergence to optimality for sufficiently small step size. However, no global bound for the step size was given in this paper, making practical interpretation difficult. This work was followed by the paper by Paganini et al. [29], wherein it was shown that with a certain set of pricing functions, a uniform bound of $\alpha < \pi/2$ on a certain gain parameter α at the source allows a proof of local stability for arbitrary topology and heterogeneous time delays. The uniform bound of $\alpha < \pi/2$ was shown to imply global stability for heterogeneous time delays in a more limited topology in [35] and [36]. All these results are discussed in more depth in Chapters 4 and 5.

Time-Delay Systems In 1963, the potential energy methods of Lagrange and Lyapunov were generalized to systems of functional differential equations by N. N. Krasovskii [20]. Since this time, many computationally tractable sufficient conditions have been given for stability of both linear and nonlinear time-delay systems, all with varying degrees of conservatism. An overview of some of these results can be obtained from survey materials such as are found in [13, 14, 19, 28]. These results can be grouped into analysis either in the frequency-domain or in the time-domain. Frequency-domain techniques can be applied to linear systems only and typically attempt to determine whether all roots of the characteristic equation of the system lie in the left half-plane. This approach is complicated by the transcendental nature of the characteristic equation, which imply the existence of a possibly infinite number of roots. Time-domain techniques generally use Lyapunov-based analysis. In the linear case, the Lyapunov approach benefits from the existence of an operator-theoretic version of Lyapunov's inequality, the existence of a positive solution to which is necessary and sufficient for stability. Computing solutions to this inequality, however, has historically been problematic. A main contribution of this thesis is to show how the operator-theoretic Lyapunov equation can be solved directly using semidefinite programming. This work is described in Chapter 7 and the references [37] and [38]. We note that one attempt to solve the Lyapunov equation by considering piecewise-linear functions was made in a series of papers by Gu et al. and is summarized in [13].

1.3 Notation

The following are used throughout this thesis.

Vector Spaces In this thesis, we use the following standard notation. $\mathbb{R}^{n\times m}$ denotes the space of real $n\times m$ matrices. $\mathbb{S}^n$ denotes the space of symmetric $n\times n$ matrices. Let $\mathbb{R}^+ := \{\, x \in \mathbb{R} \mid x \geq 0 \,\}$. Let $\mathcal{C}(I)$ denote the set of continuous functions $u : I \to \mathbb{R}^n$ where $I \subset \mathbb{R}$. We say $f \in \mathcal{C}(I)$ is *bounded* if there exists some $b \in \mathbb{R}^+$ such that $\|f(\theta)\|_2 \leq b$ for all $\theta \in I$. We use $\mathcal{C}_\tau$ to denote the Banach space of continuous functions $u \in \mathcal{C}([-\tau, 0])$ with norm $\|u\| = \sup_{t\in[-\tau,0]} \|u(t)\|_2$.

Definition 1. *For a given $\tau > 0$, $x \in \mathcal{C}([a, b))$, and $t \in [a + \tau, b]$, where $b > a + \tau$, define $x_t \in \mathcal{C}_\tau$ by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-\tau, 0]$.*

$L_2(-\infty,\infty)$ is the Hilbert space of Lebesgue measurable real vector-valued functions $x:\mathbb{R}\to\mathbb{R}^n$ with inner-product $\langle u,v\rangle_2=\int_{-\infty}^{\infty}u(t)^Tv(t)dt$. L_2 denotes $L_2[a,\infty)=\{\,x\in L_2(-\infty,\infty)\mid x(t)=0 \text{ for all } t<0\,\}$ and is a Hilbert subspace of $L_2(-\infty,\infty)$. Similarly, $L_2[a,b]$ denotes the restriction of $L_2(-\infty,\infty)$ to the interval $[a,b]$. Throughout, the dimensions of $x(t)$ for $x\in L_2$ should be clear from context and are not explicitly stated. We will occasionally also associate with $\mathcal{C}_\tau$ an inner product space equipped with the inner product associated with L_2. Thus for $x,y\in\mathcal{C}_\tau$, $\langle x,y\rangle$ denotes the L_2 inner product. $\hat{L}_2$ denotes the Hilbert space of complex vector-valued functions on the imaginary axis, $x:j\mathbb{R}\to\mathbb{C}^n$ with inner-product $\langle\hat{u},\hat{v}\rangle_2=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{u}(j\omega)^*\hat{v}(j\omega)d\omega$. $\hat{L}_\infty$ denotes the Banach space of matrix-valued functions on the imaginary axis, $\hat{G}:j\mathbb{R}\to\mathbb{C}^{m\times n}$ with norm $\|\hat{G}\|_\infty=\operatorname{ess\,sup}_{\omega\in\mathbb{R}}\bar{\sigma}\left(\hat{G}(j\omega)\right)$ where $\bar{\sigma}(\hat{G}(j\omega))$ denotes the maximum singular value of $\hat{G}(j\omega)$. For a given $\tau>0$, let M_2 denote the product space $\mathbb{R}^n\times L_2[-\tau,0]$ endowed with the inner product

$$\langle x,y\rangle := x_1^Ty_1+\langle x_2,y_2\rangle_2,$$

where we associate with $x\in M_2$ a pair (x_1,x_2) where $x_1\in\mathbb{R}^n$ and $x_2\in L_2[-\tau,0]$.

Maps and Operators An operator A on an inner-product space X is defined to be positive if $\langle x,Ax\rangle\geq 0$ for all $x\in X$. A function $x:\mathbb{R}\to\mathbb{R}$ is said to be ***absolutely continuous*** if for any integer N and any sequence $t_1,\ldots,t_N$, we have $\sum_{k=1}^{N-1}|x(t_k)-x(t_{k+1})|\to 0$ whenever $\sum_{k=1}^{N-1}|t_k-t_{k+1}|\to 0$. P_T is the truncation operator such that if $y=P_Tz$, then $y(t)=z(t)$ for all $t\leq T$ and $y(t)=0$ otherwise. L_{2e} denotes the space of functions such that for any $T>0$ and $y\in L_2e$, we have $P_Ty\in L_2$. We also make use of the space $W_2=\{y:y,\dot{y}\in L_2\}$ with inner product $\langle x,y\rangle_{W_2}=\langle x,y\rangle_{L_2}+\langle\dot{x},\dot{y}\rangle_{L_2}$ and extended space $W_{2e}=\{y:y,\dot{y}\in L_{2e}\}$. A causal operator $H:L_{2e}\to L_{2e}$ is bounded if $H(0)=0$ and if it has finite gain, defined as

$$\|H\|=\sup_{u\in L_2\neq 0}\frac{\|Hu\|}{\|u\|}$$

$\hat{u}$ denotes the either the Fourier or Laplace transform of u, depending on u. We will also make use of the following specialized set of transfer functions which define bounded linear operators on L_2. $\mathcal{A}$ is defined to be those transfer functions which are the Laplace transform of functions of the form

$$g(t) = \begin{cases} h(t) + \sum_{i=1}^{N} g_i \delta(t - t_i) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $h \in L_1$, $g_i \in \mathbb{R}$ and $t_i \geq 0$.

Real Algebraic Geometry Denote by $\mathbb{R}[x]$ the ring of scalar polynomials in variables x. Denote by $\mathbb{R}^{n\times m}[x]$ the set of n by m matrices with scalar elements in $\mathbb{R}[x]$. Let $\mathbb{S}^n[x]$ denote the set of symmetric n by n matrices with elements in $\mathbb{R}[x]$ and define $\mathbb{S}^n_d[x]$ to be the elements of $\mathbb{S}^n[x]$ of degree d or less. Let $\mathcal{P}^Y \subset \mathbb{R}[x]$ denote the convex cone of scalar polynomials which are non-negative on Y. Let $\mathcal{P}^+ \subset \mathbb{R}[x]$ denote the convex cone of globally non-negative scalar polynomials. Let $\mathcal{S}^Y_n \subset \mathbb{S}^n[x]$ denote the convex cone of elements $M \in \mathbb{S}^n[x]$ such that $M(x) \geq 0$ for all $x \in Y$. Let $\mathcal{S}^+_n$ denote the convex cone of elements $M \in \mathbb{S}^n[x]$ such that $M(x) \geq 0$ for all x. We let $Z_d[x]$ denote the $\binom{n+d}{d}$-dimensional vector of monomials in n variables x of degree d or less. Define $\bar{Z}^n_d[x] := I_n \otimes Z_d[x]$, where I_n is the identity matrix in $\mathbb{S}^n$. Finally, we note that for the sake of notational convenience, we will often use the expression $M(x) \in X$ to indicate that the function $M \in X$ for some set of functions X. This will hopefully not cause substantial amounts of confusion.

Chapter 2

Functional Differential Equations

2.1 Introduction

The best way to introduce the concept of a functional differential equation is through the use of an example.

Example 1: Perhaps the most easily understood example of a system defined by a functional differential equation is that of the dynamics of taking a shower. Specifically, define $\delta T(t)$ to be the difference between water temperature and body temperature at time t. We assume that the typical bather controls the water temperature by turning the hot-water knob at rate $\omega(t)$, proportional to the difference between water temperature and body temperature so that $\dot{\omega}(t) = -\alpha\delta T(t)$. Ideally, the position of the hot water knob is directly proportional to the temperature of the water coming from the head, $\delta T(t) = \beta\omega(t)$. In this case, we have the following ordinary differential equation.

$$\delta\dot{T}(t) = \alpha\omega(t) = -\alpha\beta\delta T(t)$$

This is a linear, time-invariant system and consequently we know that $\lim_{t\to\infty}\delta T(t) = 0$ for any any initial condition $\delta T(0)$ and any positive value of α. However, as most people know, there is occasionally a delay between action on the hot-water knob and change of temperature at the head. This delay occurs because hot water mixed at the knob must pass through a length of pipe before arrival at the shower head. We assume that the delay, τ, is constant and can be calculated as $\tau = L/v$; where L is the linear distance from the tap to the head and v is the rate of the water flowing in the pipe. The simplest description of the delayed system is given as follows.

$$\delta \dot{T}(t) = -\alpha\beta\delta T(t-\tau)$$

It can be shown[14] that a system of this form is stable in the region $\alpha\beta \in (0, \frac{\pi}{2\tau})$ and unstable outside of this region. Thus we conclude that in the presence of delay, any bather will get scalded if sufficiently impatient. The reason that a large proportional feedback gain fails in the shower example is that the temperature of the water at the head does not provide an adequate representation of the state of the system. In order to exactly predict exactly how the water temperature will evolve over time, one needs to know the water temperature, $T(\theta, t)$, at every point θ in the pipe from the knob to the head for some time t. This information precisely defines the state of the system at time t. Because the state at time t, $T(\cdot, t)$ is a function, the dynamics of a shower are defined by a function of a function or a *functional*.

In fact, the dynamics given above represent a simplification of the more difficult problem of fluid flow described by partial differential equations. We are able to represent the system using such a simple model only because the controller and observer are highly structured so that control and observation take place only at discrete points in the flow. Such simplified models are quite common and are collectively known as *time-delay systems*.

2.2 Definitions and the Concept of State

To begin, we define the following.

Definition 2. *Suppose we are given a $\tau \geq 0$ and map $f : \mathcal{C}_\tau \times \mathbb{R}^+ \to \mathbb{R}^n$. We say that a function $x \in \mathcal{C}([-\tau, b))$ is a* solution *on $[-\tau, b)$ to the functional differential equation defined by f with initial condition $x_0 \in \mathcal{C}_\tau$ if x is differentiable, $x(\theta) = x_0(\theta)$*

for $\theta \in [-\tau, 0]$ *and the following holds for* $t \geq 0$.

$$\dot{x}(t) = f(x_t, t) \tag{2.1}$$

In this thesis, we make use of two different concepts of state space. The first, and perhaps the most common is the Banach space $\mathcal{C}_\tau$, equipped with the supremum norm. In this scenario, the state of the system is simply the trajectory of the system over the past τ seconds of time. It is in this space that we will define solution maps and theorems of existence and uniqueness. An example of such a state is illustrated in Figure 2.1.

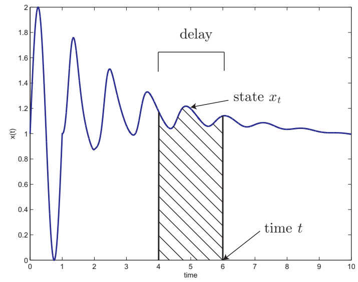

Figure 2.1: An illustration of the state $x_t \in \mathcal{C}_\tau$

Another concept of state space which we find more satisfying for applications such as stability is the space M_2, which was thoroughly investigated by such work as that by Delfour and Mitter[8]. For solution x, the state of the system, $\phi(t)$ at time t is defined as $\phi(t) = (x(t), x_t)$. ϕ lies in the subspace $H_2 \subset M_2$ defined as follows.

$$H_2 := \{y \in M_2 \, : \, y_2 \in \mathcal{C}_\tau, \; y_1 = y_2(0)\}$$

This space better reflects the dependency of the system on both the finite dimensional element $x(t)$ as well as the hereditary infinite dimensional element x_t. In addition, use of this state space allows us to embed the state space within the Hilbert space M_2, a feature useful in describing the operators used to define Lyapunov stability criteria.

Definition 3. *For a given functional,* $f : \mathcal{C}_\tau \times \mathbb{R}^+ \to \mathbb{R}^n$*, suppose that for any* $\phi \in \mathcal{C}_\tau$*, there exists a unique solution on* $[-\tau, \infty)$ *to the functional differential equation defined* *by f. Define the* ***solution map*** $G_f : \mathcal{C}_\tau \to \mathcal{C}([-\tau, \infty))$ *by* $y = G_f(\phi)$ *where* y *is the solution to the functional differential equation defined by* f *with initial condition* ϕ*.*

For the time-invariant case, where $f(x_t, t) = \tilde{f}(x_t)$, we define the following, which maps the evolution of the state.

Definition 4. *For a given solution map,* G_f*, define the* ***flow map*** $\Gamma_f : \mathcal{C}_\tau \times [0, b) \to \mathcal{C}_\tau$ *by* $\xi = \Gamma_f(\phi, \Delta t)$ *if* $\xi = y_{\Delta t}$ *where* $y = G_f(\phi)$*.*

The solution map, G_f, and flow map, Γ_f, provide convenient notation for expressing the evolution of the system over time. The following result gives conditions under which the solution map is well-defined in the time-varying case.

2.2.1 A Note on Existence of Solutions

In this subsection, we give a theorem which provides conditions for the existence and uniqueness of solutions defined on $\mathcal{C}([-\tau, \infty))$.

Theorem 5. *Suppose a functional* $f : \mathcal{C}_\tau \times \mathbb{R}^+ \to \mathbb{R}^n$ *satisfies the following.*

- *There exists a* $K_1 > 0$ *such that*

$$\|f(x,t) - f(y,t)\|_2 \leq K_1 \|x - y\|_{\mathcal{C}_\tau} \quad \text{for all } x, y \in \mathcal{C}_\tau, t \geq 0$$

- *There exists a* $K_2 > 0$ *such that*

$$\|f(0,t)\| \leq K_2 \quad \text{for all } t \geq 0$$

- $f(x,t)$ *is jointly continuous in* x *and* t*. i.e. for every* (x,t) *and* $\epsilon > 0$*, there exists a* $\eta > 0$ *such that*

$$\|x - y\|_{\mathcal{C}_\tau} + \|t - s\| \leq \eta \;\Rightarrow\; \|f(x,t) - f(y,s)\| \leq \epsilon.$$

Then for any $\phi \in \mathcal{C}_\tau$, there exists a unique $x \in \mathcal{C}[-\tau,\infty)$ such that x is differentiable for $t \geq 0$, $x(t) = \phi(t)$ for $t \in [-\tau, 0]$ and

$$\dot{x}(t) = f(x_t, t) \qquad \textit{for all } t \geq 0$$

See Appendix A for Proof.

By use of Theorem, 5, we can show that if a functional, f, satisfies a global Lipschitz continuity condition, then for any initial condition there exists a unique solution to the functional differential equation defined by f which is defined on $[-\tau, \infty)$. In this case the solution map and, in the time-invariant case, the flow map are well-defined.

2.3 Concepts of Stability

In this section we define stability of a functional differential equation.

2.3.1 Internal Stability

Two concepts of stability will be used in this thesis. The first, internal stability, defines stability as a property of solutions of a functional differential equation for arbitrary initial conditions. The second, input-output stability, is used to define stability of an operator and describes the relationship between inputs and outputs. In this subsection we give a definition of internal stability.

Definition 6. *Assume that f satisfies $f(0,t) = 0$. The solution map G_f, defined by f, is* ***stable*** *on $X \subset \mathcal{C}_\tau$ if*

(i) $G_f x$ is bounded for any $x \in X$

(ii) G_f is continuous at 0 with respect to the supremum norm on $\mathcal{C}([-\tau,\infty))$ and $\mathcal{C}_\tau$.

This is the usual notion of Lyapunov stability, which states that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|y\| < \delta$ implies $\|G_f y\| < \varepsilon$.

Definition 7. *The solution map G_f defined by f is* ***asymptotically stable*** *on $X \subset \mathcal{C}_\tau$ if it is stable on X and $y = G_f x_0$ implies $\lim_{t\to\infty} y(t) = 0$ for any $x_0 \in X$.*

Definition 8. *The solution map, G_f, is* ***globally stable*** *if it is stable on $\mathcal{C}_\tau$*

Definition 9. *The solution map, G_f, is* ***globally asymptotically stable*** *if it is asymptotically stable on $\mathcal{C}_\tau$.*

Note: For a system defined by a linear functional, global stability is equivalent to stability on any open neighborhood of the origin. See, for example [13, 19].

2.3.2 Input-Output stability

We can associate with a functional, f, an input-output system.

Definition 10. *For functional f and function g, let $y = \Psi_{f,g} u$ if $y(t) = g(x(t))$ for $t \in \mathbb{R}^+$ where $x = G_{\hat{f}}(0)$ and*

$$\hat{f}(x_t, t) = f(x_t, t) + u(t).$$

We can now define input-output stability directly in terms of properties of the operator Ψ.

Definition 11. *For normed spaces X, Y, the operator Ψ is Y stable on X if it defines a single-valued map from X to Y and there exists some β such that $\|\Psi u\|_Y \leq \beta \|u\|_X$ for all $u \in X$.*

Note: We refer to X stability on X as simply X stability.

2.4 Time-Delay Systems

So far, we have only considered the general case of a dynamic system defined by a functional. In this section, we identify a specific class of functional differential equation which will be of particular importance throughout this thesis.

Definition 12. *We say that a functional, f, defines a **Time-Delay System** if f can be represented using a function $p : \mathbb{R}^{n(K+1)+1} \to \mathbb{R}^n$ in the following way where $\tau_i > \tau_{i-1}$ for $i = 1, \dots, K$ and $\tau_0 = 0$.*

$$f(x_t) = \int_{-\tau_K}^{0} p(x_t(-\tau_0), x_t(-\tau_1), \cdots, x_t(-\tau_K), x_t(\theta), \theta) d\theta$$

Stability of Time-Delay systems are generally classified using the following definitions.

Definition 13. *A Time-Delay System is **Delay-Independent Stable** if it is stable for arbitrary $\tau_i \in \mathbb{R}$ for $i = 1 \dots K$.*

Definition 14. *A Time-Delay System is **Delay-Dependent Stable** on X if it is stable for $\tau_i \in X_i$ for $i = 1 \dots K$ where the X_i are compact subsets of $\mathbb{R}$.*

2.4.1 The Case of Linear Time-Delay Systems

In this subsection, we consider the special case of linear time-delay systems. Specifically, we consider functionals which can be expressed in the following form, where $\tau_i > \tau_{i-1}$ for $i = 1, \dots, K$ and $\tau_0 = 0$.

$$f(x_t) = \sum_{i=0}^{K} A_i x(t - \tau_i) + \int_{-\tau_K}^{0} A(\theta) x(t + \theta) d\theta \tag{2.2}$$

Here $A_i \in \mathbb{R}^{n \times n}$ and $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ is bounded on $[-\tau_K, 0]$. The following lemma, combined with Theorem 5 shows that elements of this class of system admit a unique solution for every initial condition $x_0 \in \mathcal{C}_{\tau_K}$.

Lemma 15. *Let*

$$f(x,t) := \sum_{i=1}^{K} A_j x(t - \tau_i) + \int_{-\tau_K}^{0} A(\theta) x(t - \theta) d\theta$$

Where $A(\theta)$ *is bounded on* $[-\tau, 0]$. *Then there exists some* $K > 0$ *such that*

$$\|f(x,t) - f(y,t)\|_2 \le K\|x - y\|_{\mathcal{C}_\tau}$$

See appendix for Proof.

By using Lemma 15, we can associate with linear systems of this form a well-defined solution map G_f and flow map Γ_f.

2.5 Conclusion

In this chapter, we have introduced various concepts associated with functional differential equations. These concepts include the state of the system, the solution map and stability in both the internal and input-output framework. We will make use of these definitions throughout the remainder of this thesis.

Chapter 3

Stability of Functional Differential Equations

In this chapter, we introduce two methods of proving stability of functional differential equations. The first, which can be used to prove internal stability, uses a generalization of Lyapunov theory to functional differential equations. The second method, which is used to prove input-output stability, is a generalization of the notion of passivity.

3.1 The Direct Method of Lyapunov

Consider a solution map G_f defined by a functional f. We consider the state of the system at time t to be defined by an element $x_t \in \mathcal{C}_\tau$. Standard Lyapunov theory, however, is defined using functions of the form $V(x(t))$. Such functions capture only part of the energy of the state, x_t, i.e. that part stored in $x_t(0)$. Therefore, any stability condition derived from such functions will be inherently conservative. An attempt to address this conservatism was made by Krasovskii [20] through the introduction of Lyapunov functionals which depend on elements of $\mathcal{C}_\tau$.

Definition 16. *Let $V : \mathcal{C}_\tau \to \mathbb{R}$ be a continuous function such that $V(0) = 0$. For a given flow map, Γ_f, defined by solution map, G_f, define the upper Lie derivative of V*

as follows.

$$\dot{V}(\phi) := \limsup_{h\to 0^+} \frac{1}{h}[V(\Gamma_f(\phi,h)) - V(\phi)]$$

The following theorem follows from Gu [13].

Theorem 17. *Let $\Omega \subset \mathcal{C}_\tau$ contain an open neighborhood of the origin. Let $f : \Omega \to \mathbb{R}^n$ be continuous and take bounded sets of Ω into bounded sets of $\mathbb{R}^n$. Let $V : \mathcal{C}_\tau \to \mathbb{R}$ be a continuous function such that $V(0) = 0$. Let $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous non-decreasing functions such that $u(s), v(s) > 0$ for $s \neq 0$ and $u(0) = v(0) = 0$. Suppose that the following holds for any $\phi \in \Omega$.*

$$u(\|\phi(0)\|) \leq V(\phi) \leq v(\|\phi\|)$$
$$\dot{V}(\phi) \leq -w(|\phi(0)|)$$

Then the solution map, G_f, defined by f is stable on some open neighborhood of the origin. If $w(s) > 0$ for $s > 0$, then the system is asymptotically stable on some open neighborhood of the origin. If $\Omega = \mathcal{C}_\tau$ and $\lim_{s\to\infty} = \infty$, then G_f is globally asymptotically stable.

3.1.1 Complete Quadratic Lyapunov Functionals

There have been a number of results concerning necessary and sufficient conditions for stability of linear time-delay systems in terms of the existence of quadratic functionals. These results are significant in that they allow us to restrict our search for a Lyapunov-Krasovskii functional to a specific class without introducing any conservatism. Consider a linear functional of the following form.

$$f(x_t) = \sum_{i=0}^{K} A_i x(t-\tau_i) + \int_{-\tau_K}^{0} A(\theta)x(t+\theta)d\theta \tag{3.1}$$

We now make the additional assumption that A is continuous on $[-\tau_K, 0]$. The following comes from Gu et al. [13].

Definition 18. *We say that a functional $V : \mathcal{C}_\tau \to \mathbb{R}$ is of the complete quadratic type if there exists a matrix $P \in \mathbb{S}^n$ and matrix-valued functions $Q : \mathbb{R} \to \mathbb{R}^{n\times n}$,*

$S : \mathbb{R} \to \mathbb{S}^n$ and $R : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ where $R(\theta,\eta) = R(\eta,\theta)^T$ such that the following holds.

$$V(\phi) = \phi(0)^T P\phi(0) + 2\phi(0)^T \int_{-\tau_K}^{0} Q(\theta)\phi(\theta)d\theta$$
$$+ \int_{-\tau_K}^{0} \phi(\theta)^T S(\theta)\phi(\theta)d\theta + \int_{-\tau_K}^{0}\int_{-\tau_K}^{0} \phi(\theta)^T R(\theta,\eta)\phi(\eta)d\theta d\eta$$

Theorem 19. *Suppose the system defined by a linear functional of the form* (3.1) *is asymptotically stable. Then there exists a complete quadratic functional V and $\eta > 0$ such that the following holds for all $\phi \in \mathcal{C}_\tau$.*

$$V(\phi) \ge \eta\|\phi(0)\|^2 \qquad \text{and} \qquad \dot{V}(\phi) \le -\eta\|\phi(0)\|^2$$

Furthermore, the matrix-valued functions which define V can be taken to be continuous everywhere except possibly at points $\theta, \eta = -\tau_i$ for $i = 1, \ldots, K-1$.

3.2 The Method of Integral Quadratic Contraints

The method of integral quadratic constraints of IQCs is a method of proving stability of the interconnection of stable operators.

3.2.1 Theory of Integral-Quadratic Constraints

Consider the following interconnection of operators.

Definition 20. *Let G be a linear operator with transfer function $\hat{G} \in \mathcal{A}$ and let the operator $\Delta : L_2 \to L_2$ be causal and bounded. Define inputs $f \in W_2$ and $g \in L_2$. The **interconnection** of G and Δ, denoted $\Phi_{F,G}$, is defined by $(y,u) = \Phi(f,g)$ where y and u are defined as follows.*

$$y = Gu + f$$
$$u = \Delta y + g$$

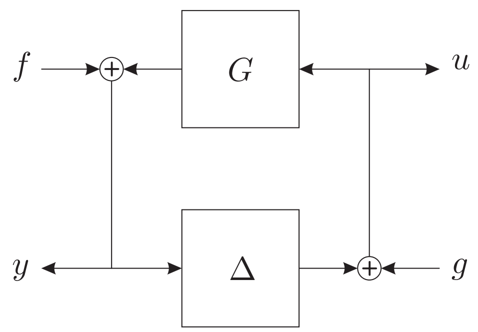

Figure 3.1: Interconnection of systems

Definition 21 (Jönsson [16], p71). *The interconnection of G and Δ, $\Phi_{G,\Delta}$, is* ***well-posed*** *if for every pair (f, g) with $f \in W_2$ and $g \in L_2$, there exists a solution $u \in L_{2e}$, $y \in W_{2e}$ and the map $(f, g) \rightarrow (y, u)$ is causal.*

If the interconnection of Δ and G is well posed, then the interconnection defines an operator $\Phi_{G,\Delta} : W_2 \times L_2 \rightarrow W_{2e} \times L_{2e}$. In this thesis, we use a result by Rantzer and Megretski [40] which can be interpreted as generalization of the classical notion of passivity. Recall the the following classical passivity theorem from e.g. Desoer and Vidyasagar(p. 182, [9])

Theorem 22. *The interconnection of Δ and G is L_2 stable on L_2 if there exists some $\epsilon > 0$ such that for any $x \in L_2$,*

$$\langle \Delta x, x \rangle \geq 0$$
$$\langle x, Gx \rangle \leq -\epsilon \|x\|$$

Now given bounded linear transformations Π_1, Π_2, define the following functional

$$\langle x, y \rangle_\Pi := \left\langle \Pi_1 \begin{bmatrix} x \\ y \end{bmatrix}, \Pi_2 \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle$$

Ignoring technical details for the moment, the result by Rantzer and Megretski states that the interconnection of Δ and G is stable if there exists an $\epsilon > 0$ such that for any $x \in L_2$,

$$\langle x, \Delta x \rangle_\Pi \geq 0$$
$$\langle Gx, x \rangle_\Pi \leq -\epsilon \|x\|$$

This idea of generalized passivity is motivated from a geometric standpoint by the topological separation argument, introduced by Safonov [41]. Consider the following definition of an operator graph.

Definition 23. *For an operator, $\rho : X \to X$, the **graph** of ρ is the set $\Phi(\rho) := \{(x, y) : y = \rho(x), x \in X\}$. The **inverse graph** of ρ is the set $\Phi_i(\rho) = \{(x, y) : x = \rho(y), y \in X\}$.*

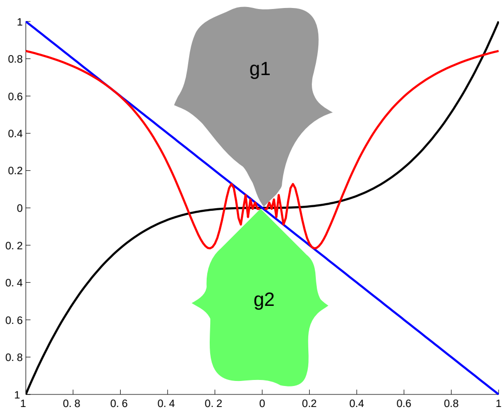

Figure 3.2: Separating functions prove separation of graphs g_1 and g_2

Two graphs are separate if they intersect only at the origin. Consider graphs g_1 and g_2. If, for some functional $\sigma : X \to \mathbb{R}$ and $\epsilon > 0$, we have $\sigma(x) \geq 0$ for all $x \in g_1$ and $\sigma(y) \leq -\epsilon \|y\|$ for all $y \in g_2$, then the graphs g_1 and g_2 can only intersect at the origin and are said to be separated by the functional σ. Now consider the interconnection of operators G and Δ when we let the input $g = 0$. We have the equation $f = (I - G\Delta)y$. The question of L_2 stability becomes the question of existence of a well-defined and bounded inverse of $(I - G\Delta)$ on L_2. Now suppose there exists a nonzero $y \in L_2$ such that $f = (I - G\Delta)y = 0$, then $(I - G\Delta)$ has a nontrivial kernel and thus can not have a bounded inverse. To ensure that $(I - G\Delta)$ has a trivial kernel, we will consider the graph of G and the inverse graph of Δ. The following well-known result shows that separation of the graph and inverse graph of interconnected operators is necessary for stability.

Theorem 24. *Let $f = (I - G\Delta)y$. The following are equivalent*

- $(I - G\Delta)y = 0 \quad \Rightarrow \quad y = 0$
- $\Phi(G) \cap \Phi_i(\Delta) = 0$

Proof. $(\Leftarrow)$ If there exists a $y \neq 0$ such that $(I - G\Delta)y = 0$ then let $x = \Delta y$. Thus

$$\begin{bmatrix} x \\ Gx \end{bmatrix} = \begin{bmatrix} \Delta y \\ y \end{bmatrix} \in \Phi(G) \cap \Phi_i(\Delta)$$

$(\Rightarrow)$ If there exists $y, x \neq 0$ such that

$$\begin{bmatrix} x \\ Gx \end{bmatrix} = \begin{bmatrix} \Delta y \\ y \end{bmatrix}$$

then $(I - G\Delta)y = y - G\Delta y = Gx - G\Delta y = G\Delta y - G\Delta y = 0$ ■

Although the separation of graphs is clearly necessary for most definitions of stability, it is unclear under what additional conditions separation may also be sufficient for L_2 stability. The result by Rantzer and Megretiski gives a particular class of functionals for which graph separation is sufficient for stability on L_2. Many classical theorems concerning the stability of the interconnection of operators can be viewed as proving the separation of graphs and inverse graphs by using these type of separating functionals. For example, the small gain theorem can be expressed using the separating functional $\sigma((x, y)) = \|x\| - k\|y\|$ for some $k > 0$. Similarly, classical passivity can be expressed using the functional $\sigma((x, y)) = \langle x, y \rangle$. The complete class of functionals shown by [40] to be sufficient for L_2 stability is given by Definition 25.

Definition 25 (Rantzer [40]). *The mapping $\sigma : L_2 \to \mathbb{R}$ is **quadratically continuous** if for every $\delta > 0$, there exists a η_δ such that the following holds for all*

$x_1, x_2 \in L_2$.

$$|\sigma(x_1) - \sigma(x_2)| \leq \eta_\delta \|x_1 - x_2\|^2 + \delta \|x_2\|^2$$

This class includes the small gain and passivity functions. Furthermore, for any bounded linear transformations Π_1, Π_2, the function $\sigma(w) = \langle \Pi_1 w, \Pi_2 w \rangle$ is quadratically continuous. In this thesis we use the following generalization of the work by Rantzer and Megretski as presented in the thesis work by Jönsson [16].

Definition 26. *Let $\Pi_B : j\mathbb{R} \to \mathbb{C}^{n\times n}$ be a bounded and measurable function that takes Hermitian values and $\lambda \in \mathbb{R}$. We say that Δ* ***satisfies the IQC*** *defined by Π_B, λ, if there exists a positive constant γ such that for all $y \in W_2$ and $v = \Delta y \in L_2$,*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix} \hat{y}(j\omega) \\ \hat{v}(j\omega) \end{bmatrix}^* \Pi_B(j\omega) \begin{bmatrix} \hat{y}(j\omega) \\ \hat{v}(j\omega) \end{bmatrix} d\omega + 2\langle v, \lambda \dot{y} \rangle \geq -\gamma |y(0)|^2$$

Theorem 27. *Assume that*

1. *G is a linear causal bounded operator with $s\hat{G}(s), \hat{G}(s) \in \mathcal{A}$*

2. *For all $\kappa \in [0, 1]$, the interconnection of $\kappa\Delta$ and G is well-posed*

3. *For all $\kappa \in [0, 1]$, $\kappa\Delta$ satisfies the IQC defined by Π_B, λ*

4. *There exists $\eta > 0$ such that for all $\omega \in \mathbb{R}$*

$$\begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix}^* \left(\Pi_B(j\omega) + \begin{bmatrix} 0 & \lambda j\omega^* \\ \lambda j\omega & 0 \end{bmatrix} \right) \begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix} \leq -\eta I$$

Then the interconnection of G and Δ is $W_2 \times L_2$ stable on $W_2 \times L_2$.

3.3 Conclusion

In this chapter, we have introduced two methods of constructing stability proofs of internal and input-output stability, respectively. In the rest of this thesis, we shall show how these methods can be applied to prove stability of different kinds of time-delay systems.

Chapter 4

Internet Congestion Control

4.1 Introduction

The analysis of Internet congestion control protocols has received much attention recently. Explicit mathematical modeling of the Internet has allowed analysis of existing protocols from a number of different theoretical perspectives and has generated some suggestions for improvement to current protocols. This work has been motivated by concern about the ability of current protocols to ensure stability and performance of the Internet as the number of users and amount of bandwidth continues to increase. Although the protocols that have been used in the past have performed remarkably well as the Internet has increased in size, analysis [22] indicates that as capacities and delays increase, instability will become a problem.

The purpose of this chapter is to outline the development of a mathematical theory of internet congestion control. Most of the recent research activity in Internet congestion control protocols can be traced back to the the paper by Kelly et al. [18], wherein the the congestion control problem was first cast as a decentralized optimization problem. Subsequently, in Low and Lapsley [23], it was shown that the dynamics of the Internet with a certain class of control algorithms could be interpreted as a decentralized implementation of the gradient projection algorithm to solve the dual to the network optimization problem, thus showing global convergence to optimality for sufficiently small step size. Finally, in Paganini et al. [29] it was shown that with a certain set of pricing functions, a bound of $\alpha < \pi/2$ on a certain parameter α at the source allows a proof of local stability for arbitrary topology and heterogeneous time delays. These fundamental results will form the background for our own work in Chapter 5, wherein we show that the parameter bound of Paganini et al. guarantees stability for a nonlinear implementation.

4.2 The Internet Optimization Problem

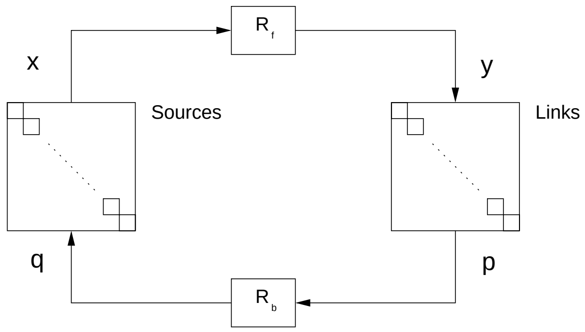

Figure 4.1: The internet is modeled as a collection of sources and links

We view the Internet as an abstract collection of sources and links. The term ***source*** refers to a connection between a user and a particular destination. The source transmits data in packets. The rate at which a source i transmits packets is dictated by the source's round-trip time, τ_i, as well as window size, w_i. The *round-trip time* is defined as the time between transmission of a packet and receipt of an acknowledgement for that packet from the destination. The *window size* is defined as the number of packets which are allowed to be simultaneously unacknowledged. In this thesis, we assume that packet losses do not affect the source transmission rates, since any lost packet will presumably be detected by the user after one round-trip time and resent. We assume that acknowledgements contribute to delay but do not contribute to congestion at the links. We assume a fixed bit size for all packets and that τ_i is known at least for the purposes of determining data rate. The packet transmission rate, x_i, at source i can be controlled by the window size according to

$$w_i = x_i \tau_i. \tag{4.1}$$

The term ***link*** refers to a single congested resource such as a router. Packets arriving at a link enter an entrance queue. A link can process packets in the queue at some rate capacity c_j. If too many data packets arrive in a given period of time, the size of the queue may grow and some packets may experience a queueing delay while in the queue. In this thesis, we assume that the dynamics from this variable queueing delay are negligible and we only model the delay due to the fixed propagation time. Links must also be able to feed back information. This can be done either through the ECN bit in the packet header, through packet dropping schemes or through measurement of variations in queueing delay at the source. The value of the congestion indicator at link j is denoted p_j. We also assume that the congestion indicator received at each source is the summation of the indicators of all links in the source's route. This value is denoted q_i.

Sources and links are related by routing tables which specify the route or set of links, J_i through which the packets from source i to its destination must pass. The rate of packets received at a link j is then the sum of the rates of all sources using that link and is denoted by y_j. The set of users for link j is denoted I_j. Ignoring delay for the moment, we have the following equations.

$$y = Rx, \quad q = R^T p,$$

where

$$R_{ji} = \begin{cases} 1 & \text{if source } i \text{ uses link } j \\ 0 & \text{otherwise} \end{cases}$$

4.3 Optimization Model

The following model for optimizing flow rates in a network was proposed by Kelly et al. [18].

$$\begin{aligned} \text{maximize} \quad & \sum_i^N U_i(x_i) \\ \text{subject to} \quad & x \geq 0, \quad Rx \leq c \end{aligned}$$

Assume that the U_i are continuously differentiable strictly concave non-decreasing functions. If all sources utilize at least one link, then the problem has a unique optimum. Note that, as N increases, the problem becomes progressively more difficult to solve using a centralized algorithm. We now consider the dual problem with dual variable $p \in \mathbb{R}^M$, where M is the number of links, which is given by

$$\begin{aligned} \text{minimize} \quad & h(p) \\ \text{subject to} \quad & p \geq 0 \end{aligned}$$

where the dual function h is given by

$$\begin{aligned} h(p) &= \max_{x \geq 0} \sum_i \big(U_i(x_i)\big) - p^T(Rx - c) \\ &= \sum_i \Big(U_i\big(x_{\text{opt},i}(p)\big)\Big) - p^T(Rx_{\text{opt}}(p) - c) \\ x_{\text{opt},i}(p) &= \max\{0, U_i'^{-1}(\sum_j R_{j,i}p_j)\} \\ &= \max\{0, U_i'^{-1}(q_i(p))\} \\ q(p) &= R^T p \end{aligned}$$

The map $U_i'^{-1} : \mathbb{R}^+ \to \{\mathbb{R} \cup \infty\}$ is well defined since $U_i' \in \mathcal{C}$ and U_i is strictly concave. We would like to construct a dynamical system which converges to the solution of the dual problem. One such system is given by the gradient projection algorithm. In discrete-time, this is

$$p_j(t+1) = \max\{0, p_j(t) - \gamma_j D_j h(p(t))\},$$

where D_j denotes the partial derivative with respect to the j'th argument and γ_j is the step size. Since the U_i are strictly concave, $h(p)$ is continuously differentiable with the following derivatives [2].

$$\begin{aligned} D_j h(p) &= c_j - \sum_{i \in I_j} x_{\text{opt},i} \\ &= c_j - y_{\text{opt},j}(p) \\ y_{\text{opt}}(p) &= R x_{\text{opt}}(p). \end{aligned}$$

If γ is sufficiently small, the discrete-time gradient projection algorithm will converge to the solution of the dual problem [23]. Because of convexity of the problem, strong duality implies that $x_{\text{opt}}(p)$ will converge to the unique optimum of the primal problem. A continuous-time implementation of this algorithm in the network framework is as follows.

$$\begin{aligned} \dot{p}_j(t) &= \begin{cases} \gamma_j(y_j(t) - c_j) & p_j(t) > 0 \\ \max\{0, \gamma_j(y_j(t) - c_j)\} & p_j(t) \leq 0 \end{cases} \\ x_i(t) &= \max\{0, U_i'^{-1}(q_i(t))\} \\ y(t) &= Rx(t), \quad q(t) = R^T p(t) \end{aligned}$$

γ_j now denotes a gain parameter, corresponding to step-size in discrete time. This algorithm has the remarkable property that it is decentralized, corresponding to the separable structure of the constraints. p_j is computed at each of M links. Link j requires only knowledge of y_j to compute this value. x_i is computed at each of N sources. Source i requires only knowledge of q_i to compute this value.

4.4 Stability Properties

To ensure that the continuous-time gradient projection algorithm will converge when implemented with the current internet framework, we must also consider the delay in transmitting packets from the source to the link and then receiving acknowledgements at the source. The delay from source i to link j is denoted τ_{ij}^f and the delay from link j to source i is denoted τ_{ij}^b. For any source i, the total round trip time is fixed, i.e. $\tau_i = \tau_{ij}^f + \tau_{ij}^b$ for all $j \in J_i$. We express these delays in the frequency domain by replacing the entries of the routing matrix R with forward and backward delay transfer functions $\hat{R}^f$ and $\hat{R}^b$, giving

$$\hat{y}(s) = \hat{R}^f(s)\hat{x}(s), \quad \hat{q}(s) = \hat{R}^b(s)^T\hat{p}(s)$$
$$\hat{R}^f_{ji}(s) = \begin{cases} e^{-\tau_{ij}^f s} & \text{if source } i \text{ uses link } j \\ 0 & \text{otherwise} \end{cases}$$
$$\hat{R}^b_{ji}(s) = \begin{cases} e^{-\tau_{ij}^b s} & \text{if source } i \text{ uses link } j \\ 0 & \text{otherwise} \end{cases}$$

The work by Paganini et al. [29] introduced a class of utility functions under which this system was shown to have a stable linearization about its positive equilibrium point for a fixed gain parameter $\gamma_j = 1/c_j$. This class was given by the set of U_i such that

$$\frac{d}{dq_i}U_i'^{-1}(q_i) = -\frac{\alpha_i}{M_i\tau_i}U_i'^{-1}(q_i),$$

where M_i is a bound on the number of links in the path of source i and $\alpha_i < \pi/2$. In particular, the choice of

$$U_i(x) = \frac{M_i\tau_i}{\alpha_i}x\left(1 - \ln\frac{x}{x_{\max,i}}\right),$$

with restricted domain $x \leq x_{\max,i}$ was suggested in [29] as a strictly concave utility function such that the function $U_i'^{-1}(q) = x_{\max,i}e^{-\frac{\alpha_i}{M_i\tau_i}q} \geq 0$ has the necessary derivative.

4.5 Recent Work

Some efforts have been made to extend this local stability result to the global case. For a single source and a single link, the paper by Wang and Paganini [46] has shown this implementation to be globally stable for $\alpha \leq f_1(x_{\max}/c)$, where

$$f_1(x) = \frac{\ln x}{x-1}.$$

In addition, the paper by Papachristodoulou [31] has shown this implementation to be stable when $\alpha \leq f_2(x_{\max}/c)$, where

$$f_2(x) = \frac{1}{x}.$$

f_1 and f_2 are illustrated in Figure 4.2. Note that when $x_{\max} = c$, both these conditions become $\alpha \leq 1$ which is more restrictive than the local stability bound of $\alpha < \pi/2$. The results presented in the next chapter attempt to eliminate the gap between local and global stability results by showing global stability for $\alpha < \pi/2 f_1(x_{\max}/c)$.

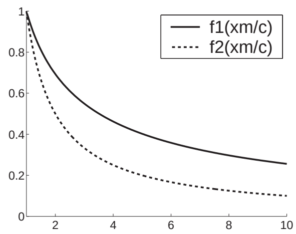

Figure 4.2: Plot of $f_1(x)$ and $f_2(x)$ vs. x

Chapter 5

Stability of Internet Congestion Control

5.1 Introduction

In this chapter, we extend some of the linear stability results of Paganini et al. [29] discussed in the previous chapter to the dynamics of a nonlinear implementation. Many algorithms have been proposed for Internet congestion control, some of which have been shown to be globally stable in the presence of delays, nonlinearities and discontinuities. These proofs can be grouped into several categories according to methodology. In particular, Lyapunov-Razumhikin theory has been used to show global stability in [49, 7, 15, 46, 47], Lyapunov-Krasovskii functionals have been used to show global stability in [1, 26, 31, 30] and an input-output approach was taken in [46, 10]. In all of these cases, stability has been proven with varying degrees of conservatism with respect to restrictions on system parameters or delays.

In general, stability analysis of nonlinear, discontinuous differential equations with delay is quite difficult. Although frequency domain techniques have been shown to be effective when applied to linear systems with delay, these tools fail in the presence of nonlinearity. In addition, although time-domain analysis of nonlinear finite dimensional systems has had some success, analysis of the infinite dimensional systems associated with delay has been more problematic. In this thesis, we are able to obtain improved results by decomposing the nonlinear, discontinuous, delayed system into an interconnection of a linear system with delay and a nonlinear, discontinuous system without delay. We analyze the subsystems separately and prove a passivity result for each. One benefit of such an approach is that it allows us to use frequency-domain arguments in addressing the infinite dimensional linear system. We can then use time-domain arguments in the analysis of the single state nonlinear system. These individual results are then be combined using the newly developed generalized passivity framework of Rantzer and Megretski as outlined in Chapter 3. This approach yields improved results by allowing us to decompose the original difficult problem into simpler subproblems, each of which may be solved with less conservatism. To apply the passivity framework of Rantzer and Megretski to the internet congestion control problem, we must first transform what is essentially a question of internal stability into a problem posed in the input-output framework. Furthermore, because input-output stability and internal stability are not equivalent, once a passivity result has been proven, we must use further analysis to show that this implies asymptotic stability. All these issues are addressed in the following sections.

5.2 Reformulation of the Problem

In this section we reformulate the question of internal stability of the proposed congestion control algorithm as a question of input-output stability of the interconnection of a linear system with delay and a nonlinear system without delay. This approach was motivated, in part, by the work of Wang[46] and Jönsson[17]. If we consider the problem of a single link and a single source, then from the development in Chapter 4, we have that $y(t) = x(t - \tau^f)$ and $q(t) = p(t - \tau^b)$ where $\tau^f + \tau^b = \tau$. Given an initial condition $x_0 \in \mathcal{C}_\tau$, the dynamics can now be summarized as $p(t) = x_0(t)$ for

$t \in [-\tau, 0]$ and the following for $t \geq 0$.

$$\dot{p}(t) = \begin{cases} \frac{x_{\max}}{c} e^{-\frac{\alpha}{\tau} p(t-\tau)} - 1 & p(t) > 0 \\ \max\{0, \frac{x_{\max}}{c} e^{-\frac{\alpha}{\tau} p(t-\tau)} - 1\} & p(t) \leq 0 \end{cases} \tag{5.1}$$

$$x(t) = x_{\max} e^{-\frac{\alpha}{\tau} p(t-\tau^b)} \tag{5.2}$$

Since the dynamics of Equation (5.1) are decoupled from those of (5.2) and stability of x follows from that of p, we need only consider stability of Equation (5.1). Now consider the equilibrium point of Equation (5.1), $p_0 = \frac{\tau}{\alpha} \ln \frac{x_{\max}}{c}$. As is customary, we change to variable z, where $z(t) = p(t) - p_0$ so that the origin is an equilibrium point. Now we have $z(t) = x_0(t) - p_0$ for $t \in [-\tau, 0]$ and the following for $t \geq 0$.

$$\dot{z}(t) = \begin{cases} e^{-\frac{\alpha}{\tau} z(t-\tau)} - 1 & z(t) > -p_0 \\ \max\{0, e^{-\frac{\alpha}{\tau} z(t-\tau)} - 1\} & z(t) \leq -p_0 \end{cases} \tag{5.3}$$

For convenience and efficiency of presentation, we will refer to the solution map defined by Equation (5.3) as $A : \mathcal{C}_\tau \to \mathcal{C}$. Implicit in these dynamics is the constraint $z(t) \geq -p_0$. If we assume that any initial condition will satisfy this constraint, we can include the constraint in the dynamics without altering the solution map. For convenience, we define the following bounded continuous functions.

$$f_1(y) = \min\big\{e^{\frac{\alpha}{\tau} y} - 1, e^{\frac{\alpha}{\tau} p_0} - 1\big\}$$
$$f_2(y) = \max\big\{0, f_1(y)\big\}$$
$$f_c(x, y) = \begin{cases} f_1(y) & \text{if } x > -p_0 \\ f_2(y) & \text{otherwise} \end{cases}$$

These functions are illustrated in Figure 5.1. We now have the following equation for $t \geq 0$.

$$\dot{z}(t) = f_c(z(t), -z(t-\tau)) \tag{5.4}$$

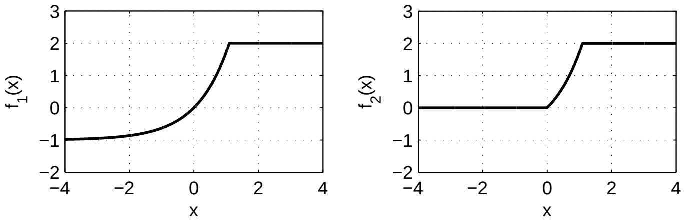

Figure 5.1: f_1 and f_2

Before proceeding, we must mention well-posedness of the solution map, A. We use a method of steps. Given any absolutely continuous solution $z(t)$ on some interval $[T_1, T_1 + \tau]$, we observe that $\dot{z}(t) = f_c(z(t), z(t-\tau)) = f_c(z(t), t)$ is a function only of time and state $z(t)$ for the interval $[T_1 + \tau, T_1 + 2\tau]$. From boundedness of the f_c, continuity with respect to $z(t-\tau)$ and noting upper semi-continuity of the associated differential inclusion with respect to $z(t)$, we can now establish via Fillipov [11][p77] the existence and uniqueness of a continuous solution $z(t)$ over the interval $[T_1 + \tau, T_1 + 2\tau]$. Assuming a continuous initial condition, this implies the existence and uniqueness of the solution map $A : \mathcal{C}_\tau \to \mathcal{C}$.

5.2.1 Separation into subsystems

Equation (5.4) is a delay-differential equation defined by a nonlinear, discontinuous function. To aid in the analysis, we will reformulate the problem as the interconnection of two subsystems where the W_2 stability on $W_2 \times L_2$ stability of this interconnection implies asymptotic stability on X of the original formulation for some set X. Define the map G by $w = Gu$ if

$$w(t) = \int_{t-\tau}^{t} u(\theta) d\theta.$$

Note that G is a linear operator which can be represented by the convolution with $g(t) = \text{step}(t) - \text{step}(t-\tau) \in L_1$. This implies that $\hat{G} \in \mathcal{A}$. Moreover, G can be represented in the frequency domain by $\hat{G}(s) = \frac{1-e^{-\tau s}}{s}$ which implies G is a bounded operator on L_2 since $\|\hat{G}(j\omega)\|_\infty = \tau$. In addition, $s\hat{G}(s) \in \mathcal{A}$ since it can be represented by convolution with $\delta(t) - \delta(t-\tau)$. Define the map Δ_z by $z = \Delta_z y$ if $z(0) = 0$ and

$$\dot{z}(t) = f_c\big(z(t), y(t) - z(t)\big).$$

We define the map Δ by $v = \Delta y$ if $v(t) = \dot{z}(t)$ where $z = \Delta_z y$. Addressing well-posedness, if $y \in W_2$, then y is absolutely continuous on any finite interval(See p. 25 in Jönsson [16]). From boundedness of f_c, continuity with respect to $y(t)$, and upper semi-continuity of the associated differential inclusion with respect to $z(t)$, we can again establish the existence and uniqueness of an absolutely continuous solution z and thus of the map Δ_z. Well-posedness of Δ follows immediately. Further properties of Δ will be derived in later sections.

If we now form the interconnection of G and $\kappa\Delta$ for $\kappa \in [0, 1]$, as defined above with a single input $f \in W_2$, we can construct a map from input f to outputs y, u. For convenience and efficiency of presentation, we will denote the interconnection map for $\kappa = 1$ by $B : W_2 \to W_{2e} \times L_{2e}$. Furthermore, for $\kappa = 1$ we denote the map from input f to internal variable z by B_z. For $t \leq 0$, we let $u(t) = y(t) = z(t) = f(t) = 0$ and for $t \geq 0$, the interconnection dynamics combine as follows.

$$\begin{aligned}
u(t) &= \kappa\dot{z}(t) \\
y(t) &= \int_{t-\tau}^{t} u(t)dt + f(t) \\
&= \kappa\big(z(t) - z(t-\tau)\big) + f(t) \\
\dot{z}(t) &= f_c(z(t), y(t) - z(t)) \\
&= f_c(z(t), f(t) - \kappa z(t-\tau) - (1-\kappa)z(t))
\end{aligned}$$

As before, from continuity with respect to $f(t)$ and $z(t-\tau)$, upper semi-continuity with respect to $z(t)$ and boundedness of f_c, we can conclude existence and uniqueness of $z \in L_{2e}$. Since z has bounded derivative and $z(t) = 0$ for $t \leq 0$, we now have that the map $B_z : W_2 \to W_{2e}$ is well-posed. Furthermore, this implies that $y \in W_{2e}$ and $u \in L_{2e}$ which yields well-posedness of the interconnection for any $\kappa \in [0, 1]$ and specifically of the map $B : W_2 \to W_{2e} \times L_{2e}$.

5.3 Input-Output Stability

In this section we use the IQC defined by Π_B, $\lambda = \frac{2}{\pi}$, where

$$\Pi_B = \begin{bmatrix} 0 & \beta \\ \beta & -\frac{4}{\pi} - 2 \end{bmatrix}$$

and $\beta = \alpha/(\alpha_{\max}\tau)$ to establish $W_2 \times L_2$ stability on W_2 of the interconnection for any $\tau \geq 0$, $0 < \alpha < \pi/2\alpha_{\max}$. Here we define

$$\alpha_{\max} = \ln(x_{\max}/c)/\big((x_{\max}/c) - 1\big).$$

5.3.1 Δ satisfies the IQC

In this subsection we show that if $\alpha > 0$, then Δ and consequently $\kappa\Delta$ are bounded and satisfy the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$ for all $\kappa \in [0, 1]$. The methods used in this subsection were motivated by those in Jönnson [17] and Wang [46]. For $\gamma = 4\beta/\pi > 0$, we prove the following for all $y \in W_2$, $v = \Delta y$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix} \hat{y}(j\omega) \\ \hat{v}(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & \beta \\ \beta & -\frac{4}{\pi} - 2 \end{bmatrix} \begin{bmatrix} \hat{y}(j\omega) \\ \hat{v}(j\omega) \end{bmatrix} + \frac{4}{\pi}\langle v, \dot{y}\rangle \geq -\gamma|y(0)|^2$$

By Parseval's equality, this is equivalent to

$$\frac{2}{\pi}\langle v, \dot{y} - v\rangle + \langle v, \beta y - v\rangle \geq -\frac{\gamma}{2}|y(0)|^2$$

A critical result used in the analysis of this section is the existence of a sector bound on the nonlinearity f_1 and consequently on f_2.

Lemma 28. $0 \leq f_i(x)x \leq \beta x^2$ *for* $i = 1, 2$ *where* $\beta = \frac{e^{\frac{\alpha}{\tau}p_0} - 1}{p_0}$.

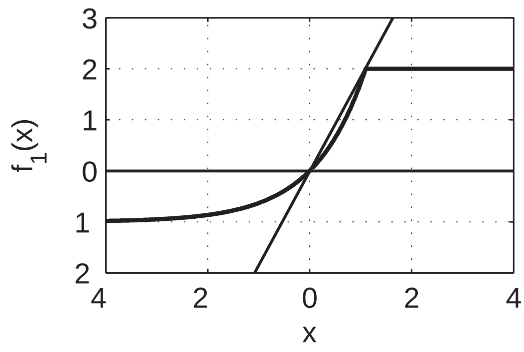

Figure 5.2: The nonlinearity f_1 satisfies a sector bound

See Appendix C for Proof.

This key feature is illustrated in Figure 5.2.

Lemma 29. *If $v = \Delta y$ with $y \in W_2$, then*

1. $v \in L_2$ *with norm bound* $\beta\|y\|$,
2. $\langle v, \beta y - v\rangle \geq 0$

Proof. We start by noting the following.

$$f_c(x,y) = \begin{cases} f_1(y) & \text{if } x > -p_0 \text{ or } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

As a consequence of the above sector bounds, we have

$$f_c(x,y)^2 \leq \beta y f_c(x,y).$$

Let $z = \Delta_z y$, then this implies

$$\begin{aligned} \dot{z}(t)^2 &= f_c\big(z(t), y(t) - z(t)\big)\dot{z}(t) \\ &\leq \beta\big(y(t) - z(t)\big)\dot{z}(t) \end{aligned}$$

Now for any $T \geq 0$, we have

$$\begin{aligned}
\|P_T v\|^2 &= \int_0^T v(t)^2 dt = \int_0^T \dot{z}(t)^2 dt \\
&\leq \beta \int_0^T \dot{z}(t)(y(t) - z(t))dt \\
&= \beta \int_0^T \dot{z}(t)y(t)dt - \frac{\beta}{2}(z(T)^2 - z(0)^2) \\
&\leq \beta \langle P_T \dot{z}, y\rangle \qquad (5.5) \\
&\leq \beta \|P_T \dot{z}\| \|y\| = \beta \|P_T v\| \|y\|
\end{aligned}$$

Therefore, $\|P_T v\| \leq \beta \|y\|$ for all $T \geq 0$. Thus $v \in L_2$ with norm bounded by $\beta\|y\|$. Statement 2 follows from line 5.5 by letting $T \to \infty$. ∎

Lemma 30. *Let $z = \Delta_z y$ with $y \in W_2$, then $\lim_{t\to\infty} z(t) = 0$.*

See Appendix C for Proof.

Lemma 31. *If $v = \Delta y$ with $y \in W_2$, then $\langle v, \dot{y} - v\rangle \geq -\beta |y(0)|^2$.*

See Appendix C for Proof.

Lemma 32. *$\kappa\Delta$ satisfies the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$ for any $\kappa \in [0, 1]$*

Proof. By Lemmas 29 and 31, we have the following.

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix} \hat{y}(j\omega) \\ \hat{v}(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & \beta \\ \beta & -\frac{4}{\pi} - 2 \end{bmatrix} \begin{bmatrix} \hat{y}(j\omega) \\ \hat{v}(j\omega) \end{bmatrix} + \frac{4}{\pi}\langle v, \dot{y}\rangle \\
&= \frac{4}{\pi}\langle v, \dot{y} - v\rangle + 2\langle v, \beta y - v\rangle \\
&\geq -\frac{4\beta}{\pi}|y(0)|^2
\end{aligned}$$

We conclude as a consequence that $\kappa\Delta$ satisfies the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$ for any $\kappa \in [0,1]$, since

$$\begin{aligned}
&\frac{2}{\pi}\langle \kappa v, \dot{y} - \kappa v\rangle + \langle \kappa v, \beta y - \kappa v\rangle \\
&\geq \kappa\big(\frac{2}{\pi}\langle v, \dot{y} - v\rangle + \langle v, \beta y - v\rangle\big) \\
&\geq -\kappa\frac{2\beta}{\pi}|y(0)|^2 \geq -\frac{2\beta}{\pi}|y(0)|^2
\end{aligned}$$

■

5.3.2 Properties of G

Recall that we define the map G as follows. $w = Gu$ if

$$w(t) = \int_{t-\tau}^{t} u(\theta)d\theta.$$

Lemma 33. *Suppose* $0 < \alpha < \pi\alpha_{max}/2$. *Then* $\hat{G}$ *satisfies condition 4 of Theorem 27.*

Proof. Recall that G can be represented as a transfer function in the frequency domain by $\hat{G} = \frac{1-e^{-j\omega\tau}}{j\omega}$. Now, examine the term

$$\begin{aligned}
&\begin{bmatrix}\hat{G}(j\omega)\\ I\end{bmatrix}^* \left(\Pi_B + \begin{bmatrix}0 & \lambda j\omega^*\\ \lambda j\omega & 0\end{bmatrix}\right)\begin{bmatrix}\hat{G}(j\omega)\\ I\end{bmatrix}\\
&= \begin{bmatrix}\frac{1-e^{-j\omega\tau}}{j\omega}\\ 1\end{bmatrix}^* \begin{bmatrix}0 & \beta + \frac{2}{\pi}j\omega^*\\ \beta + \frac{2}{\pi}j\omega & -\frac{4}{\pi} - 2\end{bmatrix}\begin{bmatrix}\frac{1-e^{-j\omega\tau}}{j\omega}\\ 1\end{bmatrix}\\
&= 2\cdot \text{Real}\left(\beta\frac{1-e^{-j\omega\tau}}{j\omega} - \frac{2}{\pi}e^{-j\omega\tau} - 1\right)\\
&= 2\left(\beta\tau\frac{\sin(\omega\tau)}{\omega\tau} - \frac{2}{\pi}\cos(\omega\tau) - 1\right) = 2p(\omega\tau)
\end{aligned}$$

Define $p_0(\omega) = p(\omega)$ for $\beta\tau = \pi/2$. The plot of $p_0(\omega) = \frac{\pi}{2}\frac{\sin(\omega)}{\omega} - \frac{2}{\pi}\cos(\omega) - 1$ is given in Figure 5.3. As one can see, the function is non-positive near the origin. In fact, for any $\beta\tau < \pi/2$, there exists an $\eta > 0$ such that $p(\omega) < -\eta$ for all ω. To see this, consider the following domains.

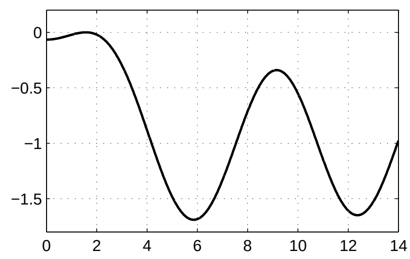

Figure 5.3: Plot of $p_0(\omega) = \frac{\pi}{2}\frac{\sin(\omega)}{\omega} - \frac{2}{\pi}\cos(\omega) - 1$ vs. ω.

$\omega > 2\pi$: We have that $|\frac{\sin(\omega)}{\omega}| \leq 1/2\pi$ for $|\omega| > 2\pi$ and $-1 - \frac{2}{\pi}\cos(\omega)) \leq -1 + 2/\pi$ for all ω. Therefore, $p(\omega) \leq \frac{1}{4} - (1 + \frac{2}{\pi}\cos(\omega)) \leq \frac{2-3\pi/4}{\pi} < -.1$.

$\pi - .1 \leq \omega \leq 2\pi$: For the interval $[\pi - .1, 2\pi]$, we have $\frac{\sin(\omega)}{\omega} \leq .04$. Therefore $p(\omega) \leq -1 + 2/\pi + .04\frac{\pi}{2} < -.3$.

$0 \leq \omega < \pi - .1$: One can see from plot of p_0 in Figure 5.3 that $p_0(\omega) \leq 0$ on the interval $[0, \pi - .1]$. Since $\beta\tau < \pi/2$, we can let $\epsilon = \pi/2 - \beta\tau > 0$. Then $p(\omega) = p_0(\omega) - \epsilon\frac{\sin(\omega)}{\omega} \leq -\epsilon\frac{\sin(\omega)}{\omega} < -.2\epsilon$ on $[0, \pi - .1]$.

Therefore, for any $\beta\tau < \frac{\pi}{2}$, let $\eta = \min\{.1, .2(\pi/2 - \beta\tau)\}$. Then $p(\omega) < -\eta$ for all $\omega \in \mathbb{R}$. Since $\beta\tau = \tau(e^{\frac{\alpha}{\tau}p_0} - 1)/p_0 = \alpha/\alpha_{\max}$, if $\alpha < \pi\alpha_{\max}/2$, we have that $\beta\tau < \pi/2$, and hence if $0 < \alpha < \pi\alpha_{\max}/2$, condition 4 of Theorem 27 is satisfied. ∎

5.3.3 Stability of the Interconnection

We conclude our discussion of input-output stability with the following Theorem concerning the stability of the the interconnection of Δ and G.

Theorem 34. *Suppose $\alpha \in (0, \pi\alpha_{max}/2)$. Then the map B, defining the interconnection of Δ and G is $W_2 \times L_2$ stable on W_2.*

Proof. We have shown that G is a linear causal bounded operator with $\hat{G}(s), s\hat{G}(s) \in \mathcal{A}$. We have also shown that the interconnection of G and $\kappa\Delta$ is well-posed for all $\kappa \in [0, 1]$ and Lemma 32 shows that $\kappa\Delta$ satisfies the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$ for all $\kappa \in [0, 1]$. Finally, by Lemma 33 we have that for all $\alpha \in (0, \pi\alpha_{\max}/2)$, condition 4 of

Theorem 27 is satisfied. We can therefore use Theorem 27 to prove $W_2 \times L_2$ stability on W_2 of the interconnection for any $\alpha \in (0, \pi\alpha_{\max}/2)$. ■

5.4 Asymptotic Stability

In this section, we show that $W_2 \times L_2$ stability on W_2 of the interconnection implies asymptotic stability of the original formulation of the congestion control protocol. Recall that the original solution map A is defined by the following.

$$\dot{z}(t) = f_c(z(t), -z(t-\tau)) \qquad t \geq 0 \tag{5.6}$$

$$z(t) = x_0(t) \qquad t \in [-\tau, 0] \tag{5.7}$$

The interconnection maps B and B_z, however, are defined by the following differential equation.

$$\dot{z}(t) = f_c(z(t), f(t) - z(t-\tau)) \qquad t \geq 0 \tag{5.8}$$

$$z(t) = 0 \qquad t \leq 0 \tag{5.9}$$

In the previous section, we have proven $W_2 \times L_2$ stability on W_2 of the map B where B represents a reformulation of the problem in the input-output framework. We would like to show, however, that for some $X \subset \mathcal{C}_\tau$ this result also implies asymptotic stability on X of the solution map A, where A represents the original formulation of the problem. This is done in the following Theorem.

Theorem 35. *Suppose $\alpha \in (0, \pi\alpha_{max}/2)$. Then the delay-differential equation* (5.1) *describing the algorithm proposed by Paganini et al. [29] is asymptotically stable on $X = \{x : x \in W_2 \cap \mathcal{C}_\tau, x(t) \geq -p_0, x(-\tau) > -p_0\}$.*

Proof. We have already shown $W_2 \times L_2$ stability on W_2 of the map B. Let $x_0 \in X$ be an arbitrary initial condition. Theorem 121 in the Appendix states that for any initial condition $x_0 \in X$, there exists some $f \in W_2$ and $T > 0$ such that $A(x_0, t) = B_z(f, t+T)$ for all $t \geq 0$. Let $(y, u) = B(f)$, then $y \in W_2$. Furthermore, recall $B_z f = \Delta_z y$ where B_z is the map to internal variable z. By Lemma 30, if $y \in W_2$, then $\lim_{t\to\infty} \Delta_z(y,t) = 0$. Therefore, $\lim_{t\to\infty} A(x_0,t) = \lim_{t\to\infty} B_z(f, t+T) = \lim_{t\to\infty} \Delta_z(y, t+T) = 0$. ■

5.5 Implementation

To implement the proposed algorithm in the Internet framework, the window size can be adjusted to deliver the required packet rate as given by equation (4.1). In implementation, the delay is unlikely to be known. In this case, a bound on the expected delay size can be used. Overestimation of the delay will result in an increased stability margin.

Modification of the link can take many forms. Price information from the link must be fed back to the source. Since queues themselves integrate excess rate, price of a congested resource can be computed directly using the queueing delay. However, this approach results in non-empty equilibrium queues and the possibility of unmodeled dynamics due to variable queueing delays. If a link instead uses a virtual capacity to avoid non-empty equilibrium queues, then explicit integration of incoming packets would be required and another mechanism must be used to feed back price information. An example of direct feedback of price information using packet marking is given by ECN. In one of the proposed implementations, packets are randomly marked at each link with probability $1 - \phi^{-p_j(t)}$ for some fixed $\phi > 1$. Thus, assuming no duplications, if ν is the percentage of marked packets received at the source, then the aggregate price can be measured as $q_i(t) = -\frac{\log(1-\nu)}{\log(\phi)}$. This variant is known as random exponential marking.

5.6 Conclusion

To summarize, for the case of a single source with a single link, we have demonstrated both input-output and asymptotic stability. We have used a generalized passivity framework to decompose a difficult nonlinear, discontinuous, infinite-dimensional problem into separate linear, infinite-dimensional and nonlinear, finite-dimensional subproblems, each of which is amenable to existing analysis techniques. A key feature of the analysis of the nonlinear subsystem was the existence of a sector bound on the nonlinearity.

In addition to the case of a single source with a single link, the result presented in this chapter applies directly to the case of multiple sources with identical fixed delay. The proof may also be easily adapted to alternate implementations of the proposed linearized protocols so long as they admit a similar sector bound on the nonlinearity. Although we would like to have proven a result in the case of multiple heterogeneous delays and arbitrary topology, we have as yet not been able to construct an appropriate sector bound. The root of the problem seems to lie in the discontinuity due to the projection used in the gradient projection algorithm.

Chapter 6

Sum-of-Squares and Convex Optimization

6.1 Convexity and Tractability

Consider the following optimization problem for $f_i \in \mathbb{R}[x]$ for $i = 0 \dots n_f$.

$$\max f_0(x) :$$
$$f_i(x) \geq 0, \qquad i = 1, \dots, n_f$$

The problem as formulated is clearly, in general, not convex. In fact, many NP-hard problems can be formulated in this manner. However, if we define $Y := \{x : f_i(x) \geq 0,\ i = 1 \dots n_f\}$, then the problem can be recast as an equivalent convex optimization problem as follows.

$$\max \gamma :$$
$$f_0(x) - \gamma \in \mathcal{P}^Y$$

Although this reformulation is convex, we must conclude that it too is intractable, unless we believe that $P = NP$. The point that we make with this section is that convexity, by itself, is not a sufficient condition for tractability of a problem. What we need in addition is an efficient linear test for membership in the set $\mathcal{P}^Y[x]$. Construction of such membership tests for various sets is the goal of Chapters 6, 7, and 8.

6.2 Sum-of-Squares Decomposition

Any polynomial, $f \in \mathbb{R}[x]$, of degree d and n variables can be expressed as the linear combination of $\binom{n+d}{d}$ monomials.

$$f(x) = \sum_{i=1}^{\binom{n+d}{d}} c_i x_1^{\gamma_{i,1}} \cdots x_n^{\gamma_{i,n}} \quad \gamma_{i,j} \in \mathbb{Z}^+ \tag{6.1}$$

The question of whether $f \in \mathcal{P}^+$, that is, $f(x) \geq 0$ for all $x \in \mathbb{R}^n$, is NP hard for polynomials of degree 4 or more. Thus there is unlikely to exist a computationally tractable set membership test for $\mathcal{P}^+$ unless $P = NP$. However, there may exist some convex cone $\Sigma \subset \mathcal{P}^+$ for which the set membership test is computationally tractable. One such cone is defined as follows.

Definition 36. *A polynomial $s \in \mathbb{R}[x]$ satisfies $s \in \Sigma_s$ if it can be represented in the following form for some finite set of polynomials $g_i \in \mathbb{R}[x]$, $i = 1 \ldots m$.*

$$s(x) = \sum_{i=1}^{m} g_i(x)^2$$

An element $s \in \Sigma_s$ is referred to as a ***sum-of-squares polynomial****. The set Σ_s^d is defined as the elements of Σ_s of degree d or less.*

Lemma 37. $\Sigma_s^d \subset \mathcal{P}^+$

In [33], it was shown that the set membership test $f \in \Sigma_s$ is computationally tractable.

Lemma 38. *A degree* $2d$ *polynomial* $f \in \mathbb{R}[x]$ *satisfies* $f \in \Sigma_s$ *if and only if there exists some matrix* $Q \in \mathbb{S}^{n_z}$ *where* $n_z = \binom{n+d}{d}$, *such that* $Q \geq 0$ *and*

$$f(x) = Z_d[x]^T Q Z_d[x].$$

The equality constraint in Lemma 38 is affine in the monomial coefficients of f. Therefore, set membership in Σ_s can be tested using semidefinite programming. The question of how well Σ_s represents $\mathcal{P}^+$ has been a topic on ongoing research for some time. It was conclusively shown through use of the Motzkin polynomial that in general $\Sigma_s \neq \mathcal{P}^+$. However, if we define Σ_r to be the convex cone of sums of squares of quotients of polynomials, then Artin showed that $\Sigma_r = \mathcal{P}^+$. Not surprisingly, however, there is no tractable set membership test for Σ_r. As an interesting corollary of this result, it was shown that $f \in \mathcal{P}^+$ if and only if there exists some $g \in \Sigma_s$ such that $fg \in \Sigma_s$. This does not constitute a tractable set membership test, unfortunately, since no bound is given on the degree of g. In addition, the introduction of the auxiliary variable g destroys the convexity of the constraint since the resulting feasibility problem is bilinear. Building on this work, however, Reznick showed that if $f \in \mathcal{P}^+$, then there exists some $d \in Z^+$ such that $(x^T x)^d (f(x) + 1) \in \Sigma_s$. This result provides a test for membership of f in the interior of $\mathcal{P}^+$, but not a tractable one, as there are, in general, still no bounds on d. What the result of Reznick does provide, however, is a sequence of tractable tests for membership in the interior of $\mathcal{P}^+$, indexed by the integer d.

6.3 Positivstellensatz Results

Membership in the cone $\mathcal{P}^K$ is harder to approximate than $\mathcal{P}^+$. Consider a region $\hat{K}_f$ of the following form for $f_i \in \mathbb{R}[x]$, $i = 1, \ldots, n_K$.

$$K_f := \{x \in \mathbb{R}^n : f_i(x) \geq 0, \ i = 1, \ldots, n_K\}.$$

Then $f_0 \in \mathcal{P}^{K_f}$ if and only if the following set is empty.

$$K_f^* := \{x \in \mathbb{R}^n : -f_0(x) \geq 0,\ f_0(x) \neq 0,\ f_i(x) \geq 0,\ i = 1, \ldots, n_K\}$$

A condition for emptiness of the set K_f^* comes from the following simplified Positivstellensatz result from Stengle [44].

Theorem 39. *Let K_f be given as above and let I denote the set of subsets of $\{0, \ldots, n_K\}$. Then $-f_0 \in \mathcal{P}^{K_f}$ if and only if there exist $s \in \Sigma_s$, $s_J \in \Sigma_s$, $J \in I$ and $k \in \mathbb{Z}^+$ such that*

$$s(x) + \sum_{J \in I} s_J(x) \prod_{i \in J} f_i(x) + f_0(x)^{2k} = 0$$

The Positivstellensatz condition given by Stengle is not tractable, due to the introduction of bilinear terms $s_J f_0$ and since no degree bound exists on either k or the degrees of the s_J. In the work by Schmüdgen, the convexity problem was addressed by considering the case when K_f is compact. This result can be stated as follows.

Theorem 40. *Let $\hat{I}$ be the set of subsets of $\{1, \ldots, n_K\}$ and suppose K_f, as defined above, is compact. Suppose f lies in the interior of $\mathcal{P}^{K_f}$. Then there exist $s_J \in \Sigma_s$ for $J \in \hat{I}$, such that the following holds.*

$$f(x) - \sum_{J \in \hat{I}} s_J(x) \prod_{i \in J} f_i(x) \in \Sigma_s$$

Theorem 40 differs from Stengle's result by essentially allowing us to set $s_J = 0$ when $0 \notin J$. This means that the resulting constraint is convex. However, Schmüdgen's result still does not constitute a tractable condition since there exist no bounds on the degrees of the s_J. The theorem does, however, suggest a method for constructing a sequence of sufficient conditions for membership in the set $\mathcal{P}^{K_f}$ by considering elements of Σ_s of bounded degree.

Definition 41. *For a given degree bound, d, and compact semi-algebraic set, K_f, of the form given above and $\hat{I}$ as defined above, we define the convex cone $\Lambda_d^{K_f}$ of* *functions* $f \in \mathbb{R}[x]$ *such that there exist functions* $s_J \in \Sigma_s^d$ *for* $J \in \hat{I}$*, of degree* d *or less, such that the following holds.*

$$f(x) - \sum_{J \in \hat{I}} s_J(x) \prod_{i \in J} f_i(x) \in \Sigma_s$$

Lemma 42. *For any* $d \in \mathbb{Z}^+$ *and any* K_f*,* $\Lambda_d^{K_f} \subset \mathcal{P}^{K_f}$

Testing membership in $\Lambda_d^{K_f}$ is a tractable problem for any fixed d and K_f and thus provides a tractable sufficiency test for membership in $\mathcal{P}^{K_f}$. However, the complexity of the test scales poorly in n_k. This problem was addressed to some extent by the work of Putinar [39], which showed that under certain additional restrictions on the set K_f, that we can assume that $s_J = 0$ for all J except $J = 1, \ldots, n_K$. This theorem is stated as follows.

Definition 43. *We say that* $f_i \in \mathbb{R}[x]$ *for* $i = 1, \ldots, n_K$ *define a* ***P-compact*** *set* K_f*, if there exist* $h \in \mathbb{R}[x]$ *and* $s_i \in \Sigma_s$ *for* $i = 0, \ldots, n_K$ *such that the level set* $\{x \in \mathbb{R}^n : h(x) \geq 0\}$ *is compact and such that the following holds.*

$$h(x) - \sum_{i=1}^{n_K} s_i(x) f_i(x) \in \Sigma_s$$

The condition that a region be P-compact may be difficult to verify. However, some important special cases include:

- Any region K_f such that all the f_i are linear.
- Any region K_f defined by f_i such that there exists some i for which the level set $\{x : f_i(x) \geq 0\}$ is compact.

Thus any compact region can be made P-compact by inclusion of a redundant constraint of the form $f_{n_K+1} := \beta - x^T x$ for sufficiently large β. However, for such an inclusion, we must have some explicit bound for the size of the original region.

Theorem 44. *Suppose* K_f*, as defined above, is P-compact. Suppose* f *lies in the interior of* $\mathcal{P}^{K_f}$*. Then there exist* $s_i \in \Sigma_s$ *for* $i = 1, \ldots, n_K$*, such that the following* *holds.*

$$f(x) - \sum_{i=1}^{n_K} s_i(x) f_i(x) \in \Sigma_s$$

Theorem 44 is similar to the Positivstellensatz result of Schmüdgen in that it does not represent a tractable test for membership in $\mathcal{P}^{K_f}$ since no bounds exist for the degree of the s_i. However, Putinar's Positivestellensatz provides a basis for a less computationally intensive sequence of tractable sufficient conditions.

Definition 45. *For a given degree bound, d, and semi-algebraic set, K_f, of the form given above, we define the convex cone $\Upsilon_d^{K_f}$ of functions $f \in \mathbb{R}[x]$ such that there exist $s_i \in \Sigma_s^d$ for $i = 1, \ldots, n_K$, such that the following holds.*

$$f(x) - \sum_{i=1}^{n_K} s_i(x) f_i(x) \in \Sigma_s$$

Lemma 46. *For any $d \in \mathbb{Z}^+$ and any K_f, $\Upsilon_d^{K_f} \subset \mathcal{P}^{K_f}$*

Membership in the set $\Upsilon_d^{K_f}$ is a tractable problem for any $d \in \mathbb{Z}^+$ and K_f and can be tested using semidefinite programming. Moreover, for any fixed d, the test associated with $\Upsilon_d^{K_f}$ is less computationally difficult than that associated with $\Lambda_d^{K_f}$, especially for large values of n_K. However, such statements should be taken with a grain of salt, since, for a fixed d, $\Lambda_d^{K_f}$ may well provide a better approximation of $\mathcal{P}^{K_f}$ than $\Upsilon_d^{K_f}$.

6.4 Applications

In this section, we demonstrate some applications of the techniques provided in Sections 6.2 and 6.3 to stability of nonlinear ordinary differential equations. Our concepts of stability are the same as those defined for functional differential equations, albeit with a state space defined in $\mathbb{R}^n$ as opposed to $\mathcal{C}_\tau$.

6.4.1 Stability of Ordinary Differential Equations

Consider a time-invariant ordinary differential equation of the following form.

$$\dot{x}(t) = f(x(t))$$

Here $x(t) \in \mathbb{R}^n$ and f is a polynomial of degree m such that $f(0) = 0$.

Theorem 47. *The system defined by the polynomial function f is globally stable if there exists an $\alpha > 0$ and a Lyapunov function, $V(x)$, with continuous derivatives such that $V(0) = 0$ and the following holds.*

$$\begin{aligned} V(x) &\geq \alpha x^T x && \textit{for all } x \in \mathbb{R}^n \\ \nabla V(x)^T f(x) &\leq 0 && \textit{for all } x \in \mathbb{R}^n \end{aligned}$$

If, in addition, $V(x)^T f(x) \leq -\alpha x^T x$ for all $x \in \mathbb{R}^n$, then the system is globally asymptotically stable.

By using the methods of the Section 6.2, we can pose the question of existence of Lyapunov function proving global stability of the system as a convex feasibility problem in the cone $\mathcal{P}^+$. Then, by approximating $\mathcal{P}^+$ by Σ_s, we can construct a sequence of sufficient conditions, of increasing accuracy, indexed by integers $d, n \geq 0$, which can be expressed as semidefinite programs.

Theorem 48. *The origin is globally asymptotically stable if there exists a $V \in \mathbb{R}[x]$, of degree d, and an $\alpha > 0$, such that the following conditions hold.*

$$\begin{aligned} V(x) - \alpha x^T x &\in \Sigma_s \\ -(\nabla V(x)^T f(x) + \alpha x^T x) &\in \Sigma_s \end{aligned}$$

6.4.2 Local Stability and Parametric Uncertainty

Consider a time-invariant ordinary differential equation of the following form.

$$\dot{x}(t) = f(p, x(t))$$

Here $x(t) \in \mathbb{R}^n$, $p \in \mathbb{R}^{d_p}$ and f is a polynomial of degree m such that $f(p, 0) = 0$ for all $p \in K \subset \mathbb{R}^{d_p}$. We first give a local, parameter-dependent version of Theorem 47.

Theorem 49. *Let $\Omega \subset \mathbb{R}^n$ contain an open neighborhood of the origin and $f \in \mathbb{R}[p, x]$ satisfy $f(p, 0) = 0$ for all $p \in K$. Suppose there exists an $\alpha > 0$ and a Lyapunov function, $V(p, x)$, with continuous derivatives such that $V(p, 0) = 0$ for $p \in K$ and the following holds.*

$$\begin{aligned} V(p, x) &\geq \alpha x^T x && \textit{for all } x \in \Omega,\ p \in K \\ \nabla_x V(p, x)^T f(p, x) &\leq 0 && \textit{for all } x \in \Omega,\ p \in K \end{aligned}$$

Then, for any $p \in K$, the system defined by f is stable on some open neighborhood of the origin. If, in addition, $V(p, x)^T f(p, x) \leq -\alpha x^T x$ for all $x \in \Omega$ and $p \in K$, then for any $p \in K$, the system is asymptotically stable on some open neighborhood of the origin.

Suppose the sets Ω and K are given by K_x and K_p, respectively, where $K_x := \{x \in \mathbb{R}^n : g_i(x) \geq 0, i = 1, \ldots, n_g\}$ for $g_i \in \mathbb{R}[x]$ and $K_p := \{p \in \mathbb{R}^{d_p} : h_i(x) \geq 0, i = 1, \ldots, n_h\}$ for $h_i \in \mathbb{R}[x]$. Now let the set $K_\times$ be the intersection of the sets K_x and K_p in the product space $\mathbb{R}^{n+d_p}$, defined as follows.

$$K_\times := \{(p, x) \in \mathbb{R}^{n+d_p} : p \in K_p,\ x \in K_x\}$$

By using the methods of the Sections 6.2 and 6.3, we can pose the stability conditions associated with Theorem 49 as a convex feasibility problem in the convex cone $\mathcal{P}^{K_\times}$. Then, for a given $d \in Z^+$, by approximating $\mathcal{P}^{K_\times}$ by $\Lambda_d^{K_\times}$ or $\Upsilon_d^{K_\times}$, we can construct a sequence of sufficient conditions, of increasing accuracy, indexed by integers $d, r \geq 0$, which can be expressed as semidefinite programs.

Theorem 50. *Let f be a polynomial such that $f(p, 0) = 0$ for all $p \in K_p$ and K_x contain an open neighborhood of the origin. Suppose there exists a $V \in \mathbb{R}[p, x]$, of* *degree r, and an $\alpha > 0$ such that the following conditions hold.*

$$V(p,x) - \alpha x^T x \in \Upsilon_d^{K_\times}$$
$$-(\nabla_x V(p,x)^T f(p,x) + \alpha x^T x) \in \Upsilon_d^{K_\times}$$

Then, for any $p \in K_p$, the origin is asymptotically stable in some open region about the origin.

6.5 The SOS Representation of Matrix Functions

The previous sections discussed tractable tests for membership in the convex cones $\mathcal{P}^+$ and $\mathcal{P}^K$ of non-negativity scalar polynomials. In this section, we present an extension of these results to the convex cones $\mathcal{S}_n^+$ and $\mathcal{S}_n^K$ of non-negative matrices of polynomials. A simple test for $M \in \mathcal{S}_n^+$ or $M \in \mathcal{S}_n^K$ can be performed through the introduction of n auxiliary variables.

Lemma 51. *For $M \in \mathbb{S}^n[x]$, let $f_M \in \mathbb{R}[x,y]$ be defined as $f(x,y) := y^T M(x) y$.*

1. *$M \in \mathcal{S}_n^+$ if and only if $f_M \in \mathcal{P}^+$.*
2. *$M \in \mathcal{S}_n^K$ if and only if $f_M \in \mathcal{P}^{K'}$, where $K' := \{(x,y) : x \in K\}$.*

Thus, we can determine whether $M \in \mathcal{S}_n^+$ by testing whether $f_M \in \mathcal{P}^+$ using the techniques discussed in the previous section. The problem with this approach is that the size of the resulting semidefinite program scales very poorly in the dimension of M. This is due to the introduction of the auxiliary variables y. Instead, we will consider the following direct approximation to the cone $\mathcal{S}_n^+$.

Definition 52. *$M \in \mathbb{S}^n[x]$ satisfies $M \in \bar{\Sigma}_s$ if it can be represented in the following form for some finite set of polynomial matrices $G_i \in \mathbb{R}^{n \times n}[x]$, $i = 1, \ldots, m$.*

$$M(x) = \sum_{i=1}^{m} G_i(x)^T G_i(x)$$

An element $M \in \bar{\Sigma}_s$ is referred to as a ***sum-of-squares matrix function****. Furthermore, define $\bar{\Sigma}_s^d$ to be elements of $\bar{\Sigma}_s$ of degree d or less.*

We now provide a tractable test for membership in $\bar{\Sigma}_s$.

Lemma 53. *Suppose $M \in \mathbb{S}_{2d}^n[x]$ for $x \in \mathbb{R}^{n_2}$. $M \in \bar{\Sigma}_s$ if and only if there exists some matrix $Q \in \mathbb{S}^{n_z \cdot n}$, where $n_z = \binom{n_2+d}{d}$ such that $Q \geq 0$ and the following holds.*

$$M(x) = \left(\bar{Z}_d^n[x]\right)^T Q \bar{Z}_d^n[x]$$

See Appendix B for Proof.

The complexity of the membership test associated with Lemma 53 is considerably lower than that associated with Lemma 51. Specifically, the variables associated with the first test are of order $n\binom{n_2+d}{d}$ as opposed to order $\binom{n+n_2+d+1}{d+1}$. Furthermore, the use of the test associated with Lemma 53 does not increase the conservatively, as evidenced by the following lemma.

Lemma 54. *$M(x) \in \bar{\Sigma}_s$ if and only if $y^T M(x) y \in \Sigma_s$.*

See Appendix B for Proof.

As for Σ_s and $\mathcal{P}^+$, the question of how well $\bar{\Sigma}_s$ approximates $\mathcal{S}_n^+$ is an open question. However, in the case of a single variable, we have the following result due to Choi et al. [5].

Lemma 55. *For $x \in \mathbb{R}$, $M \in \mathbb{S}^n[x]$, the following are equivalent*

- $M \in \bar{\Sigma}_s$
- $M \in \mathcal{S}_n^+$.

Lemma 55 will find considerable application in Chapter 7.

6.6 Positivstellensatz Results for Matrix Functions

In this section, we consider membership in the set $\mathcal{S}_n^{K_f}$ for some semi-algebraic set K_f, defined by polynomials $f_i \in \mathbb{R}[x]$ as follows.

$$K_f := \{x : f_i(x) \geq 0,\ i = 1 \ldots n_f\}$$

Scherer and Hol [43] have proven the following generalization of Putinar's Positivstellensatz.

Theorem 56. *Suppose that K_f is P-compact and $M \in \mathcal{S}_n^{K_f}$. Then there exist $\eta > 0$ and $S_i \in \bar{\Sigma}_s$ for $i = 1, \ldots, n_f$ such that the following holds.*

$$M(x) - \sum_{i=1}^{n_f} f_i(x) S_i(x) - \epsilon I \in \bar{\Sigma}_s$$

Theorem 56 does not represent a tractable test for $\mathcal{S}_n^{K_f}$, since no degree bounds are given on the S_i. However, the result does suggest a sequence of sufficient conditions, indexed by $d \in \mathbb{Z}^+$, and defined as follows.

Definition 57. *We say that $M \in \bar{\Upsilon}_{n,d}^{K_f}$ if there exist $\eta > 0$ and $S_i \in \bar{\Sigma}_s^d$ for $i = 1, \ldots, n_f$ such that the following holds.*

$$M(x) - \sum_{i=1}^{n_f} f_i(x) S_i(x) \in \bar{\Sigma}_s$$

Lemma 58. *For any $d \geq 0$ and K_f, $\bar{\Upsilon}_{n,d}^{K_f} \subset \mathcal{S}_n^{K_f}$.*

Therefore, for any $d \geq 0$, we can construct a tractable test for membership in the convex cone $\bar{\Upsilon}_{n,d}^{K_f}$, expressible as a semidefinite program. Furthermore, as d increases, $\bar{\Upsilon}_{n,d}^{K_f}$ will constitute a better approximation for $\mathcal{S}_n^{K_f}$.

6.6.1 Linear ODEs with Parametric Uncertainty

The most common application for the sum-of-squares matrix representation is the case of a linear finite-dimensional system which contains uncertainty. Consider a system of ordinary differential equations of the following form where $A : \mathbb{R}^{n_y} \to \mathbb{R}^{n \times n}$.

$$\dot{x}(t) = A(y)x(t) \qquad y \in G \tag{6.2}$$

The following is a standard result in analysis of LTI systems.

Theorem 59. *The system defined by Equation 6.2 is globally asymptotically stable for all $y \in G$ if and only if there exists a matrix function $P : G \to \mathbb{S}^n$ such that the following conditions hold.*

$$\begin{aligned} P(y) - I &\in \mathcal{S}_n^G \\ -(A(y)^T P(y) + P(y)A(y) + I) &\in \mathcal{S}_n^G \end{aligned}$$

Now suppose that G is a semi-algebraic set defined by polynomials $\{g_i\}_{i=1}^{n_g}$. Then for integer $d \geq 0$, we can approximate the conditions 59 using the following.

Theorem 60. *The system defined by Equation 6.2 is globally asymptotically stable for all $y \in G$ if there exists a matrix of polynomials $P \in \mathbb{S}^n[y]$ such that the following conditions hold.*

$$\begin{aligned} P(y) - I &\in \Upsilon_{n,d}^G \\ -(A(y)^T P(y) + P(y)A(y) + I) &\in \Upsilon_{n,d}^G \end{aligned}$$

The conditions of Theorem 60 can be expressed as a semidefinite program.

6.7 Conclusion

This chapter has detailed a method for expressing may difficult problems in control as convex optimization problems. These optimization problems are, in general, not tractable due to the lack of an efficient membership test for the convex cone $\mathcal{P}^+$. However, we have shown that by approximating $\mathcal{P}^+$ by Σ_s, one can construct tractable approximations. The techniques and concepts of this chapter will play an integral role as we seek to generalize the ODE applications presented in subsections 6.4.1, 6.4.2, and 6.6.1 to differential equations which contain delay.

Chapter 7

Stability of Linear Time-Delay Systems

7.1 Introduction

The study of stability of systems of differential equations which contain delays has been an active area of research for some time. A complete summary of the results in this field is beyond the scope of this paper. However, an overview of these results can be obtained from various survey papers and books on the subject, see for example [13, 14, 19, 28]. The previous results on this subject can be grouped into analysis either in the frequency-domain or in the time-domain. Frequency-domain techniques apply only to linear systems and typically attempt to determine whether all roots of the characteristic equation of the system lie in the left half-plane. Time-domain techniques generally use Lyapunov-based analysis, an approach which was extended to infinite dimensional systems by Krasovskii in [20]. Stability results in this area are grouped into delay-dependent and delay-independent conditions. If a delay-dependent condition holds, then stability is guaranteed for a specific value or range of values of the delay. If a delay-independent condition holds, then the system is stable for all possible values of the delay. A particularly interesting result on the linear delay-independent case appears in [3].

In this chapter, we show how to compute solutions to an operator-theoretic version of the Lyapunov inequality using semidefinite programming. This inequality, defined by the derivative of the complete quadratic functional, can be posed as a convex feasibility problem over certain infinite-dimensional convex cones defined by positive operators on certain subspaces of $\mathcal{C}_\tau$. Our result is expressed as a nested sequence of sufficient conditions which are of increasing accuracy and can be tested using semidefinite programming. Our approach is based on parameterizing the convex cones mentioned above using polynomial functions of bounded degree.

7.2 Positive Operators

In this section, we present the convex cones of positive multiplier and integral operators which define the complete quadratic functional and some its derivative forms. We then use polynomials of bounded degree to parametrization certain convex subsets of these cones in terms of positive semidefinite matrices. These results will allow us to express the Lyapunov stability conditions in terms of semidefinite programming problems. To begin, consider the complete quadratic functional.

$$\begin{aligned} V(x) &= x(0)^T P x(0) + 2\int_{-\tau}^{0} x(0)^T Q(\theta) x(\theta) d\theta \\ &+ \int_{-\tau}^{0} x(\theta)^T S(\theta) x(\theta) d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} x(\theta)^T R(\theta,\omega) x(\omega) d\theta d\omega \end{aligned}$$

We can associate with V and $\epsilon > 0$ an operator $A : \mathcal{C}_\tau \to \mathcal{C}_\tau$ such that $V(x) = \langle x, Ax\rangle$, where A is defined as follows.

$$\begin{aligned} (Ay)(\theta) &= \begin{bmatrix} P - \epsilon I & Q(\theta) \\ Q(\theta)^T & S(\theta) \end{bmatrix} y(\theta) + \int_{-\tau}^{0} \begin{bmatrix} 0 & 0 \\ 0 & R(\theta,\omega) \end{bmatrix} y(\omega) d\omega \\ &= (A_1 y)(\theta) + (A_2 y)(\theta), \end{aligned}$$

The operator A is a combination of multiplier operator A_1 and integral operator A_2. The complete quadratic functional, V satisfies the positivity condition of the stability theorem with $u(\phi(0)) = \epsilon\phi(0)^2$ if the integral operator, A_2, is positive on $\mathcal{C}_\tau$

and there exists some $\epsilon > 0$ such that the multiplication operator, A_1, is positive on the subspace $X \subset \mathcal{C}_\tau$ where

$$X := \{x \in \mathcal{C}_\tau \mid , x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } x_1(\theta) = x_2(0) \text{ for all } \theta\}.$$

We now define the specific sets of operators which define the complete quadratic functional and its derivative forms.

Definition 61. *For a matrix-valued function $M : \mathbb{R} \to \mathbb{S}^n$, we denote by A_M the multiplication operator such that*

$$(A_M x)(\theta) = M(\theta)x(\theta)$$

Definition 62. *For a matrix-valued function $R : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n}$ such that $R(\theta, \omega) = R(\omega, \theta)^T$, we denote by B_R the integral operator such that*

$$(B_R x)(\theta) = \int_{-\tau}^{0} R(\theta, \omega)x(\omega)d\omega$$

Definition 63. *For a continuous matrix-valued function M, we say $M \in H_1^+$ if $\langle x, A_M x\rangle \geq 0$ for all $x \in X$.*

Definition 64. *For a continuous matrix-valued function $R : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n}$ such that $R(\theta, \omega) = R(\omega, \theta)^T$, we say $R \in H_2^+$ if $\langle x, B_R x\rangle \geq 0$ for all $x \in \mathcal{C}_\tau$.*

Definition 65. *For a piecewise-continuous matrix-valued function M, we say $M \in \tilde{H}_1^+$ if $\langle x, A_M x\rangle \geq 0$ for all $x \in X$.*

Definition 66. *For a piecewise-continuous matrix-valued function $R : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n}$ such that $R(\theta, \omega) = R(\omega, \theta)^T$, we say $R \in \tilde{H}_2^+$ if $\langle x, B_R x\rangle \geq 0$ for all $x \in \mathcal{C}_\tau$.*

Definition 67. *For a continuous matrix-valued function M, we say $M \in H_3^+$ if*

$\langle x, A_M x\rangle \geq 0$ *for all* $x \in X_3$*, where*

$$X_3 := \{x \in \mathcal{C}_\tau \mid , x = \begin{bmatrix} x_1^T & x_2^T & x_3^T \end{bmatrix}^T \text{ and } x_1(\theta) = x_3(0) \text{ and } \\ x_2(\theta) = x_3(-\tau) \text{ for all } \theta\}.$$

Definition 68. *For a piecewise-continuous matrix-valued function* M*, we say* $M \in \tilde{H}_3^+$ *if* $\langle x, A_M x\rangle \geq 0$ *for all* $x \in \tilde{X}_3$*, where*

$$\tilde{X}_3 := \Big\{x \in \mathcal{C}_\tau \mid , x = \begin{bmatrix} x_1^T & \dots & x_{K+2}^T \end{bmatrix}^T \text{ and } \\ x_i(\theta) = x_{K+2}(-\tau_{i-1}) \text{ for all } \theta, \; i = 1, \dots, K+1\Big\}.$$

7.2.1 Multiplier Operators and Spacing Functions

In this subsection, we consider the convex cones H_1^+ and H_3^+ which define positive multiplication operators on X_1 and X_3 respectively. We use the following theorem.

Theorem 69. *Suppose* $M : \mathbb{R} \to \mathbb{S}^{2n}$ *is a continuous matrix-valued function. Then the following are equivalent*

1. *There exists some* $\epsilon > 0$ *such that the following holds for all* $x \in \mathcal{C}_\tau$*.*

$$\int_{-\tau}^0 \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta \geq \epsilon \|x\|_2^2$$

2. *There exists some* $\epsilon' > 0$ *and some continuous matrix-valued function* $T : \mathbb{R} \to \mathbb{S}^n$ *such that the following holds.*

$$\int_{-\tau}^0 T(\theta) d\theta = 0$$

$$M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \geq 0 \qquad \text{for all } \theta \in [-\tau, 0]$$

See Appendix D for Proof.

Definition 70. *Given n and τ, we refer to any piecewise-continuous matrix-valued function $T : \mathbb{R} \to \mathbb{R}^n$ such that $\int_{-\tau}^0 T(\theta)d\theta = 0$ as a* ***spacing function****, denoted $T \in \Omega$.*

This theorem states that any element of H_1^+ which satisfies a certain additional positivity condition can be represented by the combination of a positive semidefinite matrix-valued function and a spacing function. This allows us to search for elements of H_1^+ by simultaneously searching over the set of positive semidefinite matrix-valued functions and the set of spacing functions. For any $d \in \mathbb{Z}^+$, the following gives a tractable condition for membership in H_1^+.

Definition 71. *Given n and τ, define Ω_p^d to be the set of functions $T \in \mathbb{S}_d^n[\theta]$ such that the following holds.*

$$\int_{-\tau}^0 T(\theta)d\theta = 0$$

Definition 72. *For a given $d \geq 0$, $M \in \mathbb{S}^{2n}[\theta]$ satisfies $M \in G_1^d$ if there exists a $T \in \Omega_p^d$ such that*

$$M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{P}^+.$$

Membership in the convex cone G_1^d can be implemented as a semidefinite programming constraint by noting that the constraint that a polynomial integrate to 0 is affine in the coefficients and that for the single variable case, $\bar{\Sigma}_s = \mathcal{P}^+$. We now consider H_3^+. We can use a simple extension of the previous theorem.

Lemma 73. *Let $M : \mathbb{R} \to \mathbb{S}^{3n}$ be a continuous matrix-valued function. Then the following are equivalent.*

1. *There exists an $\epsilon > 0$ such that the following holds for all $x \in \mathcal{C}_\tau$.*

$$\int_{-\tau}^0 \begin{bmatrix} x(0) \\ x(-\tau) \\ x(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} x(0) \\ x(-\tau) \\ x(\theta) \end{bmatrix} \geq \epsilon \|x\|_2^2$$

2. *There exists some* $\epsilon' > 0$ *and a continuous matrix-valued function* $T : \mathbb{R} \to \mathbb{S}^{2n}$ *such that the following holds.*

$$\int_{-\tau}^{0} T(\theta) d\theta = 0$$

$$M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \geq 0 \qquad \text{for all } \theta \in [-\tau, 0]$$

See Appendix D for Proof.

Definition 74. *For a given* $d \geq 0$*,* $M \in \mathbb{S}^{3n}[\theta]$ *satisfies* $M \in G_3^d$ *if there exists a* $T \in \mathbb{S}^{2n}[\theta]$ *such that* $T \in \Omega_p^d$ *and*

$$M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{P}^+.$$

Similar to Theorem 69, Lemma 73 allows us to use $G_3^d \subset H_3^+$ to replace $M \in H_3^+$ with a semidefinite programming constraint.

Piecewise-Continuous Spacing Functions

We now consider the sets $\tilde{H}_1^+$ and $\tilde{H}_3^+$ of piecewise continuous functions which define positive multiplication operators on X_1 and $\tilde{X}_3$ respectively. The following lemma is a generalization of Theorem 69.

Lemma 75. *Suppose* $S_i : \mathbb{R} \to \mathbb{S}^{2n}$*,* $i = 1, \ldots, K$ *are continuous symmetric matrix-valued functions with domains* $[-\tau_i, -\tau_{i-1}]$ *where* $\tau_i > \tau_{i-1}$ *for* $i = 1, \ldots, K$ *and* $\tau_0 = 0$*. Then the following are equivalent.*

1. *There exists an* $\epsilon > 0$ *such that the following holds for all* $x \in \mathcal{C}_{\tau_K}$*.*

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T S_i(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta \geq \epsilon \|x\|_2^2$$

2. *There exists an* $\epsilon' > 0$ *and continuous symmetric matrix valued functions,* $T_i : \mathbb{R} \to \mathbb{S}^n$, *such that*

$$S_i(\theta) + \begin{bmatrix} T_i(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \geq 0 \qquad \text{for } \theta \in [-\tau_i, -\tau_{i-1}],\ i = 1, \ldots, K \text{ and}$$

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} T_i(\theta) = 0.$$

See Appendix D for Proof.

Lemma 75 allows us to represent the constrain $S \in \tilde{H}_1^+$ by considering a piecewise-continuous function S to be defined by $S(\theta) := S_i(\theta)$ for $\theta \in [-\tau_i, -\tau_{i-1}]$.

Definition 76. *Given* n *and* τ, *define* $\tilde{\Omega}_p^d$ *to be the set of piecewise-continuous functions* $T : \mathbb{R} \to \mathbb{S}^n$ *such that* $T(\theta) = T_i(\theta)$ *for* $T_i \in \mathbb{S}_d^n[\theta]$ *and* $\theta \in [-\tau_i, -\tau_{i-1}]$ *where* $i = 1, \ldots, K$ *and such that*

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} T_i(\theta) = 0.$$

Definition 77. *For a given* $d \geq 0$, *we say that* $M : \mathbb{R} \to \mathbb{S}^{2n}$ *satisfies* $M \in \tilde{G}_1^d$ *if there exists* $M_i \in \mathbb{S}_d^{2n}[\theta]$ *such that* $M(\theta) = M_i(\theta)$ *for* $\theta \in [-\tau_i, -\tau_{i-1}]$ *and there exists a* $T \in \tilde{\Omega}_p^d$, *defined by* $T_i \in \mathbb{S}_d^n[\theta]$ *on the interval* $[-\tau_i, -\tau_{i-1}]$, *and such that*

$$M_i(\theta) + \begin{bmatrix} T_i(\theta) & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{P}^+ \qquad i = 1, \ldots, K.$$

Similar to Theorem 69, Lemma 75 allows us to use $\tilde{G}_1^d \subset \tilde{H}_1^+$ to replace $M \in \tilde{H}_1^+$ with a semidefinite programming constraint. Now for $\tilde{H}_3^+$, we have the following lemma.

Lemma 78. *Suppose* $S_i : \mathbb{R} \to \mathbb{S}^{n(K+2)}$ *are continuous matrix valued functions with domains* $[-\tau_i, -\tau_{i-1}]$ *for* $i = 1, \ldots, K$ *where* $\tau_i > \tau_{i-1}$ *for* $i = 1, \ldots, K$ *and* $\tau_0 = 0$. *Then the following are equivalent.*

1. *There exists an* $\epsilon > 0$ *such that the following holds for all* $x \in \mathcal{C}_{\tau_K}$.

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(-\tau_0) \\ \vdots \\ x(-\tau_K) \\ x(\theta) \end{bmatrix}^T S_i(\theta) \begin{bmatrix} x(-\tau_0) \\ \vdots \\ x(-\tau_K) \\ x(\theta) \end{bmatrix} d\theta \geq \epsilon \|x\|_2^2$$

2. *There exists an* $\epsilon' > 0$ *and continuous matrix valued functions* $T_i : \mathbb{R} \to \mathbb{S}^{n(K+1)}$ *such that*

$$S_i(\theta) + \begin{bmatrix} T_i(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \geq 0 \qquad \text{for } \theta \in [-\tau_i, -\tau_{i-1}],\ i = 1, \dots, K \text{ and}$$

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} T_i(\theta) = 0.$$

See Appendix D for Proof.

Definition 79. *For a given* $d \geq 0$*, we say that* $M : \mathbb{R} \to \mathbb{S}^{n(K+2)}$ *satisfies* $M \in \tilde{G}_3^d$ *if there exist* $M_i \in \mathbb{S}_d^{n(K+2)}[\theta]$ *such that* $M(\theta) = M_i(\theta)$ *on* $[-\tau_i, -\tau_{i-1}]$ *and* $T_i \in \mathbb{S}_d^{n(K+1)}[\theta]$ *such that* $T(\theta) = T_i(\theta)$ *on* $[-\tau_i, -\tau_{i-1}]$ *implies* $T \in \tilde{\Omega}_p^d$ *and such that*

$$M_i(\theta) + \begin{bmatrix} T_i(\theta) & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{P}^+ \qquad i = 1, \dots, K.$$

Similar to Theorem 69, Lemma 78 allows us to use $\tilde{G}_3^d \subset \tilde{H}_3^+$ to replace $M \in \tilde{H}_3^+$ with a semidefinite programming constraint.

Parameter-Dependent Spacing Functions

In this subsection, we consider the case which arises when the dynamics of the system depend on some uncertain time-invariant vector of parameters.

Lemma 80. *For a given $d \in \mathbb{Z}^+$ and $S \in \mathbb{S}_d^{2n}[\theta, y]$, suppose there exists some $T \in \mathbb{S}_d^n[\theta, y]$ such that the following holds.*

$$\int_{-\tau}^{0} T(\theta, y)d\theta = 0$$

$$S(\theta, y) + \begin{bmatrix} T(\theta, y) & 0 \\ 0 & 0 \end{bmatrix} \in \bar{\Sigma}_s$$

Then $S(\cdot, y) \in H_1^+$ for all $y \in \mathbb{R}^p$.

Now suppose we wish to impose the condition that $S(\cdot, y) \in H_1^+$ for all $y \in \{y : p_i(y) \geq 0, i = 1, \ldots, N\}$ for scalar polynomials p_i.

Lemma 81. *Suppose there exist functions S_i such that $S_i(\cdot, y) \in H_1^+$ for $i = 0, \ldots, N$ for all y and such that the following holds.*

$$S(\theta, y) = S_0(\theta, y) + \sum_{i=1}^{N} p_i(y) S_i(\theta, y),$$

Then $S(\cdot, y) \in H_1^+$ for all $y \in \{y : p_i(y) \geq 0, i = 1, \ldots, N\}$

The purpose of these lemmas is to show how our representation of H_1^+ can be used to prove stability in the presence of parametric uncertainty. Clearly, these results are motivated by the construction $\bar{\Upsilon}_d^K$ presented in Chapter 6. The generalization of Lemmas 80 and 81 to H_3^+, $\tilde{H}_1^+$ and $\tilde{H}_3^+$ is similar and therefore not explicitly discussed.

7.2.2 Positive Finite-Rank Integral Operators

We now consider the set H_2^+ of continuous matrix-valued kernel functions which define positive integral operators on $\mathcal{C}_\tau$. These operators are compact and satisfy the following inequality for all $x \in \mathcal{C}_\tau$.

$$\int_{-\tau}^{0} \int_{-\tau}^{0} x(\theta)^T k(\theta, \omega) x(\omega) d\theta d\omega \geq 0$$

Because $\mathcal{C}_\tau$ is an infinite dimensional space, we cannot fully parameterize the operators which are positive on this space using a finite-dimensional set of positive semidefinite matrices. Instead, we will consider the subset of finite-rank operators.

Theorem 82. *Let A be a compact Hermitian operator which is positive on $Y_m \subset \mathcal{C}_\tau$, where*

$$\begin{aligned} Y_d := \{p \in \mathcal{C}_\tau : p \text{ is an } n\text{-dimensional vector of} \\ \text{univariate polynomials of degree } d \text{ or less.}\} \end{aligned}$$

Then there exists a positive semidefinite matrix $Q \in \mathbb{S}^{n\times(d+1)}$ such that

$$\begin{aligned} \langle x, Ax\rangle &= \int_{-\tau}^{0}\int_{-\tau}^{0} x(\theta)^T k(\theta,\omega)x(\omega)d\theta d\omega \qquad \forall\, x \in Y_m \\ k(\theta,\omega) &:= \begin{bmatrix} Z_d[\theta] & & \\ & \ddots & \\ & & Z_d[\theta] \end{bmatrix}^T Q \begin{bmatrix} Z_d[\omega] & & \\ & \ddots & \\ & & Z_d[\omega] \end{bmatrix} \\ &= \bar{Z}_d^n(\theta)^T Q \bar{Z}_d^n(\omega), \end{aligned}$$

where recall $Z_d[\theta]$ is the $(d+1)$-dimensional vector of powers of θ of degree d or less.

Before proving the theorem, we quote the following lemma which follows directly from the Spectral Theorem [50].

Lemma 83. *Let $\{e_i\}_{i=1}^p$ be a basis for some finite dimensional subspace Y of an inner product space Z. Let A be some compact Hermitian operator which is positive on Y. Then there exists some $K \geq 0$ such that the following holds for all $x \in Y$.*

$$\langle x, Ax\rangle = \sum_{i,j=1}^{p} K_{ij}\langle e_i, x\rangle\langle e_j, x\rangle$$

Proof. Define e_i to be the transpose of the i^{th} row of $\bar{Z}_d^n$, then $\{e_i\}_{i=1}^{n(d+1)}$ forms a basis for the finite dimensional subspace $Y_m \subset \mathcal{C}_\tau$. By Lemma 83, there exists some

$Q \geq 0$ such that the following holds where $p = n(d+1)$.

$$\begin{aligned}
\langle x, Ax \rangle &= \sum_{i,j=1}^{p} Q_{ij} \langle e_i, x \rangle \langle e_j, x \rangle \\
&= \int_{-\tau}^{0} \int_{-\tau}^{0} \sum_{i,j=1}^{p} Q_{ij} e_i(\theta)^T x(\theta) e_j(\omega)^T x(\omega) d\theta d\omega \\
&= \int_{-\tau}^{0} \int_{-\tau}^{0} x(\theta)^T \sum_{i,j=1}^{p} \left(e_i(\theta) Q_{ij} e_j(\omega)^T \right) x(\omega) d\theta d\omega \\
&= \int_{-\tau}^{0} \int_{-\tau}^{0} x(\theta)^T \bar{Z}_d^n[\theta]^T Q \bar{Z}_d^n[\omega] x(\omega) d\theta d\omega
\end{aligned}$$

■

Definition 84. *For a given integer $d \geq 0$, we denote the set of compact Hermitian operators which are positive on Y_d by G_2^{2d}*

This theorem shows how we can use semidefinite programming to represent the subset G_2^{2d} which consists of finite rank elements of the convex cone H_2^+ with polynomial eigenvectors of bounded degree. Naturally, as we increase the degree bound, d, the rank of the operators in G_2^{2d} will increase, as will the computational complexity of the problem.

Piecewise-Continuous Positive Finite-Rank Operators

In this subsection, we consider the set $\tilde{H}_2^+$ of matrix-valued kernel functions which are discontinuous only at points $\theta, \omega = \{-\tau_i\}_{i=1}^{K-1}$ and which define positive integral operators on $\mathcal{C}_\tau$. Although the results of the previous section can be applied directly, the kernel functions defined in such a manner will necessarily be continuous since they are constructed using a finite number of continuous functions. In order to allow for the construction of operators defined by piecewise-continuous matrix valued functions, we use the following lemma.

Lemma 85. *Let M be a matrix valued function $M : \mathbb{R}^2 \to \mathbb{S}^n$ which is discontinuous only at points $\theta, \omega = -\tau_i$ for $i = 1, \ldots, K-1$ where the τ_i are increasing and $\tau_0 = 0$.*

Then $M \in \tilde{H}_2^+$ if and only if there exists some continuous matrix valued function $R : \mathbb{R}^2 \to \mathbb{R}^{nK\times nK}$ such that $R \in H_2^+$ and the following holds where $I_i = [-\tau_i, -\tau_{i-1}]$, $\Delta_i = \tau_i - \tau_{i-1}$.

$$\begin{aligned} M(\theta,\omega) &= M_{ij}(\theta,\omega) \qquad \text{for all} \;\; \theta \in I_i, \quad \omega \in I_j \\ M_{ij}(\theta,\omega) &= R_{ij}\left(\frac{\tau_K}{\Delta_i}\theta + \tau_{i-1}\frac{\tau_K}{\Delta_i}, \frac{\tau_K}{\Delta_j}\omega + \tau_{j-1}\frac{\tau_K}{\Delta_j}\right) \\ R(\theta,\omega) &= \begin{bmatrix} R_{11}(\theta,\omega) & \dots & R_{1K}(\theta,\omega) \\ \vdots & & \vdots \\ R_{K1}(\theta,\omega) & \dots & R_{KK}(\theta,\omega) \end{bmatrix} \end{aligned}$$

See Appendix D for Proof.

Definition 86. *For an integer $d \geq 0$, we denote by $\tilde{G}_2^d$ the set of piecewise continuous matrix valued functions $M : \mathbb{R}^2 \to \mathbb{S}^n$ such that there exist some $R : \mathbb{R}^2 \to \mathbb{R}^{nK\times nK}$ such that $R \in G_2^d$ and the following holds where I_i, Δ_i are as defined above.*

$$\begin{aligned} M(\theta,\omega) &= M_{ij}(\theta,\omega) \qquad \text{for all} \;\; \theta \in I_i, \quad \omega \in I_j \\ M_{ij}(\theta,\omega) &= R_{ij}\left(\frac{\tau_K}{\Delta_i}\theta + \tau_{i-1}\frac{\tau_K}{\Delta_i}, \frac{\tau_K}{\Delta_j}\omega + \tau_{j-1}\frac{\tau_K}{\Delta_j}\right) \\ R(\theta,\omega) &= \begin{bmatrix} R_{11}(\theta,\omega) & \dots & R_{1K}(\theta,\omega) \\ \vdots & & \vdots \\ R_{K1}(\theta,\omega) & \dots & R_{KK}(\theta,\omega) \end{bmatrix} \end{aligned}$$

Lemma 85 allows us to use elements of the set G_2^d to represent the subset of $\tilde{H}_2$ of finite-rank operators with piecewise-continuous eigenvectors. Since the transformation from G_2^d to $\tilde{G}_2^d$ is affine, this allows us to replace $R \in \tilde{H}_3$ with a semidefinite programming constraint.

7.2.3 Parameter Dependent Positive Finite-Rank Operators

We now briefly discuss the extension of the results of the previous two subsections when the dynamics contain parametric uncertainty.

Lemma 87. *For a given $d \in \mathbb{Z}^+$, suppose there exists a $P \in \bar{\Sigma}_s^d$ such that*

$$M(\theta, \omega, \alpha) = \bar{Z}_n^d[\theta]^T P(\alpha) \bar{Z}_n^d[\omega]$$

Then $M(\cdot, \cdot, \alpha) \in H_2^+$ for all $\alpha \in \mathbb{R}^n$

Define $K_p := \{\alpha : p_i(\alpha) \geq 0, i = 1, \ldots, l\}$.

Lemma 88. *For a given $d \in \mathbb{Z}^+$, suppose there exist $P \in \bar{\Upsilon}_{n(d+1),d}^{K_p}$ such that*

$$M(\theta, \omega, \alpha) = \bar{Z}_n^d[\theta]^T P(\alpha) \bar{Z}_n^d[\omega].$$

Then $M(\cdot, \cdot, \alpha) \in H_2^+$ for all $\alpha \in K_p$.

These lemmas allow us to search for parameter dependent positive integral operators using the set of positive semidefinite matrix functions. The extension of these lemmas to the set $\tilde{H}_2^+$ should be clear and is not explicitly discussed.

7.3 Results

We now turn our attention to the following three classes of time-delay systems which we will explicitly consider.

1. $\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau)$
2. $\dot{x}(t) = A_0 x(t) + \sum_{i=1}^n A_i x(t - \tau_i)$
3. $\dot{x}(t) = A_0 x(t) + \int_{-\tau}^0 A(\theta) x(t + \theta) d\theta$

Here $x(t) \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n \times n}$ and $A \in \mathbb{R}^{n \times n}[\theta]$ is a matrix of polynomials. These classes contain constant, finite aftereffect. We refer to system 1) as the case of single delay, system 2) as the case of multiple delays and system 3) as the case of distributed delay. In this section, we show that in each of these cases, the derivative of the complete quadratic functional along trajectories of the system can be represented by a quadratic functional defined by the integral and multiplier operators parameterized in the previous section. Moreover, the transformation from the coefficients of the polynomials defining the complete quadratic functional to those defining its derivative is affine.

7.3.1 The Single Delay Case

In this section, we consider the first subclass of time-delay systems which are given in the following form for matrices $A, B \in \mathbb{R}^{n\times n}$

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) \tag{7.1}$$

The following theorem gives a condition for stability of the system.

Theorem 89. *For integer $d \geq 0$, the solution map, G, defined by Equation* (7.1) *is asymptotically stable if there exists a constant $\epsilon > 0$, matrix $P \in \mathbb{S}^n$, and continuous matrix-valued functions $S : \mathbb{R} \to \mathbb{S}^n$, $Q : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $R : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ where $R(\theta,\eta) = R(\eta,\theta)^T$ and such that*

$$\begin{aligned}
&\begin{bmatrix} P-\epsilon I & \tau Q \\ \tau Q^T & \tau S \end{bmatrix} \in G_1^d & \qquad & R \in G_2^d \\
&-D_1 \in G_3^d & & M \in G_2^d
\end{aligned}$$

$$\begin{aligned}
D_1(\theta) &= \begin{bmatrix} D_{11} & PB - Q(-\tau) & \tau(A^TQ(\theta) - \dot{Q}(\theta) + R(0,\theta)) \\ *^T & -S(-\tau) & \tau(B^TQ(\theta) - R(-\tau,\theta)) \\ *^T & *^T & -\tau\dot{S}(\theta) \end{bmatrix} \\
M(\theta,\omega) &= \frac{d}{d\theta}R(\theta,\omega) + \frac{d}{d\omega}R(\theta,\omega) \\
D_{11} &= PA + A^TP + Q(0) + Q(0)^T + S(0) + \epsilon I
\end{aligned}$$

Proof. We use the following complete quadratic functional.

$$
\begin{aligned}
V(\phi) &= \phi(0)^T P\phi(0) + 2\phi(0)^T \int_{-\tau}^{0} Q(\theta)\phi(\theta)d\theta \\
&+ \int_{-\tau}^{0} \phi(\theta)^T S(\theta)\phi(\theta)d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(\theta)^T R(\theta,\omega)\phi(\omega)d\theta d\omega \\
&= \frac{1}{\tau}\int_{-\tau}^{0} \begin{bmatrix}\phi(0)\\ \phi(\theta)\end{bmatrix}^T \begin{bmatrix} P-\epsilon I & \tau Q(\theta)\\ \tau Q(\theta)^T & \tau S(\theta)\end{bmatrix} \begin{bmatrix}\phi(0)\\ \phi(\theta)\end{bmatrix} d\theta \\
&+ \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(\theta)^T R(\theta,\omega)\phi(\omega)d\theta d\omega + \epsilon\|\phi(0)\|^2 \geq \epsilon\|\phi(0)\|^2
\end{aligned}
$$

All that is required for asymptotic stability is strict negativity of the the derivative. The Lie derivative of this functional along trajectories of the system is given by the following.

$$
\begin{aligned}
&\dot{V}(\Gamma(\phi,0)) = \phi(0)^T P(A\phi(0) + B\phi(-\tau)) + (\phi(0)^T A^T + \phi(-\tau)^T B^T)P\phi(0) \\
&+ 2\int_{-\tau}^{0} \phi(\theta)^T Q(\theta)^T (A\phi(0) + B\phi(-\tau))d\theta \\
&+ 2\phi(0)^T \left(Q(0)\phi(0) - Q(-\tau)\phi(-\tau) - \int_{-\tau}^{0} \dot{Q}(\theta)\phi(\theta)d\theta \right) \\
&+ \phi(0)^T S(0)\phi(0) - \phi(-\tau)^T S(-\tau)\phi(-\tau) - \int_{-\tau}^{0} \phi(\theta)^T \dot{S}(\theta)\phi(\theta)d\theta \\
&+ \int_{-\tau}^{0} \left(\phi(0)^T R(0,\theta)\phi(\theta) - \phi(-\tau)^T R(-\tau,\theta)\phi(\theta)\right) d\theta \\
&+ \int_{-\tau}^{0} \left(\phi(\theta)^T R(\theta,0)\phi(0) - \phi(\theta)^T R(\theta,-\tau)\phi(-\tau)\right) d\theta \\
&- \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(\theta) \left(\frac{d}{d\theta}R(\theta,\omega) + \frac{d}{d\omega}R(\theta,\omega) \right) \phi(\omega)d\theta d\omega \\
&= \frac{1}{\tau}\int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(\theta) \end{bmatrix}^T D_1(\theta) \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(\theta) \end{bmatrix} - \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(\theta)M(\theta,\omega)\phi(\omega)d\theta d\omega - \epsilon\|\phi(0)\|^2 \\
&\le -\epsilon\|\phi(0)\|^2
\end{aligned}
$$

Therefore, if the conditions of the theorem hold, then the functional is strictly decreasing along trajectories of the system, proving asymptotic stability. ∎

7.3.2 Examples of a Single Delay

Example 1: In this example, we compare our results with the discretized Lyapunov functional approach used by Gu et al. in [13] in the case of a system with a single delay. Although numerous other papers have also given sufficient conditions for stability of time-delay systems, e.g. [12, 27, 24], we use the approach introduced by Gu since it has demonstrated a particularly high level of precision. When we are comparing with the piecewise linear approach here and throughout this chapter, we will only consider examples which have been presented in the work [13] and we will compare our results with the numbers that are cited therein. We use SOSTools [32] and SeDuMi [45] for solution of all semidefinite programming problems. Now consider the following system of delay differential equations.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

The problem is to estimate the range of τ for which the differential equation remains stable. Using the method presented in this paper and by sweeping τ in increments of .1, we estimate the range of stability to be an interval. We then use a bisection method to find the minimum and maximum stable delay. Our results are also summarized in Table 7.1 and are compared to the analytical limit and the results obtained in [13].

Our results			Piecewise Functional		
d	$\tau_{\min}$	$\tau_{\max}$	N_2	$\tau_{\min}$	$\tau_{\max}$
1	.10017	1.6249	1	.1006	1.4272
2	.10017	1.7172	2	.1003	1.6921
3	.10017	1.71785	3	.1003	1.7161
Analytic	.10017	1.71785			

Table 7.1: τ_{max} and τ_{min} for discretization level N_2 using the piecewise-linear Lyapunov functional and for degree d using our approach and compared to the analytical limit

Clearly, the results for this test case illustrate a high rate of convergence to the analytical limit. However, although the results presented here give reasonable estimates for the interval of stability, they do not prove stability over any interval, but rather only at the specific values of τ for which the algorithm was tested. To provide a more rigorous analysis, we now include τ as an uncertain parameter and search for parameter dependent Lyapunov functionals which prove stability over an interval. The results from this test are given in Table 7.3.

Example 2: In this example, we illustrate the flexibility of our algorithm through a simplistic control design and analysis problem. Suppose we wish to control a simple inertial mass remotely using a PD controller. Now suppose that the derivative control

d in τ	d in θ	τ_{min}	τ_{max}
1	1	.1002	1.6246
1	2	.1002	1.717
Analytic		.10017	1.71785

Table 7.2: Stability on the interval $[\tau_{min}, \tau_{max}]$ vs. degree using a parameter-dependent functional

is half of the proportional control. Then we have the following dynamical system.

$$\ddot{x}(t) = -ax(t) - \frac{a}{2}\dot{x}(t)$$

It is easy to show that this system is stable for all positive values of a. However, because we are controlling the mass remotely, some delay may be introduced due to, for example, the fixed speed of light. We assume that this delay is known and changes sufficiently slowly so that for the purposes of analysis, it may be taken to be fixed. Now we have the following delay-differential equation with uncertain, time-invariant parameters a and τ.

$$\ddot{x}(t) = -ax(t-\tau) - \frac{a}{2}\dot{x}(t-\tau)$$

Whereas before the system was stable for all positive values of a, now, for any fixed value of a, there exists a τ for which the system will be unstable. In order to determine which values of a are stable for any fixed value of τ, we divide the parameter space into regions of the form $a \in [a_{\min}, a_{\max}]$ and $\tau \in [\tau_{\min}, \tau_{\max}]$. This type of region is compact and can be represented as a semi-algebraic set using the polynomials $p_1(a) = (a - a_{\min})(a - a_{\max})$ and $p_2(\tau) = (\tau - \tau_{\min})(\tau - \tau_{\max})$. By using these polynomials, we are able to construct parameter dependent Lyapunov functionals which prove stability over a number of parameter regions. These regions are illustrated in Figure 7.1.

7.3.3 Multiple Delay Case

We now consider the case of multiple delays. The system is now defined by the following functional differential equation where $A_i \in \mathbb{R}^{n\times n}$, $\tau_i > \tau_{i-1}$ for $i = 1, \ldots, K$,

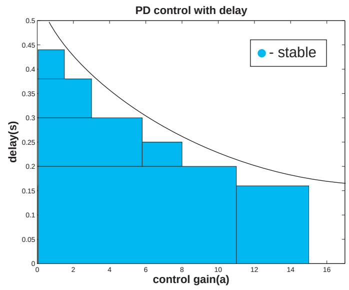

Figure 7.1: Regions of Stability for Example 2

and $\tau_0 = 0$.

$$\dot{x}(t) = \sum_{i=0}^{K} A_i x(t - \tau_i) \tag{7.2}$$

The difference between the analysis of the case of a single delay and that of multiple delays is that the matrix-valued functions defining the complete quadratic functional necessary for stability may now contain discontinuities at discrete points given by the values of the delay. In this case, we express the the complete quadratic functional in the following form where $Q_i : \mathbb{R} \to \mathbb{R}^{n\times n}$, $S_i : \mathbb{R} \to \mathbb{S}^n$ and $R_{ij} : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ for $i, j = 1, \ldots, K$ are continuous matrix-valued functions, $P \in \mathbb{S}^n$ and $R_{ij}(\theta, \omega) = R_{ji}(\omega, \theta)^T$.

$$V(\phi) = \phi(0)^T P\phi(0) + 2\sum_{j=1}^{K} \phi(0)^T \int_{-\tau_i}^{-\tau_{i-1}} Q_i(\theta)\phi(\theta)d\theta$$
$$+ \sum_{j=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \phi(\theta)^T S_i(\theta)\phi(\theta)d\theta + \sum_{i=1}^{K}\sum_{j=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \int_{-\tau_j}^{-\tau_{j-1}} \phi(\theta)^T R_{ij}(\theta, \omega)\phi(\omega)d\theta d\omega$$

Theorem 90. *For integer $d \geq 0$, suppose $P \in \mathbb{S}^n$, $\eta > 0$, $Q_i : \mathbb{R} \to \mathbb{R}^{n\times n}$, $S_i : \mathbb{R} \to \mathbb{S}^n$ and $R_{ij} : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ where $R_{ij}(\theta, \omega) = R_{ji}(\omega, \theta)^T$ are continuous for $i, j = 1, \ldots, K$. Let $R(\theta, \omega) = R_{ij}(\theta, \omega)$ for $\theta \in I_i$, $\omega \in I_j$ where $I_i = [-\tau_i, -\tau_{i-1}]$. Then the solution map defined by equation (7.2) is asymptotically stable if the following holds.*

$$M \in \tilde{G}_1^d \qquad R \in \tilde{G}_2^d$$
$$-D \in \tilde{G}_3^d \qquad L \in \tilde{G}_2^d$$

$$M(\theta) = \begin{bmatrix} P - \eta I & \tau Q_i(\theta) \\ \tau Q_i(\theta)^T & \tau S_i(\theta) \end{bmatrix} \qquad \theta \in I_i$$

$$L(\theta, \omega) = \frac{\delta}{\delta\theta} R_{ij}(\theta, \omega) + \frac{\delta}{\delta\theta} R_{ij}(\theta, \omega) \qquad \theta \in I_i, \omega \in I_j$$

$$D(\theta) = \begin{bmatrix} D11 & \tau D12_i(\theta) \\ \tau D12_i(\theta)^T & \tau D22_i(\theta) \end{bmatrix} \qquad \textit{for } \theta \in I_i$$

$$D11_{11} = PA_0 + A_0^T P + Q_1(0) + Q_1(0)^T + S_1(0) - \eta I$$

$$D11_{ij} =$$

$$\begin{cases} PA_{i-1} - Q_{i-1}(-\tau_{i-1}) + Q_i(-\tau_{i-1}) & i, j = 1, 2 \dots K \\ S_i(-\tau_{i-1}) - S_{i-1}(-\tau_{i-1}) & i = j = 2 \dots K \\ PA_K - Q_K(-\tau_K) & i, j = 1, K+1 \\ -S_K(-\tau_K) & i = j = K+1 \\ 0 & \textit{otherwise} \end{cases}$$

$$D12_j(\theta) =$$

$$\begin{bmatrix} R_{1j}(0, \theta) + A_0^T Q_j(\theta) - \dot{Q}_j(\theta) \\ R_{2j}(-\tau_1, \theta) - R_{1j}(-\tau_1, \theta) + A_1^T Q_j(\theta) \\ \vdots \\ R_{(K)j}(-\tau_{K-1}, \theta) - R_{(K-1)j}(-\tau_{K-1}, \theta) + A_1^T Q_j(\theta) \\ A_K^T Q_j(\theta) - R_{Kj}(-\tau_K, \theta) \end{bmatrix}$$

$$D22_i(\theta) = -\dot{S}_i(\theta)$$

Proof. We use the following complete quadratic functional.

$$V(\phi) = \frac{1}{\tau} \int_{-\tau_K}^{0} \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix}$$
$$+ \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} \phi(\theta)^T R(\theta, \omega) \phi(\omega) d\theta d\omega + \eta \|\phi(0)\|^2 \geq \eta \|\phi(0)\|^2$$

The system is asymptotically stable if the derivative of the functional is strictly negative. The Lie derivative of the functional along trajectories of the systems is given by the following.

$$\dot{V}(\Gamma(\phi,0)) = \frac{1}{\tau}\int_{-\tau_K}^{0} \begin{bmatrix} \phi(0) \\ \vdots \\ \phi(-\tau_K) \\ \phi(\theta) \end{bmatrix}^T D(\theta) \begin{bmatrix} \phi(0) \\ \vdots \\ \phi(-\tau_K) \\ \phi(\theta) \end{bmatrix} d\theta$$

$$+ \int_{-\tau_K}^{0} \phi(\theta)^T L(\theta,\omega)\phi(\omega)d\theta d\omega - \eta\|\phi(0)\|^2 \le -\eta\|\phi(0)\|^2$$

Thus if the conditions of the theorem hold, then the derivative of the functional is strictly negative, which implies asymptotic stability. ∎

7.3.4 Examples of Multiple Delay

Example 3: Consider the following system of delay-differential equations.

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{9}{10} \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \left[\frac{1}{20}x(t-\frac{\tau}{2}) + \frac{19}{20}x(t-\tau) \right]$$

Again, the system is stable when τ lies on some interval. The problem is to search for the minimum and maximum value of τ for which the system remains stable. In applying the methods of this paper, we again use a bisection method to find the minimum and maximum value of τ for which the system remains stable. Our results are summarized in Table 7.3 and are compared to the analytical limit as well as piecewise-linear functional method. For the piecewise functional method, N_2 is the level of both discretization and subdiscretization.

7.3.5 Distributed Delay Case

We now consider the case of distributed delay where the dynamics are given by the following functional differential equation where $A_0 \in \mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{n\times n}[\theta]$.

$$\dot{x}(t) = A_0 x(t) + \int_{-\tau}^{0} A(\theta)x(t+\theta)d\theta \tag{7.3}$$

Our Approach			Piecewise Functional		
d	$\tau_{\min}$	$\tau_{\max}$	N_2	$\tau_{\min}$	$\tau_{\max}$
1	.20247	1.354	1	.204	1.35
2	.20247	1.3722	2	.203	1.372
Analytic	.20246	1.3723			

Table 7.3: τ_{max} and τ_{min} using the piecewise-linear Lyapunov functional of Gu et al. and our approach and compared to the analytical limit

We can again assume that the matrix-valued functions defining the complete quadratic functional are continuous. This leads to the following theorem.

Theorem 91. *For integer $d \geq 0$, the solution map, G, defined by equation (7.3) is asymptotically stable if there exists a constant $\eta > 0$, a matrix $P \in \mathbb{S}^n$ and matrix functions $Q : \mathbb{R} \to \mathbb{R}^{n\times n}$, $S : \mathbb{R} \to \mathbb{S}^n$, $R : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ where $R(\theta,\omega) = R(\omega,\theta)^T$ and such that the following holds*

$$\begin{bmatrix} P-\eta I & \tau Q \\ \tau Q^T & \tau S \end{bmatrix} \in G_1^d \qquad -D \in G_3^d$$

$$R \in G_2^d \qquad M \in G_2^d$$

where

$$D(\theta) = \begin{bmatrix} D_{11}+\eta I & \tau D_{12}(\theta) \\ \tau D_{12}(\theta)^T & -\tau \dot{S}(\theta) \end{bmatrix}$$

$$D_{11} = \begin{bmatrix} PA_0 + A_0^T P + Q(0) + Q(0)^T + S(0) & -Q(-\tau) \\ -Q(-\tau)^T & -S(-\tau) \end{bmatrix}$$

$$D_{12}(\theta) = \begin{bmatrix} A_0^T Q(\theta) + PA(\theta) - \frac{d}{d\theta}Q(\theta) + R(0,\theta) \\ -R(-\tau,\theta) \end{bmatrix}$$

$$M(\theta,\omega) = \frac{d}{d\theta}R(\theta,\omega) + \frac{d}{d\omega}R(\theta,\omega) - A(\theta)^T Q(\omega) - Q(\theta)^T A(\omega)$$

Proof. We consider the complete quadratic functional

$$V(\phi) = \frac{1}{\tau}\int_{-\tau}^{0} \begin{bmatrix}\phi(0)\\ \phi(\theta)\end{bmatrix}^T \begin{bmatrix}P-\eta I & \tau Q(\theta)\\ \tau Q(\theta)^T & \tau S(\theta)\end{bmatrix} \begin{bmatrix}\phi(0)\\ \phi(\theta)\end{bmatrix}^T d\theta$$
$$+\int_{-\tau}^{0}\int_{-\tau}^{0} \phi(\theta)^T R(\theta,\omega)\phi(\omega)d\theta d\omega + \eta\|\phi(0)\|^2 \geq \eta\|\phi(0)\|^2$$

The system is asymptotically stable if the Lie derivative is strictly negative. The Lie derivative of the functional along trajectories of the system is given by

$$\dot{V}(\Gamma(\phi,0)) = \int_{-\tau}^{0} \begin{bmatrix}\phi(0)\\ \phi(-\tau)\\ \phi(\theta)\end{bmatrix}^T D(\theta) \begin{bmatrix}\phi(0)\\ \phi(-\tau)\\ \phi(\theta)\end{bmatrix} d\theta$$
$$-\int_{-\tau}^{0}\int_{-\tau}^{0} \phi(\theta)^T M(\theta,\omega)\phi(\omega)d\omega d\theta - \eta\|\phi(0)\|^2 \leq -\eta\|\phi(0)\|^2$$

Thus, if the conditions of the theorem hold, then the system is asymptotically stable. ■

7.4 Conclusion

In this chapter, we have shown how to compute solutions to an operator-theoretic version of the Lyapunov inequality. Our approach is to combine results from real algebraic geometry with functional analysis in order to parameterize certain convex cones of positive operators using the convex cone of positive semidefinite matrices. We then show that the operator inequalities can be expressed using affine constraints on these matrices. This allows us to compute solutions using semidefinite programming, for which there exist efficient numerical algorithms. We have further extended our results to the case when the dynamics contain parametric uncertainty using a construction based on certain results of Putinar and others. The numerical examples given in this Chapter demonstrate a quick convergence to the analytic limit of stability.

We conclude by mentioning that the methods of this chapter can be used to construct full-rank solutions of the Lyapunov inequality. Furthermore, we have developed methods for constructing the inverses of these full-rank operators. By computing solutions to the Lyapunov inequality for the adjoint system constructed by Delfour and Mitter [8], this invertibility result seems to imply that one can construct stabilizing controllers for linear time-delay systems. This work is ongoing, however, and is therefore not detailed in this thesis.

Chapter 8

Stability of Nonlinear Time-Delay Systems

8.1 Introduction

The question of stability of nonlinear functional differential equations is complicated by the lack of a complete Lyapunov functional structure whose existence is necessary for stability of a general nonlinear time-delay system. For this reason all results given in this section will only be sufficient. Furthermore, we cannot claim, as was done in the previous chapter, that these conditions will generally approach necessity in any sense. Given these limitations, we felt that the best approach was to construct a sequence of conditions that would approach necessity in at least one special case, that of linear systems. For this reason, the conditions presented in this chapter are a generalization of those in the previous one and the Lyapunov functionals reduce to the complete quadratic functional in the case where the degree of x is restricted to 2. This chapter differs from the presentation of the linear case in that it includes a section on delay-independent stability, a topic not previously discussed.

8.2 Delay-Dependent Stability

Consider a nonlinear discrete time-delay system defined by a functional of the following form for $\tau_i < \tau_{i-1}$ for $i = 1, \ldots, K$ and $\tau_0 = 0$.

$$f(x_t) = f(x(t), x(t-\tau_1), \cdots, x(t-\tau_K))$$

In this section, we present conditions for stability based on the use of a generalization of the compete quadratic functional. The generalized Lyapunov functional has the following form.

$$\begin{aligned} V(\phi) := &\int_{-\tau_K}^{0} Z_d[\phi(0), \phi(\theta)]^T M(\theta) Z_d[\phi(0), \phi(\theta)] d\theta \\ &+ \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} Z_d[\phi(\theta)]^T R(\theta, \omega) Z_d[\phi(\omega)] d\theta d\omega \end{aligned}$$

8.2.1 Single Delay Case

In this subsection, we consider the special case of a single delay.

$$\dot{x}(t) = f(x(t), x(t-\tau))$$

Here we assume $x(t) \in \mathbb{R}^n$ and f is continuous. We first introduce some notation.

Definition 92. *We say a continuous function $f : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}$ satisfies $f \in K_1$ if there exists some $\alpha > 0$ such that the following holds for all $\phi \in \mathcal{C}_\tau$.*

$$\int_{-\tau}^{0} f(\phi(0), \phi(\theta), \theta) d\theta \geq \alpha \|\phi(0)\|^2$$

Definition 93. *We say a continuous function $f : \mathbb{R}^{3n} \times \mathbb{R}^2 \to \mathbb{R}$ satisfies $f \in K_2$ if the following holds for all $\phi \in \mathcal{C}_\tau$.*

$$\int_{-\tau}^{0} \int_{-\tau}^{0} f(\phi(\theta), \phi(\omega), \theta, \omega) d\theta d\omega \geq 0$$

Definition 94. *We say a continuous function* $f : \mathbb{R}^{3n} \times \mathbb{R} \to \mathbb{R}$ *satisfies* $f \in K_3$ *if there exists some* $\alpha > 0$ *such that the following holds for all* $\phi \in \mathcal{C}_\tau$.

$$\int_{-\tau}^{0} f(\phi(0), \phi(-\tau), \phi(\theta), \theta) d\theta \geq \alpha \|\phi(0)\|^2$$

The sets K_1, K_2, and K_3 are used to define a positive Lyapunov functional and some of its derivative forms. For a given degree bound, these sets can be parameterized by the space of positive semidefinite matrices using the following subsets.

Definition 95. *Denote* $f \in \Xi_1^d \subset K_1$ *if* f *is a polynomial of degree* d *or less such that there exists a polynomial function* $t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, *of degree* d *or less, and* $\alpha > 0$ *such that the following holds.*

$$f(x_1, x_2, \theta) - t(x_1, \theta) - \alpha\|x_1\|^2 \in \Sigma_s$$
$$\int_{-\tau}^{0} t(x_1, \theta) d\theta = 0$$

Definition 96. *Denote* $f \in \Xi_2^{2d} \subset K_2$ *if* f *is a polynomial of degree* $2d$ *or less and there exists a polynomial matrix function* $R : \mathbb{R}^2 \to \mathbb{R}^{\binom{n+d}{d} \times \binom{n+d}{d}}$ *such that* $R \in G_2^{2d}$ *(defined in Chapter 7) and the following holds.*

$$f(x_1, x_1, \theta, \omega) = Z_d(x_1)^T R(\theta, \omega) Z_d(x_2) \tag{8.1}$$

Definition 97. *Denote* $f \in \Xi_3^d \subset K_3$ *if* f *is a polynomial of degree* d *or less such that there exist polynomial function* $t : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}$, *of degree* d *or less, and* $\alpha > 0$ *such that the following holds.*

$$f(x_1, x_2, x_3, \theta) - t(x_1, x_2, \theta) - \alpha\|x_1\|^2 \in \Sigma_s$$
$$\int_{-\tau}^{0} t(x_1, x_2, \theta) d\theta = 0$$

We now state the stability theorem.

Theorem 98. *For a given integer* $d \geq 0$, *suppose there exist functions* $g \in \Xi_1^d$,

$h, \hat{h} \in \Xi_2^d$, and $-\hat{g} \in \Xi_3^d$ such that the following conditions hold.

$$\begin{aligned}
&\hat{g}(x_t(0), x_t(-\tau), x_t(\theta), \theta) \\
&\quad = g(x_t(0), x_t(0), 0) - g(x_t(0), x_t(-\tau), -\tau) \\
&\quad + \tau \nabla_{x_t(0)} g(x_t(0), x_t(\theta), \theta)^T f(x_t(0), x_t(-\tau)) - \tau \frac{\delta}{\delta\theta} g(x_t(0), x_t(\theta), \theta) \\
&\quad + \tau h(x_t(0), x_t(\theta), 0, \theta) - \tau h(x_t(-\tau), x_t(\theta), -\tau, \theta) \\
&\quad + \tau h(x_t(\theta), x_t(0), \theta, 0) - \tau h(x_t(\theta), x_t(-\tau), \theta, -\tau) \\
&\hat{h}(x_t(\theta), x_t(\omega), \theta, \omega) = \frac{\delta}{\delta\theta} h(x_t(\theta), x_t(\omega), \theta, \omega) + \frac{\delta}{\delta\omega} h(x_t(\theta), x_t(\omega), \theta, \omega)
\end{aligned}$$

Then the time-delay system defined by f is globally asymptotically stable.

Proof. Consider the Lyapunov functional defined as follows

$$V(x_t) := \int_{-\tau}^{0} g(x_t(0), x_t(\theta), \theta) d\theta + \int_{-\tau}^{0} \int_{-\tau}^{0} h(x_t(\theta), x_t(\omega), \theta, \omega) d\theta d\omega \geq \alpha \|x_t(0)\|^2$$

The derivative of this functional along trajectories of the systems can be expressed as follows.

$$\begin{aligned}
\dot{V}(x_t) :=& g(x_t(0), x_t(0), 0) - g(x_t(0), x_t(-\tau), -\tau) \\
&+ \int_{-\tau}^{0} \Big(\nabla_{x_t(0)} g(x_t(0), x_t(\theta), \theta)^T p(x_t(0), x_t(-\tau)) - \frac{\delta}{\delta\theta} g(x_t(0), x_t(\theta), \theta) \Big) d\theta \\
&+ \int_{-\tau}^{0} \Big(h(x_t(0), x_t(\theta), 0, \theta) - h(x_t(-\tau), x_t(\theta), -\tau, \theta) \Big) d\theta \\
&+ \int_{-\tau}^{0} \Big(h(x_t(\theta), x_t(0), \theta, 0) - h(x_t(\theta), x_t(-\tau), \theta, -\tau) \Big) d\theta \\
&- \int_{-\tau}^{0} \int_{-\tau}^{0} \Big(\frac{\delta}{\delta\theta} h(x_t(\theta), x_t(\omega), \theta, \omega) + \frac{\delta}{\delta\omega} h(x_t(\theta), x_t(\omega), \theta, \omega) \Big) d\theta d\omega \\
=& \frac{1}{\tau} \int_{-\tau}^{0} \hat{g}(x_t(0), x_t(-\tau), x_t(\theta), \theta) d\theta + \int_{-\tau}^{0} \int_{-\tau}^{0} \hat{h}(x_t(\theta), x_t(\omega), \theta, \omega) d\theta d\omega \\
\leq& -\alpha \|x_t(0)\|^2
\end{aligned}$$

Thus, we have that if the conditions of the theorem are satisfied, then the derivative of the Lyapunov functional is negative definite, proving global asymptotic stability of the functional differential equation defined by f. ∎

As with the linear case, the conditions contained in Theorem 98 can be expressed as a semidefinite program.

8.2.2 Multiple Delays

In this section, we consider stability of nonlinear time-delay systems defined by functionals of the following form for $\tau_i < \tau_{i-1}$ for $i = 1, \ldots, K$ and $\tau_0 = 0$.

$$\dot{x}(t) = f(x(t), x(t-\tau_1), \cdots, x(t-\tau_K))$$

As before, $x(t) \in \mathbb{R}^n$, f is continuous, and we introduce some notation.

Definition 99. *We say a function* $f : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}$ *satisfies* $f \in \tilde{K}_1$ *if* $f(x_1, x_2, \theta)$ *is continuous except possibly at points* $\theta \in \{-\tau_i\}_{i=1}^{K-1}$ *and there exists some* $\alpha > 0$ *such that the following holds for all* $\phi \in \mathcal{C}_\tau$.

$$\int_{-\tau_K}^{0} f(\phi(0), \phi(\theta), \theta) d\theta \geq \alpha \|\phi(0)\|^2$$

Definition 100. *We say a function* $f : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$ *satisfies* $f \in \tilde{K}_2$ *if* $f(x_1, x_2, \theta, \omega)$ *is continuous except possibly at points* $\theta, \omega \in \{-\tau_i\}_{i=1}^{K-1}$ *and the following holds for all* $\phi \in \mathcal{C}_\tau$.

$$\int_{-\tau_K}^{0} \int_{-\tau_K}^{0} f(\phi(\theta), \phi(\omega), \theta, \omega) d\theta d\omega \geq 0$$

Definition 101. *We say a function* $f : \mathbb{R}^{n(K+2)} \times \mathbb{R} \to \mathbb{R}$ *satisfies* $f \in \tilde{K}_3$ *if* f *is continuous except possibly at points* $\theta \in \{-\tau_i\}_{i=1}^{K}$ *and there exists some* $\alpha > 0$ *such that the following holds for all* $\phi \in \mathcal{C}_\tau$.

$$\int_{-\tau_K}^{0} f(\phi(0), \phi(-\tau_1), \cdots, \phi(-\tau_K), \phi(\theta), \theta) d\theta \geq \alpha \|\phi(0)\|^2$$

The sets $\tilde{K}_1$, $\tilde{K}_2$, and $\tilde{K}_3$ are used to define positive Lyapunov functionals and some of its derivative forms in the case of multiple delays. For a given degree bound, these sets can be parameterized by the space of positive semidefinite matrices using the following subsets.

Definition 102. *Denote* $f \in \tilde{\Xi}_1^d \subset \tilde{K}_1$ *if there exist polynomial functions* $f_i : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}$, $t_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $i = 1 \cdots K$ *of degree* d *or less and* $\alpha > 0$ *such that the following holds for all* $x_i \in \mathbb{R}^n$.

$$\begin{aligned} & f_i(x_1, x_2, \theta) - t_i(x_1, \theta) - \alpha \|x_1\|^2 \in \Sigma_s \\ & f(x_1, x_2, \theta) = f_i(x_1, x_2, \theta) \quad \forall \theta \in [-\tau_i, -\tau_{i-1}] \\ & \sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} t_i(x_1, \theta) d\theta = 0 \end{aligned}$$

Definition 103. *Denote* $f \in \tilde{\Xi}_2^{2d} \subset \tilde{K}_2$ *if* f *is a polynomial of degree* $2d$ *or less such there exists a matrix function* $R : \mathbb{R}^2 \to \mathbb{R}^{\binom{n+d}{d} \times \binom{n+d}{d}}$ *such that* $R \in \tilde{G}_2^{2d}$ *and the following holds.*

$$f(x_1, x_1, \theta, \omega) = Z_d(x_1)^T R(\theta, \omega) Z_d(x_2) \tag{8.2}$$

Definition 104. *Denote* $f \in \tilde{\Xi}_3^d \subset \tilde{K}_3$ *if there exist functions* $f_i : \mathbb{R}^{n(K+2)} \times \mathbb{R} \to \mathbb{R}$, $t_i : \mathbb{R}^{n(K+2)} \times \mathbb{R} \to \mathbb{R}$, $i = 1 \cdots K$, *of degree* d *or less, and* $\alpha > 0$ *such that the following holds.*

$$\begin{aligned} & f_i(x_1, \cdots, x_{K+1}, x_{K+2}, \theta) - t_i(x_1, \cdots, x_{K+1}, \theta) - \alpha \|x_1\|^2 \in \Sigma_s \\ & f(x_1, \cdots, x_{K+1}, x_{K+2}, \theta) = f_i(x_1, \cdots, x_{K+1}, x_{K+2}, \theta) \quad \forall \theta \in [-\tau_i, -\tau_{i-1}] \\ & \sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} t_i(x_1, \cdots, x_{K+1}, \theta) d\theta = 0 \end{aligned}$$

We now state the stability theorem in the case of multiple delays.

Theorem 105. *For a given integer* $d \geq 0$, *suppose there exist functions* $h \in \tilde{\Xi}_1^d$, $g, \hat{g} \in \tilde{\Xi}_2^d$, *and* $\hat{h} \in \tilde{\Xi}_3^d$ *such that the following conditions hold where* $I_i := [-\tau_i, -\tau_{i-1}]$

and $\Delta\tau_i := \tau_i - \tau_{i-1}$.

$$\begin{aligned}
&\hat{g}_i(x_t(0), \cdots, x_t(-\tau_K), x_t(\theta), \theta) \\
&\quad = g_i(x_t(0), x_t(-\tau_{i-1}), -\tau_{i-1}) - g_i(x_t(0), x_t(-\tau_i), -\tau_i) \\
&\quad + \Delta\tau_i \nabla_{x_t(0)} g_i(x_t(0), x_t(\theta), \theta)^T f(x_t(0), \cdots, x_t(-\tau_K)) - \Delta\tau_i \frac{\delta}{\delta\theta} g_i(x_t(0), x_t(\theta), \theta) \\
&\quad + \sum_{j=1}^{K} \Big(h_{ij}(x_t(-\tau_{i-1}, x_t(\theta), -\tau_{i-1}, \theta) - h_{ij}(x_t(-\tau_i), x_t(\theta), -\tau_i, \theta) \Big) \\
&\quad + \sum_{j=1}^{K} \Big(h_{ji}(x_t(\theta), x_t(-\tau_{j-1}), \theta, -\tau_{j-1}) - h_{ji}(x_t(\theta), x_t(-\tau_j), \theta, -\tau_j) \Big) \\
&\hat{h}_{ij}(x_t(\theta), x_t(\omega), \theta, \omega) = \frac{\delta}{\delta\theta} h_{ij}(x_t(\theta), x_t(\omega), \theta, \omega) + \frac{\delta}{\delta\omega} h_{ij}(x_t(\theta), x_t(\omega), \theta, \omega)
\end{aligned}$$

Where

$$\begin{aligned}
&\Delta\tau_i g_i(x_t(0), x_t(\theta), \theta) := g(x_t(0), x_t(\theta), \theta) \quad \forall\theta \in I_i \\
&\hat{g}_i(x_t(0), \cdots, x_t(-\tau_K), x_t(\theta), \theta) := \hat{g}(x_t(0), \cdots, x_t(-\tau_K), x_t(\theta), \theta) \quad \forall\theta \in I_i \\
&h_{ij}(x_t(\theta), x_t(\omega), \theta, \omega) := h(x_t(\theta), x_t(\omega), \theta, \omega) \quad \forall\theta, \omega \in I_i \\
&\hat{h}_{ij}(x_t(\theta), x_t(\omega), \theta, \omega) := \hat{h}(x_t(\theta), x_t(\omega), \theta, \omega) \quad \forall\theta, \omega \in I_i
\end{aligned}$$

Then the time-delay system defined by f is globally asymptotically stable.

Proof. Consider the Lyapunov functional defined as follows.

$$\begin{aligned}
&V(x_t) := \int_{-\tau_K}^{0} g(x_t(0), x_t(\theta), \theta) d\theta + \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} h(x_t(\theta), x_t(\omega), \theta, \omega) d\theta d\omega \\
&= \sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \Delta\tau_i g_i(x_t(0), x_t(\theta), \theta) d\theta + \sum_{i,j=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \int_{-\tau_j}^{-\tau_{j-1}} h_{ij}(x_t(\theta), x_t(\omega), \theta, \omega) d\theta d\omega \\
&\geq \alpha \|x_t(0)\|^2
\end{aligned}$$

The inequality holds for some $\alpha > 0$ by definition of the set $\tilde{K}_1$. The derivative of this functional along trajectories of the system can be expressed as follows.

$$\dot{V}(x_t) := \sum_{i=1}^{K} \Delta\tau_i \Big(g_i(x_t(0), x_t(-\tau_{i-1}), -\tau_{i-1}) - g_i(x_t(0), x_t(-\tau_i), -\tau_i) \Big)$$
$$+ \sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \Delta\tau_i \Big(\nabla_{x_t(0)} g_i(x_t(0), x_t(\theta), \theta)^T f(x_t(0), \cdots, x_t(-\tau_K)) - \frac{\delta}{\delta\theta} g_i(x_t(0), x_t(\theta), \theta) \Big) d\theta$$
$$+ \sum_{i,j=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \Big(h_{ij}(x_t(-\tau_{j-1}, x_t(\omega), -\tau_{j-1}, \omega) - h_{ij}(x_t(-\tau_j), x_t(\omega), -\tau_j, \omega) \Big) d\omega$$
$$+ \sum_{i,j=1}^{K} \int_{-\tau_j}^{-\tau_{j-1}} \Big(h_{ij}(x_t(\theta), x_t(-\tau_{i-1}), \theta, -\tau_{i-1}) - h_{ij}(x_t(\theta), x_t(-\tau_i), \theta, -\tau_i) \Big) d\theta$$
$$- \sum_{i,j=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \int_{-\tau_j}^{-\tau_{j-1}} \Big(\frac{\delta}{\delta\theta} h_{ij}(x_t(\theta), x_t(\omega), \theta, \omega) + \frac{\delta}{\delta\omega} h_{ij}(x_t(\theta), x_t(\omega), \theta, \omega) \Big) d\theta d\omega$$
$$= \frac{1}{\tau_K} \sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \hat{g}_i(x_t(0), \cdots, x_t(-\tau_K), x_t(\theta), \theta) d\theta$$
$$+ \sum_{i,j=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \int_{-\tau_j}^{-\tau_{j-1}} \hat{h}_{ij}(x_t(\theta), x_t(\omega), \theta, \omega) d\theta d\omega$$
$$= \int_{-\tau}^{0} \hat{g}(x_t(0), \cdots, x_t(-\tau_K), \theta) d\theta + \int_{-\tau}^{0} \int_{-\tau}^{0} \hat{h}(x_t(\theta), x_t(\omega), \theta, \omega) d\theta d\omega \le -\alpha \|x_t(0)\|^2$$

The final inequality follows from the definition of $\tilde{K}_2$ and $\tilde{K}_3$. Thus the conditions of the theorem imply global asymptotic stability of the time-delay system defined by f.

■

Similar to the case of a single delay, the conditions of Theorem 105 can be expressed as a semidefinite program.

8.3 Delay-Independent Stability

We now briefly discuss conditions for delay-independent stability of time-delay systems. Recall that a system defined by a functional of the following form is stable independent of delay if it is stable for any finite values of $\{\tau_i\}_{i=1}^K$.

$$\hat{f}(x_t) := f(x_t(0), x_t(-\tau_1), \cdots, x_t(-\tau_K))$$

Here we assume f is continuous and $x(t) \in \mathbb{R}^n$. The theorems of the previous section can be applied directly to the problem of delay-independent stability by considering τ_i to be an uncertain parameter and searching for a parameter-dependent Lyapunov functional. However, a less computationally intensive approach is to consider specific Lyapunov functional forms where τ_i does not explicitly appear in the stability conditions. One such Lyapunov functional form is given as follows.

$$V(\phi) = p_0(\phi(0)) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} p_i(\phi(\theta))\, d\theta$$

This functional has an upper Lie derivative defined as follows.

$$\dot{V}(\phi) = \nabla p_0(\phi(0))^T f(x_t(0), x_t(-\tau_1), \cdots, x_t(-\tau_K)) + \sum_{i=1}^{K} p_i(\phi(0)) - p_2(\phi(-\tau_i))$$

Since τ_i does not appear explicitly in the derivative of the functional, we can prove stability for arbitrary τ_i using the following theorem.

Theorem 106. *For a given integer d, suppose there exist functions $\{p_i\}_{i=0}^K$ and constant $\alpha > 0$ such that $p_i(0) = 0$ for $i = 0, \ldots, K$ and the following conditions hold for all $x_i \in \mathbb{R}^n$.*

$$\begin{aligned} &p_0(x_0) - \alpha\|x_0\|^2 \in \Sigma_s^d \\ &p_i(x_i) \in \Sigma_s^d \qquad i = 1, \ldots, K \\ &-\nabla p_0(x_0)^T f(x_0, x_1, \cdots, x_K) - \sum_{i=1}^{K} \Big(p_i(x_0) - p_2(x_i)\Big) - \alpha\|x_0\|^2 \in \Sigma_s \end{aligned}$$

Then the system defined by f is globally asymptotically stable for any finite values of τ_i.

Proof. Consider the following Lyapunov functional.

$$V(x_t) = p_0(x_t(0)) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} p_i(x_t(\theta))\, d\theta \geq \alpha \|x_t(0)\|^2$$

The Lyapunov functional is positive definite by the conditions of the theorem. The derivative of the functional along trajectories of the system is given as follows.

$$\dot{V}(\phi) = \nabla p_0(\phi(0))^T f(x_t(0), x_t(-\tau_1), \cdots, x_t(-\tau_K)) + \sum_{i=1}^{K} p_i(\phi(0)) - p_2(\phi(-\tau_i)) \leq -\alpha \|x_t(0)\|^2$$

Therefore the derivative is negative definite by the conditions of the theorem. Therefore we have that the system defined by f is globally asymptotically stable for any finite values of τ_i. ■

The computational burden associated with the conditions expressed in Theorem 106 is significantly less than that associated with proving delay-independent stability using Theorem 105. This reduction occurs because in this case, we have explicitly chosen a Lyapunov functional structure where the delays τ_i do not explicitly appear in the derivative. Naturally, the use of a more specialized Lyapunov functional structure may result in an increased level of conservativity. For this reason, when testing delay-independent stability, it is recommended that Theorem 106 be used as a first approximation and that Theorem 105 be used if additional accuracy is required.

8.3.1 Numerical Examples

The contribution of this chapter is to show that Lyapunov functions to prove stability of arbitrary non-linear polynomial systems may be computed algorithmically. To illustrate this, we compare our results with those derived through less algorithmic analysis.

Example 1: Reflection Dynamics

To begin, we compare our results with a solution given by Hale [14] and attributed

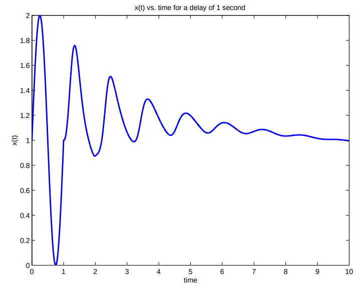

Figure 8.1: Example of a trajectory of $x(t)$ for $a = -1$, $b = .9$

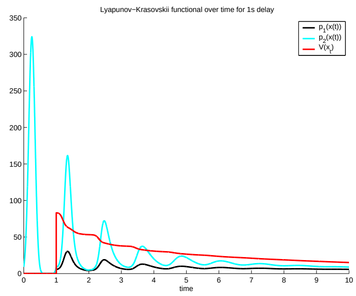

Figure 8.2: Example of a trajectory of $V(x_t)$ for $a = -1$, $b = .9$

to LaSalle for the following dynamics.

$$\dot{x}(t) = ax^3(t) + bx^3(t - \tau) \tag{8.3}$$

The following Lyapunov function is used by Hale to show that the system is stable for any $a < 0$, $|b| < |a|$.

$$V(x_t) = -\frac{x_t^4(0)}{2a} + \int_{-\tau}^{0} x_t^6(\theta)\, d\theta \tag{8.4}$$

Our own computation of various points give similar results. For example, for $a = -1$, $b = .9$, we can prove asymptotic stability using the following function

$$V(x_t) = -3.92x_t^2(0) + 1.91x_t^4(0) + \int_{-\tau}^{0} 5.35x_t^4(\theta) + 3.72x_t^6(\theta)\, d\theta$$

The purpose here is not to construct this particular Lyapunov function, but rather to demonstrate that such functions can be numerically computed without the insight needed to manually derive the results. To illustrate this expanded flexibility, suppose we introduce a cross-term, such as might arise due to interference, then we have

$$\dot{x}(t) = ax^3(t) + c(x(t)x(t-\tau))^2 + bx^3(t-\tau) \tag{8.5}$$

We find the following Lyapunov function in the case when $a = -1$, $b = 0.5$, $c = 0.2$.

$$V(x_t) = 4.233x_t^2(0) - 0.2147x_t^3(0) + 1.856x_t^4(0) \int_{-\tau}^{0} 4.107x_t^4(\theta) + 0.1525x_t^5(\theta) + 3.454x_t^6(\theta)\, d\theta$$

Example 2: Epidemiology

We now consider the following epidemiological model. Consider a human population subject to non-lethal infection by a virus. Assume the disease has incubation period τ. Cooke [6] models the percentage of infected population, $y(t)$ using the following dynamics.

$$\dot{y}(t) = -ay(t) + by(t-\tau)\big(1 - y(t)\big)$$

Where

- a is the recovery rate for the infected population.
- b is the rate of infection for those exposed to the virus.

The model is nonlinear and equilibria exist at $y^* = 0$ and $y^* = (b-a)/b$. Cooke used the following Lyapunov functional to prove delay-independent stability of the 0 equilibrium for $a > b > 0$.

$$V(\phi) = \frac{1}{2}\phi(0)^2 + \frac{1}{2}\int_{-\tau}^{0} a\phi(\theta)^2 d\theta$$

Using semidefinite programming, we were also able to prove delay-independent stability for $a > b > 0$ using the following functional.

$$V(\phi) = 1.75\phi(0)^2 + \int_{-\tau}^{0} (1.47a + .28b)\phi(\theta)^2 d\theta$$

The point of these examples is obviously not to trivialize the work of Cooke and Hale, but rather to show the broad applicability and utility of the algorithms presented in this thesis.

8.4 Conclusion

This chapter provides a generalization of the stability analysis results for linear time-delay systems presented in Chapter 7 to the case of nonlinear dynamics. To this end, we have proposed a sequence of sufficient conditions for both delay-dependent and delay independent stability of nonlinear systems with multiple delays. The numerical examples presented illustrate that many previous results on stability analysis can be derived using our algorithmic framework. Unfortunately, unlike in the linear stability case, we are not optimistic that the results of this chapter will lead to algorithms for the construction of stabilizing controllers for any broad class of nonlinear time-delay systems. This is because the results of this chapter to not fit as neatly into the operator-theoretic framework of the linear case.

Appendix A

Appendix to Chapter 2

Additional Notation: Denote the complete metric space

$$V_\alpha := \{x \in \mathcal{C}[-\tau, \infty) : x(\theta) = \phi(\theta) \text{ for } \theta \in [-\tau, 0],\ \sup_{t \geq -\tau} \|x(t)\| e^{-\alpha t} < \infty\},$$

which has the following metric.

$$d(x, y)_{V_\alpha} = \sup_{t \geq -\tau} \|x(t) - y(t)\| e^{-\alpha t}$$

Define $\mathcal{C}_\tau$ to be the space of continuous functions defined on $[-\tau, 0]$ with norm

$$\|x\|_{\mathcal{C}_\tau} = \sup_{t \in [-\tau, 0]} \|x(t)\|_2.$$

For a given functional $f : \mathcal{C}_\tau \times \mathbb{R}^+ \to \mathbb{R}^n$, let Λ be defined by

$$(\Lambda x)(t) = \phi(0) + \int_0^t f(x_s, s) ds \qquad \text{for all } t \geq 0$$

and $(\Lambda x)(t) = \phi(t)$ for $t \in [-\tau, 0)$.

Definition 107. *We say that a functional $f : \mathcal{C}_\tau \times \mathbb{R}^+ \to \mathbb{R}^n$ satisfies **Assumptions 1** if*

- *There exists a* $K_1 > 0$ *such that*

$$\|f(x,t) - f(y,t)\|_2 \le K_1 \|x - y\|_{\mathcal{C}_\tau} \quad \text{for all } x, y \in \mathcal{C}_\tau, t \ge 0$$

- *There exists a* $K_2 > 0$ *such that*

$$\|f(0,t)\| \le K_2 \quad \text{for all } t \ge 0$$

- $f(x,t)$ *is jointly continuous in* x *and* t*. i.e. for every* (x,t) *and* $\epsilon > 0$*, there exists a* $\eta > 0$ *such that*

$$\|x - y\|_{\mathcal{C}_\tau} + \|t - s\| \le \eta \;\Rightarrow\; \|f(x,t) - f(y,s)\| \le \epsilon.$$

Definition 108. *Given a functional* $f : \mathcal{C}_\tau \times \mathbb{R}^+ \to \mathbb{R}^n$*, we say that a function* $x \in \mathcal{C}[-\tau, \infty)$ *is a **solution to Problem 1** with initial condition* $\phi \in \mathcal{C}_\tau$ *if* x *is differentiable for* $t \ge 0$*,* $x(t) = \phi(t)$ *for* $t \in [-\tau, 0]$ *and*

$$\dot{x}(t) = f(x_t, t) \qquad \text{for all } t \ge 0$$

Results:

Lemma 109. *If* $f : \mathcal{C}_\tau \times \mathbb{R}^+ \to \mathbb{R}^n$ *satisfies Assumptions 1, and* x *is a continuous function, then* $f(x_t, t)$ *is a continuous function in* t*.*

Proof. If $f : \mathcal{C}_\tau \times \mathbb{R}^+ \to \mathbb{R}^n$ satisfies Assumptions 1, then for any t_1, $\epsilon > 0$, there exists a $\eta_1 > 0$ such that

$$\|x_{t_1} - x_{t_2}\|_{\mathcal{C}_\tau} + \|t_1 - t_2\| \le \eta_1 \;\Rightarrow\; \|f(x_{t_1}, t_1) - f(x_{t_2}, t_2)\| \le \epsilon.$$

Now if $x(t)$ is a continuous function of t, then for any $t_1 \ge 0$, $x(t)$ is uniformly continuous on the interval $I_{t_1} := [\max\{-\tau, t_1 - \tau - .1\}, t_1 + .1]$. Hence, for any $\epsilon_2 > 0$, there exists a $\eta_2 > 0$ such that for $s_1, s_2 \in I_{t_1}$, we have that

$$|s_1 - s_2| \le \eta_2 \;\Rightarrow\; \|x(s_1) - x(s_2)\|_2 \le \epsilon_2.$$

Now for a given $t_1 \geq 0$ and $\epsilon > 0$, let η_1 be defined as above. Let $\epsilon_2 = \eta_1/2$ define η_2. Then if $\eta = \min\{.1, \eta_2, \eta_1/2\}$, we have that $|t_1 - t_2| \leq \eta$ implies that

$$\begin{aligned}\|x_{t_1} - x_{t_2}\|_{\mathcal{C}_\tau} &= \sup_{\theta\in[-\tau,0]} \|x_{t_1}(\theta) - x_{t_2}(\theta)\|_2 \\ &= \sup_{\theta\in[-\tau,0]} \|x(t_1+\theta) - x(t_2+\theta)\|_2 \leq \epsilon_2 = \eta_1/2.\end{aligned}$$

This holds since $\|(t_1+\theta) - (t_2+\theta)\| = \|t_1 - t_2\| \leq \eta_2$ means $\|x(t_1+\theta) - x(t_2+\theta)\|_2 \leq \epsilon_2$ since $t_1 + \theta \in I_{t_1}$ for $\theta \in [-\tau, 0]$ and $t_2 + \theta \in I_{t_1}$ for $\|t_1 - t_2\| < .1$, $\theta \in [-\tau, 0]$ and $t_2 \geq 0$. Therefore we have that

$$\|x_{t_1} - x_{t_2}\|_{\mathcal{C}_\tau} + \|t_1 - t_2\| \leq \eta_1/2 + \eta_1/2 = \eta_1.$$

This implies that

$$\|f(x_{t_1}, t_1) - f(x_{t_2}, t_2)\| \leq \epsilon.$$

Thus we have continuity of $f(x_t, t)$. ■

Lemma 110. *Suppose $f : \mathcal{C}_\tau \times \mathbb{R}^+ \to \mathbb{R}^n$ satisfies Assumptions 1, then $\Lambda : V_\alpha \to V_\alpha$.*

Proof. Suppose $x \in V_\alpha$. Let $y = \Lambda x$. Then $y(t) = \phi(t)$ for $t \in [-\tau, 0]$, y is continuous, and the following holds for $t \geq 0$.

$$\begin{aligned}
\|y(t)\|_2 &= \|\phi(0) + \int_0^t f(x_s, s)ds\|_2 \\
&\leq \|\phi(0)\|_2 + \|\int_0^t f(x_s, s)ds\|_2 \\
&\leq \|\phi(0)\|_2 + \int_0^t \|f(x_s, s)\|_2 ds \\
&= \|\phi(0)\|_2 + \int_0^t \|f(x_s, s) - f(0, s) + f(0, s)\|_2 ds \\
&\leq \|\phi(0)\|_2 + \int_0^t \|f(x_s, s) - f(0, s)\|_2 + \|f(0, s)\|_2 ds \\
&\leq \|\phi(0)\|_2 + \int_0^t K_1 \|x_s\|_{\mathcal{C}_\tau} + K_2 ds \\
&= \|\phi(0)\|_2 + K_1 \int_0^t \sup_{\theta \in [-\tau, 0]} \|x_s(\theta)\|_2 ds + K_2 t \\
&= \|\phi(0)\|_2 + K_1 \int_0^t \sup_{\theta \in [-\tau, 0]} \|x(s + \theta)\|_2 ds + K_2 t
\end{aligned}$$

Now, since $x \in V_\alpha$, there exists some $c > 0$ such that $\|x(t)\|_2 \leq ce^{\alpha t}$ for all $t \geq -\tau$. Therefore, $\sup_{\theta \in [-\tau, 0]} \|x(t + \theta)\| \leq ce^{\alpha t}$ for all $t \geq 0$. Therefore, we have the following for $t \geq 0$.

$$\begin{aligned}
\|y(t)\|_2 &\leq \|\phi(0)\|_2 + K_1 \int_0^t \sup_{\theta \in [-\tau, 0]} \|x(s + \theta)\|_2 ds + K_2 t \\
&\leq \|\phi(0)\|_2 + K_1 \int_0^t ce^{\alpha s} ds + K_2 t \\
&= \|\phi(0)\|_2 + \frac{cK_1}{\alpha}(e^{\alpha t} - 1) + K_2 t
\end{aligned}$$

Thus

$$\begin{aligned}\sup_{t\geq 0}\|y(t)\|_2 e^{-\alpha t} &\leq \sup_{t\geq 0}\left(\|\phi(0)\|_2 e^{-\alpha t} + \frac{cK_1}{\alpha}(1-e^{-\alpha t}) + K_2 t e^{-\alpha t}\right)\\ &\leq \|\phi(0)\|_2 + \frac{cK_1}{\alpha} + \frac{eK_2}{\alpha} = c_1.\end{aligned}$$

Now let $c_2 = \sup_{\theta\in[-\tau,0]}\|\phi(\theta)\|$ and $\tilde{c} = \max\{c_1, c_2\}$. Then $\sup_{t\geq -\tau}\|y(t)\|_2 e^{-\alpha t} \leq \tilde{c}$. Thus $y \in V_\alpha$. ■

Lemma 111. *Suppose $f : \mathcal{C}_\tau \times \mathbb{R}^+ \to \mathbb{R}^n$ satisfies Assumptions 1, and $K_1 < \alpha$. Then Λ is a contraction on V_α.*

Proof. Suppose $x, y \in V_\alpha$. Then we have the following for $t \geq 0$.

$$\begin{aligned}\|\Lambda x(t) - \Lambda y(t)\|_2 &= \|\int_0^t f(x_s, s) - f(y_s, s) ds\|_2\\ &\leq \int_0^t \|f(x_s, s) - f(y_s, s)\|_2 ds\\ &\leq \int_0^t K_1\|x_s - y_s\|_{\mathcal{C}_\tau} ds\\ &= K_1 \int_0^t \sup_{\theta\in[-\tau,0]} \|x_s(\theta) - y_s(\theta)\|_2 ds\\ &= K_1 \int_0^t \sup_{\theta\in[-\tau,0]} \|x(s+\theta) - y(s+\theta)\|_2 ds\\ &= K_1 \int_0^t \sup_{\theta\in[s-\tau,s]} \|x(\theta) - y(\theta)\|_2 ds\end{aligned}$$

Now recall that $d(x,y)_{V_\alpha} = \sup_{t\geq -\tau}\|x(t) - y(t)\|e^{-\alpha t}$. Therefore, $\|x(t) - y(t)\|_2 \leq d(x,y)_{V_\alpha} e^{\alpha t}$ for all $t \geq -\tau$. Thus $\sup_{\theta\in[t-\tau,t]}\|x(\theta) - y(\theta)\|_2 \leq d(x,y)e^{\alpha t}$ for all $t \geq 0$.

Thus we have the following for all $t \geq 0$.

$$\begin{aligned}\|\Lambda x(t) - \Lambda y(t)\|_2 &\leq K_1 \int_0^t \sup_{\theta \in [s-\tau, s]} \|x(\theta) - y(\theta)\|_2 ds \\ &\leq K_1 \int_0^t d(x,y)_{V_\alpha} e^{\alpha s} ds \\ &= \frac{K_1}{\alpha} d(x,y)_{V_\alpha} (e^{\alpha t} - 1)\end{aligned}$$

Therefore

$$\begin{aligned}d(\Lambda x, \Lambda y)_{V_\alpha} &= \sup_{t \geq -\tau} \|\Lambda x(t) - \Lambda y(t)\|_2 e^{-\alpha t} \\ &= \sup_{t \geq 0} \|\Lambda x(t) - \Lambda y(t)\|_2 e^{-\alpha t} \\ &\leq \sup_{t \geq 0} \frac{K_1}{\alpha} d(x,y)_{V_\alpha} (1 - e^{-\alpha t}) \\ &\leq \frac{K_1}{\alpha} d(x,y)_{V_\alpha} < d(x,y)_{V_\alpha}\end{aligned}$$

Therefore Λ is a contraction on V_α. ∎

Lemma 112 (Existence). *Suppose f satisfies Assumptions 1, $x \in V_\alpha$, and $x = \Lambda x$. Then x is a solution of Problem 1. i.e. x is differentiable , $x(t) = \phi(t)$ for $t \in [-\tau, 0]$ and*

$$\dot{x}(t) = f(x_t, t) \qquad \textit{for all } t \geq 0.$$

Proof. Since $x \in V_\alpha$, x is continuous and $x(t) = \phi(t)$ for $t \in [-\tau, 0]$. Furthermore, since $x = \Lambda x$, we have that

$$x(t) = \phi(0) + \int_0^t f(x_s, s) ds.$$

Since x is continuous and f satisfies Assumptions 1, by Lemma 109 we have that $f(x_s, s)$ is a continuous and integrable function on $[0, t]$. Now, since $x(t) - x(0) = \int_0^t f(x_s, s) ds$, we have by the fundamental theorem of Calculus(v1) that $x(t) - x(0)$

is differentiable and

$$\dot{x}(t) = f(x_t, t).$$

∎

Lemma 113 (Uniqueness). *Suppose f satisfies Assumptions 1, then any solution to Problem 1 is unique.*

Proof. Suppose x and y are solutions to Problem 1. Define p as follows for $t \geq 0$.

$$\begin{aligned}
p(t) &= \sup_{\theta\in[t-\tau,t]} \|x(\theta) - y(\theta)\|_2 \\
&= \sup_{\theta\in[t-\tau,t],\theta\geq 0} \|x(\theta) - y(\theta)\|_2 \\
&= \sup_{\theta\in[t-\tau,t],\theta\geq 0} \|\int_0^\theta f(x_s, s) - f(y_s, s) ds\|_2 \\
&\leq \sup_{\theta\in[t-\tau,t],\theta\geq 0} \int_0^\theta \|f(x_s, s) - f(y_s, s)\|_2 ds \\
&\leq \sup_{\theta\in[t-\tau,t],\theta\geq 0} K_1 \int_0^\theta \|x_s - y_s\|_{\mathcal{C}_\tau} ds \\
&= \sup_{\theta\in[t-\tau,t],\theta\geq 0} K_1 \int_0^\theta \sup_{\omega\in[s-\tau,s]} \|x(\omega) - y(\omega)\|_2 ds \\
&= \sup_{\theta\in[t-\tau,t],\theta\geq 0} K_1 \int_0^\theta p(s) ds
\end{aligned}$$

Now let $q(t) = \int_0^t p(s)ds$. Then $q(0) = 0$ and $\dot{q}(t) = p(t) \geq 0$ for all $t \geq$. Therefore, q is non-negative and monotonically increasing. Thus we have the following.

$$\begin{aligned}
\dot{q}(t) &\leq \sup_{\theta\in[t-\tau,t],\theta\geq 0} K_1 \int_0^\theta p(s) ds \\
&\leq \sup_{\theta\in[t-\tau,t],\theta\geq 0} K_1 q(\theta) = K_1 q(t)
\end{aligned}$$

Therefore, we have that $\dot{q}(t) - K_1 q(t) \leq 0$ for all $t \geq 0$. Which implies the following for $g(t) = q(t)e^{-K_1 t}$, $t \geq 0$.

$$\dot{g}(t) = \frac{d}{dt}\left(q(t)e^{-K_1 t}\right) = e^{-K_1 t}\left(\dot{q}(t) - K_1 q(t)\right) \leq 0$$

Since $g(0) = q(0) = 0$ and $\dot{g}(t) \leq 0$ for all $t \geq 0$, we have that $g(t) \leq 0$ for all $t \geq 0$. However, $g(t) \leq 0$ implies $q(t) \leq 0$, and so we have that $q(t) = 0$ for all $t \geq 0$. Therefore, since $q(t) = \dot{p}(t) = 0$ for all $t \geq 0$ and $p(0) = 0$, we have that $p(t) = 0$ for all $t \geq 0$. This implies that $x(t) = y(t)$ for all $t \geq 0$, which implies that $x = y$. Thus any solution to problem 1 is unique. ■

Theorem 114. *Suppose a functional $f : \mathcal{C}_\tau \times \mathbb{R}^+ \to \mathbb{R}^n$ satisfies the following.*

- *There exists a $K_1 > 0$ such that*

$$\|f(x,t) - f(y,t)\|_2 \leq K_1 \|x - y\|_{\mathcal{C}_\tau} \quad \textit{for all } x, y \in \mathcal{C}_\tau, t \geq 0.$$

- *There exists a $K_2 > 0$ such that*

$$\|f(0,t)\| \leq K_2 \quad \textit{for all } t \geq 0.$$

- *$f(x,t)$ is jointly continuous in x and t. i.e. for every (x,t) and $\epsilon > 0$, there exists a $\eta > 0$ such that*

$$\|x - y\|_{\mathcal{C}_\tau} + \|t - s\| \leq \eta \;\Rightarrow\; \|f(x,t) - f(y,s)\| \leq \epsilon.$$

Then for any $\phi \in \mathcal{C}_\tau$, there exists a unique $x \in \mathcal{C}[-\tau, \infty)$ such that x is differentiable for $t \geq 0$, $x(t) = \phi(t)$ for $t \in [-\tau, 0]$ and

$$\dot{x}(t) = f(x_t, t) \qquad \textit{for all } t \geq 0.$$

Proof. We have already shown by Lemmas 110 and 111 that Λ, as defined by f, is a contraction on V_α for any $\alpha > K$. Therefore, there exists some $\alpha > K$, $x \in V_\alpha$ such that $x = \Lambda x$. Therefore, by Lemma 112, we have that x is a solution. By Lemma 113, we have that this solution is unique. ■

Lemma 115. *Let*

$$f(x,t) := \sum_{j=1}^{N} A_j x(t-\tau_j) + \int_{-\tau}^{0} A(\theta)x(t-\theta)d\theta,$$

where $A(\theta)$ is bounded on $[-\tau, 0]$. Then there exists some $K > 0$ such that

$$\|f(x,t) - f(y,t)\|_2 \le K\|x-y\|_{\mathcal{C}_\tau}.$$

Proof. For $A \in \mathbb{R}^{n\times m}$, let $\bar{\sigma}(A)$ denote the maximum singular value of A. Then we have the following.

$$\begin{aligned}
&\|f(x,t) - f(y,t)\|_2 \\
&= \|\sum_{j=1}^{N} A_j(x(t-\tau_j) - y(t-\tau_j)) + \int_{-\tau}^{0} A(\theta)(x(t-\theta) - y(t-\theta))d\theta\|_2 \\
&\le \sum_{j=1}^{N} \|A_j(x(t-\tau_j) - y(t-\tau_j))\|_2 + \int_{-\tau}^{0} \|A(\theta)(x(t-\theta) - y(t-\theta))\|_2 d\theta \\
&\le \sum_{j=1}^{N} \bar{\sigma}(A_j)\|x(t-\tau_j) - y(t-\tau_j)\|_2 + \int_{-\tau}^{0} \bar{\sigma}(A(\theta))\|x(t-\theta) - y(t-\theta)\|_2 d\theta \\
&\le \sum_{j=1}^{N} \bar{\sigma}(A_j)\|x-y\|_{\mathcal{C}_\tau} + \int_{-\tau}^{0} \bar{\sigma}(A(\theta))d\theta\|x-y\|_{\mathcal{C}_\tau} \\
&= \left(\sum_{j=1}^{N} \bar{\sigma}(A_j) + \int_{-\tau}^{0} \bar{\sigma}(A(\theta))d\theta\right)\|x-y\|_{\mathcal{C}_\tau}
\end{aligned}$$

■

Appendix B

Appendix to Chapter 6

Lemma 116. *$M \in \bar{\Sigma}_s \bigcap \mathbb{S}^n_{2d}[x]$ if and only if there exists some matrix $Q \in \mathbb{R}^{n_z n \times n_z n}$, where $n_z = \binom{n+d}{d}$ such that $Q \geq 0$ and the following holds.*

$$M(x) = \left(\bar{Z}^n_d[x]\right)^T Q \bar{Z}^n_d[x]$$

Proof. Since $M \in \bar{\Sigma}_s$, we know that there exist $G_i \in \mathbb{R}^{n \times n}[x]$ for $i = 1 \ldots m$ such that

$$M(x) = \sum_{i=1}^{m} G_i(x)^T G_i(x)$$

Now suppose that there exists some i' and i_0, j_0 such that element $[G_{i'}]_{i_0,j_0}$ of $G_{i'}$ is of degree $\hat{d} > d$. Then, since $M_{i_0,i_0}(x) = \sum_{i=1}^m \sum_k [G_i]_{k,i_0}(x)^2$, we have that M_{i_0,i_0} is of at least degree $2\hat{d}$. This is because a finite sum of squares of $\mathbb{R}[x]$ is of degree $2\hat{d}$, where $\hat{d}$ is the maximum degree of the squared elements. However, since $M \in \mathbb{S}^n_d$ by assumption, we have a contradiction. Therefore $G_i \in \mathbb{R}^{n \times n}_d[x]$. Now, for any element $G_i \in \mathbb{R}^{n \times n}_d[x]$, there exists a $B \in \mathbb{R}^{n \times n(d_z)}$ where d_z is the length of $Z_d[x]$ such that $G_i(x) = B\bar{Z}^n_d[x]$. Therefore we have that

$$M(x) = \sum_{i=1}^{m} \left(\bar{Z}^n_d[x]\right)^T B^T B \bar{Z}^n_d[x] = \left(\bar{Z}^n_d[x]\right)^T Q \bar{Z}^n_d[x],$$

where $Q = \sum_{i=1}^m B^T B \geq 0$. ∎

Lemma 117. *$M \in \bar{\Sigma}_s$ if and only if $y^T M(x) y \in \Sigma_s$.*

Proof. ($\Leftarrow$) Suppose $f(x) = y^T M(x) y \in \Sigma_s$. Then $f(x) = \sum_{i=1}^m g_i(x,y)^2 = \sum_{i=1}^m (b_i(x)^T y)^2$ where the b_i are some vectors of functions. This follows since the g_i must be homogeneous of degree 1 in y. Therefore, we have the following.

$$y^T M(x) y = \sum_{i=1}^m y^T b_i(x) b_i(x)^T y$$

Thus $M(x) = \sum_{i=1}^m b_i(x) b_i(x)^T$. Therefore $M(x) \in \bar{\Sigma}_s$.

($\Rightarrow$) Now Suppose $M \in \bar{\Sigma}_s$. Then for some $G_i(x) \in \mathbb{R}^{n\times n}[x]$ for $i = 1 \dots m$, we have that $f(x) = y^T M(x) y = \sum_{i=1}^m y^T G_i^T(x) G_i(x) y = \sum_{i=1}^m g_i(x,y)^T g_i(x,y)$, where $g_i(x,y) = G_i(x) y$. Therefore, $f \in \Sigma_s$. ∎

Appendix C

Appendix to Chapter 5

Lemma 118. $0 \leq f_1(x)x \leq \beta x^2$ *where* $\beta = \frac{e^{\frac{\alpha}{\tau}p_0 - 1}}{p_0}$.

Proof. Recall $f_1(y) = \min\left\{e^{\frac{\alpha}{\tau}y} - 1, e^{\frac{\alpha}{\tau}p_0} - 1\right\}$.Let

$$c_1(y) = \frac{e^{\frac{\alpha}{\tau}y} - 1}{y}.$$

We show that $\dot{c}_1(y) \geq 0$ for all y.

$$\dot{c}_1(y) = \frac{(\frac{\alpha}{\tau}y - 1)e^{\frac{\alpha}{\tau}y} + 1}{y^2}.$$

$\dot{c}_1(y) \geq 0$ for all y is and only if $c_2 := \dot{c}_1(y)y^2 \geq 0$ for all y.

$$\dot{c}_2(y) = (\frac{\alpha}{\tau}ye^{\frac{\alpha}{\tau}y}.$$

Thus $\dot{c}_2(y) < 0$ for all $y < 0$ and $\dot{c}_2(y) > 0$ for all $y > 0$, therefore c_2 has a global minimum at $y = 0$. Since $c_2(0) = 0$, we conclude that $c_2(y) \geq 0$ for all y and thus $\dot{c}_1(y) \geq 0$ for all y. Since c_1 is monotone increasing, we have the following for all $y \leq p_0$.

$$\frac{e^{\frac{\alpha}{\tau}y} - 1}{y} \leq \frac{e^{\frac{\alpha}{\tau}p_0} - 1}{p_0}.$$

Thus for $y \leq p_0$.

$$(e^{\frac{\alpha}{\tau}y} - 1)y \leq \beta y^2.$$

Furthermore, $\lim_{y\to-\infty} c_1(y) = 0$ therefore, $c_1(y) \geq 0$ for all y, thus

$$(e^{\frac{\alpha}{\tau}y} - 1)y \geq 0.$$

Therefore, by the definition of f_1, we have that $0 \leq f_1(x)x \leq \beta x^2$. ∎

Lemma 119. *Let $z = \Delta_z y$ with $y \in W_2$, then $\lim_{t\to\infty} z(t) = 0$.*

Proof. Let $v = \dot{z} = \Delta y$. Suppose that $T_2 > T_1 > 0$ and let $H = P_{T_2} - P_{T_1}$. Then

$$\begin{aligned}
\|Hv\|_2^2 &= \int_{T_1}^{T_2} \dot{z}(t)^2 dt \\
&\leq \beta \int_{T_1}^{T_2} \dot{z}(t)y(t)\,dt - \beta \int_{T_1}^{T_2} \dot{z}(t)z(t)\,dt \\
&= \beta\langle Hv, Hy\rangle - \frac{\beta}{2}(z(T_2)^2 - z(T_1)^2) \\
&\leq \beta\|Hv\|_2\|Hy\|_2 - \frac{\beta}{2}\big(z(T_2)^2 - z(T_1)^2\big)
\end{aligned}$$

Hence

$$\begin{aligned}
z(T_2)^2 - z(T_1)^2 &\leq 2\|Hv\|_2\|Hy\|_2 - \frac{2}{\beta}\|Hv\|_2^2 \\
&\leq 2\|Hv\|_2\|Hy\|_2
\end{aligned}$$

By Lemma 29, $v \in L_2$. Since $\|v\|$ and $\|y\|$ exist, we can use the Cauchy criterion and the above inequality to establish that for any $\delta > 0$, there exists a T_δ such that $T_2 > T_1 > T_\delta$ implies $(z(T_2)^2 - z(T_1)^2) < \delta$. It is shown in Lemma 122 that this implies that for any infinite increasing sequence $\{T_i\}$, $\{z(T_i)^2\}$ is a Cauchy sequence and therefore $z(t)^2$ converges to a limit. Since z is continuous, this implies that $z(t)$ also converges to a limit, z_∞. Since $y \in W_2$, we have $\lim_{t\to\infty} y(t) = y_\infty = 0$. Recall $f_c(a, b) = f_1(b)$ at points such that $a > -p_0$ or $b > 0$. Suppose $z_\infty \neq 0$. If $z_\infty < 0$, then

$y_\infty - z_\infty > 0$ and $f_c(a,b) = f_1(b)$ in some neighborhood of $(z_\infty, y_\infty - z_\infty)$. If $z_\infty > 0$, then $z_\infty > 0 \geq -p_0$ and $f_c(a,b) = f_1(b)$ in some neighborhood of $(z_\infty, y_\infty - z_\infty)$. Since f_1 is continuous, we have $\lim_{t\to\infty} \dot{z}(t) = \lim_{t\to\infty} f_c(z(t), y(t) - z(t)) = f_1(y_\infty - z_\infty) = f_1(-z_\infty)$. By inspection of the function f_1, we see that $z_\infty \neq 0$ implies that $\dot{z}$ has a nonzero limit. However, since $\dot{z} \in L_2$, it cannot have a nonzero limit. Thus we conclude by condradiction that $z_\infty = 0$. ■

Lemma 120. *If $v = \Delta y$ with $y \in W_2$, then $\langle v, \dot{y} - v\rangle \geq -\beta|y(0)|^2$.*

Proof. Let $z = \Delta_z v$ and define the variable $r(t) = y(t) - z(t)$ and the set $M = \{t : z(t) > -p_0 \ \text{ or } \ r(t) \geq 0\}$, then

$$\begin{aligned}\langle v, \dot{y} - v\rangle = \langle \dot{z}, \dot{y} - \dot{z}\rangle &= \int_0^\infty \dot{z}(t)\dot{r}(t)dt \\ &= \int_M f_1(r(t))\dot{r}(t)dt \leq \beta\|y\|\|\dot{y}\| + \beta^2\|v\|^2\end{aligned}$$

Since $y \in W_2$, we have that y is absolutely continuous and thus r is absolutely continuous. Since r, z are absolutely continuous functions and since by Lemma 30, we have $z(t) \to 0$, we can partition the set M into the countable union of sequential disjoint intervals $\bigcup_i I_i \bigcup I_f$ where $I_i = [T_{a,i}, T_{b,i})$ with $\{T_{a,i}\}, \{T_{b,i}\} \subset \mathbb{R}^+$ and $I_f = [T_{a,f}, \infty)$. To see that the intervals are closed on the left, suppose I_i were open on the left. Then, since $T_{a,i} \notin M$, $z(T_{a,i}) = -p_0$ and $r(T_{a,i}) < 0$. However, since r is continuous, $r(T_{a,i} + \eta) < 0$ for η sufficiently small. Since $r(t) < 0$ implies $\dot{z}(t) \leq 0$, we have that $z(T_{a,i} + \eta) \leq -p_0$ and thus $T_{a,i} + \eta \notin M$ for η sufficiently small, which is a contradiction. Thus all the intervals are closed on the left. Similarly, one can show that all the intervals are open on the right.

Now, consider time $T_a > 0$, where $T_a \in M$ defines the start of one of the intervals described above. If $z(T_a) > -p_0$, then since z is continuous, $z(T_a - \eta) > -p_0$ for all η sufficiently small. Therefore $T_a - \eta \in M$ for all η sufficiently small. This contradicts the statement that the intervals are disjoint. We thus conclude $z(T_a) = -p_0$ and consequently $r(T_a) \geq 0$ by definition of M. Now suppose $r(T_a) > 0$. Since r is continuous, $r(T_a - \epsilon) > 0$ and consequently $T_a - \epsilon \in M$ for all ϵ sufficiently small,

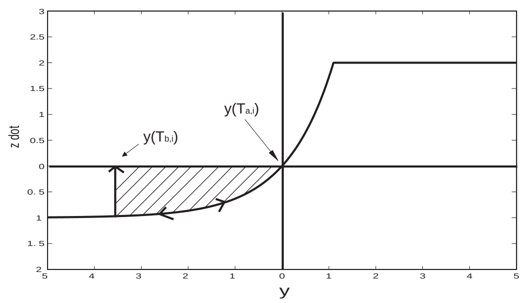

Figure C.1: Value of $y, \dot{z}$ at times $T_{a,i}$ and $T_{b,i}$

which contradicts the statement that the intervals are disjoint. Therefore we conclude $r(T_a) = 0$ if $T_a \neq 0$. Then

$$\begin{aligned}\langle v, \dot{y} - v\rangle &= \sum_i \int_{I_i} f_1(r(t))\dot{r}(t)dt + \int_{T_{a,f}}^{\infty} f_1(r(t))\dot{r}(t)dt \\ &= \sum_i \int_{T_{a,i}}^{T_{b,i}} f_1(r(t))\dot{r}(t)dt + \int_{T_{a,f}}^{\infty} f_1(r(t))\dot{r}(t)dt\end{aligned}$$

We will assume that $T_{a,1} = 0$. If $T_{a,1} \neq 0$, we have $r(T_{a,1}) = 0$ and the proof becomes simpler. Since $f_1(r)$ is continuous in r and $r(t)$ is absolutely continuous in time, by the substitution rule we have

$$\begin{aligned}\langle v, \dot{y} - v\rangle &= \sum_i \int_{r(T_{a,i})}^{r(T_{b,i})} f_1(r)dr \\ &= \int_{r(0)}^{r(T_{b,1})} f_1(r)dr + \sum_{i\neq 1} \int_0^{r(T_{b,i})} f_1(r)dr \\ &= \int_{r(0)}^{0} f_1(r)dr + \sum_i \int_0^{r(T_{b,i})} f_1(r)dr\end{aligned}$$

Since $f_1 \in \text{sector}[0, \beta]$, $\int_0^{r(T_{b,i})} f_1(r)dr \geq 0$ for any $r(T_{b,i}) \in \mathbb{R}$. The summation converges since it is bounded, increasing. Furthermore, since $r(0) = y(0) - z(0) = y(0)$

and $|\int_0^y f_1(r)dr| \leq f_1(y)y \leq \beta y^2$ for any y, we have

$$\langle v, \dot{y} - v\rangle = \int_{y(0)}^{0} f_1(r)dr + \sum_i \int_0^{r(T_{b,i})} f_1(r)dr \geq -\beta|y(0)|^2$$

■

Theorem 121. *For any initial condition $x_0 \in W_2$ with $x_0(-\tau) > -p_0$, $x_0(\theta) \geq -p_0$, there exists a $f \in W_2$ and $T > 0$ such that $A(x_0, t) = B_z(f, t+T)$ for $t \geq 0$.*

Proof. Recall the map B_z is defined by the following dynamics for $z = B_z y$. $z(t) = 0$ for $t \leq 0$ and for $t \geq 0$

$$\dot{z}(t) = f_c\big(z(t), y(t) - z(t-\tau)\big)$$

Now suppose we let $y(t) = f(t) + z(t-\tau)$ on a finite interval $[0, T']$ for some $f \in W_2$. The system is still well-posed and $y \in W_2$ if $f \in W_2$ since the derivative of z is bounded. The dynamics are now given by $z(t) = 0$ for $t \leq 0$ and the following for $t \geq 0$

$$\dot{z}(t) = f_c\big(z(t), f(t)\big)$$

Part 1: The first part of the proof is to construct a $f \in W_2$ that drives the state $z(t)$ to $z(T') = x_0(0)$ for some $T' \geq 0$. Furthermore, for continuity with Part 2, we require that $f(T') = -x_0(-\tau)$. If $x_0(-\tau) = 0$ and $x_0(0) = 0$, then we are done. Otherwise, there are 4 cases to consider.

$x_0(-\tau) < 0, x_0(0) > 0$ First, let $f(t) = \epsilon t$ until $T_1 = -\frac{x_0(-\tau)}{\epsilon}$ for $\epsilon > 0$. Let ϵ be sufficiently large so that $z(T_1) < x_0(0)$. Such an ϵ exists since $\dot{z}$ is bounded. Let $f(t) = -x_0(-\tau) > 0$ until time T' such that $z(T') = x_0(0)$. Such T' exists since $z(t)$ is now linearly increasing.

$x_0(-\tau) > 0, x_0(0) < 0$ This case is handled similarly to the previous one.

$x_0(-\tau) < 0, x_0(0) < 0$ For this case, let $f(t) = -\epsilon t$ until time $T_1 = \frac{1}{\epsilon}$. Then let $f(t) = -1$ until time T_2 such that $z(T_2) = x_0(0) - \gamma$ for some $\gamma > 0$. Such time exists for any $\gamma < p_0 - x_0(0)$ since $z(t)$ is linearly decreasing for $z(t) \geq -p_0$. Then let $f(t) = x_0(0) - \gamma + \lambda t$ for time $\Delta t = \frac{1}{\lambda}(x_0(-\tau) + 1)$. Make λ sufficiently large so that $z(T_2 + \Delta t) < x_0(0)$. This is possible since $\dot{z}$ is bounded. Finally let $f(t) = -x_0(-\tau) > 0$ until time T' when $z(T') = x_0(0)$. Such T' exists since $z(t)$ is now linearly increasing.

$x_0(-\tau) > 0, x_0(0) > 0$ **or** $x_0(0) = 0$ **or** $x_0(-\tau) = 0$ These cases are handled similarly to the previous one.

Part 2: For time $t \in [T', T' + \tau]$, let $f(t) = -x_0(t - T')$. We then have $y(T' + \tau) = x_0(t - T') - z(T') = x_0(-\tau) - x_0(-\tau) = 0$. Let $y(t) = 0$ for $t \geq T' + \tau$. Therefore $y \in W_2$ and we conclude that the dynamics of the interconnection for time $t \in [T', T' + \tau]$ are given by $z(T') = x_0(0)$

$$\dot{z}(t) = f_c(z(t), y(t) + z(t - \tau)) \quad \text{(C.1)}$$

$$= f_c(z(t), x_0(t - \tau)) \quad \text{(C.2)}$$

And for $t \geq T' + \tau$, we have

$$\dot{z}(t) = f_c(z(t), y(t) + z(t - \tau)) \quad \text{(C.3)}$$

$$= f_c(z(t), -z(t - \tau)) \quad \text{(C.4)}$$

Therefore, we have that $A(y, t + T') = B(x_0, t)$.

■

Lemma 122. *Suppose that for any $\delta > 0$, there exists a T_δ such that $T_2 > T_1 > T_\delta$ implies $z(T_2)^2 - z(T_1)^2 < \delta$. Then for any infinite, increasing sequence, $\{T_i\}$, $\{z(T_i)^2\}$ is a Cauchy sequence.*

Proof. Proof by contradiction. Suppose that $\{z(T_i)^2\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ such that for any $N > 0$, there exists $i, j > N$ such that $|z(T_i)^2 - z(T_j)^2| > \epsilon$.

Let $\delta = \frac{\epsilon}{2}$. By assumption there exists a T_δ such that for all $T_2 > T_1 > T_\delta$,

$$z(T_2)^2 - z(T_1)^2 < \frac{\epsilon}{2} \tag{C.5}$$

Since $\{T_i\}$ is strictly increasing, infinite, there exists a $N > 0$ such that $T_N > T_\delta$. Since $\{z(T_i)^2\}$ is not Cauchy, there exists $i_1, j_1 > N$ such that $|z(T_{i_1})^2 - z(T_{j_1})^2| > \epsilon$. Without loss of generality, we may assume $i_1 > j_1$. We now have one of the two possibilities.

- $z(T_{i_1})^2 - z(T_{j_1})^2 > \epsilon$
- $z(T_{i_1})^2 - z(T_{j_1})^2 < -\epsilon$

However, by Equation (C.5), the first option is not possible since $T_i > T_j > T_N > T_\delta$. Therefore $z(T_{i_1})^2 < z(T_{j_1})^2 - \epsilon$. Thus for all $k > i_1$, by (C.5) and since $T_k > T_{i_1} > T_{j_1} > T_N \geq T_\delta$, we have

$$z(T_k)^2 < z(T_{i_1})^2 + \frac{\epsilon}{2} < z(T_{j_1})^2 - \epsilon + \frac{\epsilon}{2} = z(T_{j_1})^2 - \frac{\epsilon}{2}$$

Now, let $N = i_1$. Again, since $\{z(T_i)^2\}$ is not Cauchy, there exists $i_2, j_2 > i_1$ such that $|z(T_{i_2})^2 - z(T_{j_2})^2| > \epsilon$. Using the method above, for all $k > i_2$ since $T_k > T_{i_2} > T_{j_2} > T_{i_1} > T_\delta$, by (C.5) we have

$$z(T_k)^2 < z(T_{i_2})^2 + \frac{\epsilon}{2} < z(T_{j_2})^2 - \frac{\epsilon}{2} < z(T_{j_1})^2 - 2\frac{\epsilon}{2}$$

Repeating n times, we find a i_n such that for all $k > i_n$, $z(T_k)^2 < z(T_{j_1})^2 - n\frac{\epsilon}{2}$. Since we can assume $z(T_{j_1})^2$ is finite, let n be sufficiently large and we have the existence of an i such that that $z(T_i)^2 < 0$, which is a contradiction. Thus we have that $\{z(T_i)^2\}$ is a Cauchy sequence. ■

Appendix D

Appendix to Chapter 7

Theorem 123. *Suppose $M : \mathbb{R} \to \mathbb{S}^{2n}$ is a continuous matrix-valued function. Then the following are equivalent*

1. *There exists some $\epsilon > 0$ such that the following holds for all $x \in \mathcal{C}_\tau$.*

$$\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta \geq \epsilon \|x\|_2^2$$

2. *There exists some $\epsilon' > 0$ and some continuous matrix-valued function $T : \mathbb{R} \to \mathbb{S}^n$ such that the following holds.*

$$\int_{-\tau}^{0} T(\theta) d\theta = 0$$

$$M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \geq 0 \qquad \textit{for all } \theta \in [-\tau, 0]$$

Proof. $(2 \Rightarrow 1)$ Suppose statement 2 is true, then

$$\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta - \epsilon' \|x\|_2^2$$
$$= \int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta \geq 0.$$

$(1 \Rightarrow 2)$ Suppose that statement 1 holds for some M. Write M as

$$M(\theta) = \begin{bmatrix} M_{11}(\theta) & M_{12}(\theta) \\ M_{12}(\theta)^T & M_{22}(\theta) \end{bmatrix}.$$

We first prove that $M_{22}(\theta) \geq \epsilon I$ for all $\theta \in [-\tau, 0]$. By statement 1, we have that

$$\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T \begin{bmatrix} M_{11}(\theta) & M_{12}(\theta) \\ M_{12}(\theta)^T & M_{22}(\theta) - \epsilon I \end{bmatrix} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta \geq 0.$$

Now suppose that $M_{22}(\theta) - \epsilon I$ is not positive semidefinite for all $\theta \in [-\tau, 0]$. Then there exists some $x_0 \in \mathbb{R}^n$ and $\theta_1 \in [-\tau, 0]$ such that $x_0^T(M_{22}(\theta_1) - \epsilon I)x_0 < 0$. By continuity of M_{22}, if $\theta_1 = 0$ or $\theta_1 = -\tau$, then there exists some $\theta_1' \in (-\tau, 0)$ such that $x_0^T(M_{22}(\theta_1') - \epsilon I)x_0 < 0$. Thus assume $\theta_1 \in (-\tau, 0)$. Now, since M_{22} is continuous, there exists some x_1 and $\delta > 0$ where $\theta_1 + \delta < 0$, $\theta_1 - \delta > -\tau$ and such that $x_1^T(M_{22}(\theta) - \epsilon I)x_1 \leq -1$ for all $\theta \in [\theta_1 - \delta, \theta_1 + \delta]$. Then for $\beta > \max\{1/(-\theta_1 - \delta), 1/(\tau + \theta_1 - \delta)\}$, let

$$x(\theta) = \begin{cases} (1 + \beta(\theta - (\theta_1 - \delta)))x_1 & \theta \in [\theta_1 - \delta - 1/\beta, \theta_1 - \delta] \\ x_1 & \theta \in [\theta_1 - \delta, \theta_1 + \delta] \\ (1 - \beta(\theta - (\theta_1 + \delta)))x_1 & \theta \in [\theta_1 + \delta, \theta_1 + \delta + 1/\beta] \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in \mathcal{C}_\tau$, $x(0) = 0$ and $\|x(\theta)\| \leq \|x_1\|$ for all $\theta \in [-\tau, 0]$. Now, since M_{22} is continuous, it is bounded on $[-\tau, 0]$. Therefore, there exists some $\epsilon_2 > 0$ such that

$M_{22}(\theta) - \epsilon I \leq \epsilon_2 I$. Then let $\beta \geq 2\epsilon_2\|x_1\|^2/\delta$ and we have the following.

$$\begin{aligned}
&\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta - \epsilon\|x\|^2 \\
&= \int_{-\tau}^{0} x(\theta)^T (M_{22}(\theta) - \epsilon I) x(\theta) d\theta \\
&= \int_{\theta_1-\delta}^{\theta_1+\delta} x_1^T (M_{22}(\theta) - \epsilon I) x_1 d\theta \\
&+ \int_{\theta_1-\delta-1/\beta}^{\theta_1-\delta} x(\theta)^T (M_{22}(\theta) - \epsilon I) x(\theta) d\theta \\
&+ \int_{\theta_1+\delta}^{\theta_1+\delta+1/\beta} x(\theta)^T (M_{22}(\theta) - \epsilon I) x(\theta) d\theta \\
&\leq -2\delta + \epsilon_2 \int_{\theta_1-\delta-1/\beta}^{\theta_1-\delta} \|x(\theta)\|^2 d\theta + \epsilon_2 \int_{\theta_1+\delta}^{\theta_1+\delta+1/\beta} \|x(\theta)\|^2 d\theta \\
&\leq -2\delta + 2\epsilon_2\|x_1\|^2/\beta \leq -\delta
\end{aligned}$$

Therefore, by contradiction, we have that $M_{22}(\theta) \geq \epsilon I$ for all $\theta \in [-\tau, 0]$. Now we define $\epsilon' = \epsilon/2$ and $\tilde{M}_{22}(\theta) = M_{22}(\theta) - \epsilon' I \geq \epsilon' I$. We now show that $\tilde{M}_{22}(\theta)^{-1}$ is continuous. We first note that $\tilde{M}_{22}(\theta)^{-1}$ is bounded, since

$$\tilde{M}_{22}(\theta) \geq \epsilon' I \quad \Rightarrow I \geq \epsilon' \tilde{M}_{22}(\theta)^{-1} \quad \Rightarrow \tilde{M}_{22}(\theta)^{-1} \leq \frac{1}{\epsilon'} I$$

Now since $\tilde{M}_{22}$ is continuous, for any $\beta > 0$, there exists a $\delta > 0$ such that

$$|\theta_1 - \theta_2| \leq \delta \Rightarrow \|\tilde{M}_{22}(\theta_1) - \tilde{M}_{22}(\theta_2)\| \leq \beta$$

Therefore, for any $\beta' > 0$, let δ be defined as above for $\beta = \beta'\epsilon'^2$. Then $|\theta_1 - \theta_2| \le \delta$ implies

$$\begin{aligned}
&\|\tilde{M}_{22}(\theta_1)^{-1} - \tilde{M}_{22}(\theta_2)^{-1}\| \\
&= \|\tilde{M}_{22}(\theta_1)^{-1}\tilde{M}_{22}(\theta_2)\tilde{M}_{22}(\theta_2)^{-1} - \tilde{M}_{22}(\theta_1)^{-1}\tilde{M}_{22}(\theta_1)\tilde{M}_{22}(\theta_2)^{-1}\| \\
&= \|\tilde{M}_{22}(\theta_1)^{-1}\left(\tilde{M}_{22}(\theta_2) - \tilde{M}_{22}(\theta_1)\right)\tilde{M}_{22}(\theta_2)^{-1}\| \\
&\le \|\tilde{M}_{22}(\theta_1)^{-1}\|\|\tilde{M}_{22}(\theta_2) - \tilde{M}_{22}(\theta_1)\|\|\tilde{M}_{22}(\theta_2)^{-1}\| \le \frac{1}{\epsilon'^2}\|\tilde{M}_{22}(\theta_2) - \tilde{M}_{22}(\theta_1)\| \le \beta'.
\end{aligned}$$

Therefore $\tilde{M}_{22}(\theta)^{-1}$ is continuous. We now prove statement 2 by construction. Suppose M satisfies statement 1. Let

$$T(\theta) = T_0 - (M_{11}(\theta) - M_{12}(\theta)\tilde{M}_{22}^{-1}(\theta)M_{12}(\theta)^T).$$

Here

$$T_0 = \frac{1}{\tau}\int_{-\tau}^{0}(M_{11}(\theta) - M_{12}(\theta)\tilde{M}_{22}^{-1}(\theta)M_{12}(\theta)^T)d\theta.$$

Then we have that T is continuous and

$$\int_{-\tau}^{0} T(\theta)d\theta = \tau T_0 - \tau T_0 = 0.$$

This implies that $T \in \Omega$. We now prove that $T_0 \ge 0$. For any vector $z_0 \in \mathbb{R}^n$, suppose z is a continuous function such that $z(0) = (I + \tilde{M}_{22}(0)^{-1}M_{12}(0)^T)z_0$. Then let $x(\theta) = z(\theta) - \tilde{M}_{22}(\theta)^{-1}M_{12}(\theta)^T z_0$. Then x is continuous, $x(0) = z_0$ and by statement 1 we have the following.

$$\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T \begin{bmatrix} M_{11}(\theta) & M_{12}(\theta) \\ M_{12}(\theta)^T & \tilde{M}_{22}(\theta) \end{bmatrix} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta$$

$$= \int_{-\tau}^{0} \begin{bmatrix} z_0 \\ z(\theta) \end{bmatrix}^T \begin{bmatrix} I & 0 \\ -\tilde{M}_{22}(\theta)^{-1} M_{12}(\theta)^T & I \end{bmatrix}^T \begin{bmatrix} M_{11}(\theta) & M_{12}(\theta) \\ M_{12}(\theta)^T & \tilde{M}_{22}(\theta) \end{bmatrix} \begin{bmatrix} I & 0 \\ -\tilde{M}_{22}(\theta)^{-1} M_{12}(\theta)^T & I \end{bmatrix} \begin{bmatrix} z_0 \\ z(\theta) \end{bmatrix} d\theta$$

$$= \int_{-\tau}^{0} \begin{bmatrix} z_0 \\ z(\theta) \end{bmatrix}^T \begin{bmatrix} M_{11}(\theta) - M_{12}(\theta)\tilde{M}_{22}^{-1}(\theta)M_{12}(\theta)^T & 0 \\ 0 & \tilde{M}_{22}(\theta) \end{bmatrix} \begin{bmatrix} z_0 \\ z(\theta) \end{bmatrix} d\theta$$

$$= z_0^T (\int_{-\tau}^{0} M_{11}(\theta) - M_{12}(\theta)\tilde{M}_{22}^{-1}(\theta)M_{12}(\theta)^T d\theta) z_0$$

$$+ \int_{-\tau}^{0} z(\theta)^T \tilde{M}_{22}(\theta) z(\theta) d\theta$$

$$= z_0^T T_0 z_0 + \int_{-\tau}^{0} z(\theta)^T \tilde{M}_{22}(\theta) z(\theta) d\theta \geq \epsilon' \|x\|_2^2$$

We now show that this implies that $T_0 \geq 0$. Suppose there exists some y such that $y^T T_0 y < 0$. Then there exists some z_0 such that $z_0 T_0 z_0 = -1$. Now let $\alpha > 1/\tau$ and

$$z(\theta) = \begin{cases} (I + \tilde{M}_{22}(0)^{-1} M_{12}(0)^T) z_0 (1 + \alpha\theta) & \theta \in [-1/\alpha, 0] \\ 0 & \text{otherwise.} \end{cases}$$

Then z is continuous, $z(0) = (I + \tilde{M}_{22}(0)^{-1} M_{12}(0)^T) z_0$ and $\|z(\theta)\|^2 \leq \|z(0)\|^2$ for all $\theta \in [-\tau, 0]$. Recall that $\tilde{M}_{22}(\theta) \leq (\epsilon_2 + \epsilon')I = \epsilon_3 I$. Let $\alpha > 2\epsilon_3 \|z(0)^2\|$ and then we have the following.

$$z_0^T T_0 z_0 + \int_{-\tau}^{0} z(\theta)^T M_{22}(\theta) z(\theta) d\theta \leq -1 + \epsilon_3 \int_{-1/\alpha}^{0} \|z(\theta)\|^2$$

$$\leq -1 + \epsilon_3 \|z(0)\|^2/\alpha < -\frac{1}{2}$$

However, this contradicts the previous relationship. Therefore, we have by contradiction that $T_0 \geq 0$. Now by using the invertibility of $\tilde{M}_{22}(\theta) = M_{22}(\theta) - \epsilon' I \geq \epsilon' I$ and the Schur complement transformation, we have that statement 2 is equivalent to the following for all $\theta \in [-\tau, 0]$.

$$\begin{aligned}
&\tilde{M}_{22}(\theta) \geq 0 \\
&M_{11}(\theta) + T(\theta) - M_{12}(\theta)\tilde{M}_{22}(\theta)^{-1}M_{12}(\theta)^T \geq 0
\end{aligned}$$

We have already proven the first condition. Finally, we have the following.

$$M_{11}(\theta) + T(\theta) - M_{12}(\theta)\tilde{M}_{22}(\theta)^{-1}M_{12}(\theta)^T = T_0 \geq 0$$

Thus we have shown that statement 2 is true. ∎

Lemma 124. *Let $M : \mathbb{R} \mapsto \mathbb{S}^{3n}$ be a continuous matrix-valued function. Then the following are equivalent.*

1. *There exists an $\epsilon > 0$ such that the following holds for all $x \in \mathcal{C}_\tau$.*

$$\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(-\tau) \\ x(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} x(0) \\ x(-\tau) \\ x(\theta) \end{bmatrix} \geq \epsilon \|x\|_2^2$$

2. *There exists some $\epsilon' > 0$ and a continuous matrix valued function $T : \mathbb{R} \mapsto \mathbb{S}^{2n}$ such that the following holds.*

$$\begin{aligned}
&\int_{-\tau}^{0} T(\theta) d\theta = 0 \\
&M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \geq 0 \qquad \text{for all } \theta \in [-\tau, 0]
\end{aligned}$$

Proof. ($2 \Rightarrow 1$) Suppose there exists some function $T \in S_0$ such that

$$M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \geq 0 \qquad \text{for all } \theta \in [-\tau, 0].$$

Then we have the following.

$$\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(-\tau) \\ x(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} x(0) \\ x(-\tau) \\ x(\theta) \end{bmatrix} d\theta - \epsilon' \|x\|_2^2$$
$$= \int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(-\tau) \\ x(\theta) \end{bmatrix}^T \left(M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \right) \begin{bmatrix} x(0) \\ x(-\tau) \\ x(\theta) \end{bmatrix} d\theta \geq 0$$

($1 \Rightarrow 2$) Suppose that statement 1 holds for some M. Write M as

$$M(\theta) = \begin{bmatrix} M_{11}(\theta) & M_{12}(\theta) \\ M_{12}(\theta)^T & M_{22}(\theta) \end{bmatrix},$$

where $M_{22} : \mathbb{R} \mapsto \mathbb{S}^n$. We first prove that $M_{22}(\theta) \geq \epsilon I$ for all $\theta \in [-\tau, 0]$. By statement 1, we have that

$$\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(-\tau) \\ x(\theta) \end{bmatrix}^T \begin{bmatrix} M_{11}(\theta) & M_{12}(\theta) \\ M_{12}(\theta)^T & M_{22}(\theta) - \epsilon I \end{bmatrix} \begin{bmatrix} x(0) \\ x(-\tau) \\ x(\theta) \end{bmatrix} d\theta \geq 0.$$

Now suppose that $M_{22}(\theta) - \epsilon I$ is not positive semidefinite for all $\theta \in [-\tau, 0]$. Then there exists some x_0 and $\theta_1 \in [-\tau, 0]$ such that $x_0^T (M_{22}(\theta_1) - \epsilon I) x_0 < 0$. By continuity of M_{22}, if $\theta_1 = 0$ or $\theta_1 = -\tau$, then there exists some $\theta_1' \in (-\tau, 0)$ such that $x_0^T (M_{22}(\theta_1') - \epsilon I) x_0 < 0$. Thus assume $\theta_1 \in (-\tau, 0)$. Now, since M_{22} is continuous, there exists some x_1 and $\delta > 0$ where $\theta_1 + \delta < 0$, $\theta_1 - \delta > -\tau$ and such that $x_1^T (M_{22}(\theta) - \epsilon I) x_1 \leq -1$ for $\theta \in [\theta_1 - \delta, \theta_1 + \delta]$. Then for $\beta >$

$\max\{1/(-\theta_1-\delta), 1/(\tau+\theta_1-\delta)\}$, let

$$x(\theta) = \begin{cases} \beta(\theta-(\theta_1-\delta-1/\beta))x_1 & \theta\in[\theta_1-\delta-1/\beta, \theta_1-\delta] \\ x_1 & \theta\in[\theta_1-\delta,\theta_1+\delta] \\ (1-\beta(\theta-(\theta_1+\delta)))x_1 & \theta\in[\theta_1+\delta,\theta_1+\delta+1/\beta] \\ 0 & \text{otherwise.} \end{cases}$$

Then $x\in\mathcal{C}_\tau$, $x(-\tau)=x(0)=0$ and $\|x(\theta)\|^2\le\|x_1\|^2$ for all $\theta\in[-\tau,0]$. Now, since M_{22} is continuous, it is bounded on $[-\tau,0]$. Therefore, there exists some $\epsilon_2>0$ such that $M_{22}(\theta)-\epsilon I\le\epsilon_2 I$ for $\theta\in[-\tau,0]$. Let $\beta\ge 2\epsilon_2\|x_1\|^2/\delta$ and we have the following.

$$\begin{aligned}
&\int_{-\tau}^{0}\begin{bmatrix}x(0)\\x(-\tau)\\x(\theta)\end{bmatrix}^T M(\theta)\begin{bmatrix}x(0)\\x(-\tau)\\x(\theta)\end{bmatrix}d\theta-\epsilon\|x\|^2\\
&=\int_{-\tau}^{0}x(\theta)^T(M_{22}(\theta)-\epsilon I)x(\theta)d\theta\\
&=\int_{\theta_1-\delta}^{\theta_1+\delta}x_1^T(M_{22}(\theta)-\epsilon I)x_1 d\theta+\int_{\theta_1-\delta-1/\beta}^{\theta_1-\delta}x(\theta)^T(M_{22}(\theta)-\epsilon I)x(\theta)d\theta\\
&+\int_{\theta_1+\delta}^{\theta_1+\delta+1/\beta}x(\theta)^T(M_{22}(\theta)-\epsilon I)x(\theta)d\theta\\
&\le -2\delta+\epsilon_2\int_{\theta_1-\delta-1/\beta}^{\theta_1-\delta}\|x(\theta)\|^2 d\theta+\epsilon_2\int_{\theta_1+\delta}^{\theta_1+\delta+1/\beta}\|x(\theta)\|^2 d\theta\\
&\le -2\delta+2\epsilon_2\|x_1\|^2/\beta\le-\delta
\end{aligned}$$

Therefore, by contradiction, we have that $M_{22}(\theta)\ge\epsilon I$ for all $\theta\in[-\tau,0]$. Now we define $\epsilon'=\epsilon/2$ and $\tilde{M}_{22}(\theta)=M_{22}(\theta)-\epsilon' I\ge\epsilon' I$. We now show that $\tilde{M}_{22}(\theta)^{-1}$ is continuous. We first note that $\tilde{M}_{22}(\theta)^{-1}$ is bounded, since

$$\begin{aligned}\tilde{M}_{22}(\theta) &\geq \epsilon' I \\ &\Rightarrow I \geq \epsilon' \tilde{M}_{22}(\theta)^{-1} \\ &\Rightarrow \tilde{M}_{22}(\theta)^{-1} \leq \frac{1}{\epsilon'} I\end{aligned}$$

Now since $\tilde{M}_{22}$ is continuous, for any $\beta > 0$, there exists a $\delta > 0$ such that

$$|\theta_1 - \theta_2| \leq \delta \Rightarrow \|\tilde{M}_{22}(\theta_1) - \tilde{M}_{22}(\theta_2)\| \leq \beta.$$

Therefore, for any $\beta' > 0$, let δ be defined as above for $\beta = \beta'\epsilon'^2$. Then $|\theta_1 - \theta_2| \leq \delta$ implies

$$\begin{aligned}&\|\tilde{M}_{22}(\theta_1)^{-1} - \tilde{M}_{22}(\theta_2)^{-1}\| \\ &= \|\tilde{M}_{22}(\theta_1)^{-1}\tilde{M}_{22}(\theta_2)\tilde{M}_{22}(\theta_2)^{-1} - \tilde{M}_{22}(\theta_1)^{-1}\tilde{M}_{22}(\theta_1)\tilde{M}_{22}(\theta_2)^{-1}\| \\ &= \|\tilde{M}_{22}(\theta_1)^{-1}\left(\tilde{M}_{22}(\theta_2) - \tilde{M}_{22}(\theta_1)\right)\tilde{M}_{22}(\theta_2)^{-1}\| \\ &\leq \|\tilde{M}_{22}(\theta_1)^{-1}\|\|\tilde{M}_{22}(\theta_2) - \tilde{M}_{22}(\theta_1)\|\|\tilde{M}_{22}(\theta_2)^{-1}\| \\ &\leq \frac{1}{\epsilon'^2}\|\tilde{M}_{22}(\theta_2) - \tilde{M}_{22}(\theta_1)\| \leq \beta'.\end{aligned}$$

Therefore $\tilde{M}_{22}(\theta)^{-1}$ is continuous.

We now prove statement 2 by construction. Suppose M satisfies statement 1. Let

$$T(\theta) = T_0 - (M_{11}(\theta) - M_{12}(\theta)\tilde{M}_{22}^{-1}(\theta)M_{12}(\theta)^T)$$

Where

$$T_0 = \frac{1}{\tau}\int_{-\tau}^{0}(M_{11}(\theta) - M_{12}(\theta)\tilde{M}_{22}^{-1}(\theta)M_{12}(\theta)^T)d\theta$$

Then we have that T is continuous and

$$\int_{-\tau}^{0} T(\theta)d\theta = \tau T_0 - \tau T_0 = 0$$

Which implies that $T \in \Omega$. We now show that $T_0 \geq 0$. For any constant vector $z_0 = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T$, suppose that z is a continuous function such that

$$z(0) = z_1 + \tilde{M}_{22}(0)^{-1} M_{12}(0)^T z_0$$
$$z(-\tau) = z_2 + \tilde{M}_{22}(-\tau)^{-1} M_{12}(-\tau)^T z_0.$$

Then, let $x(\theta) = z(\theta) - \tilde{M}_{22}(\theta)^{-1} M_{12}(\theta)^T z_0$. Then x is continuous, $\begin{bmatrix} x(0) & x(-\tau) \end{bmatrix}^T = z_0$ and by statement 1 we have the following.

$$\begin{aligned}
&\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(-\tau) \\ x(\theta) \end{bmatrix}^T \begin{bmatrix} M_{11}(\theta) & M_{12}(\theta) \\ M_{12}(\theta)^T & \tilde{M}_{22}(\theta) \end{bmatrix} \begin{bmatrix} x(0) \\ x(-\tau) \\ x(\theta) \end{bmatrix} d\theta \\
&= \int_{-\tau}^{0} \begin{bmatrix} z_0 \\ z(\theta) \end{bmatrix}^T \begin{bmatrix} I & 0 \\ -\tilde{M}_{22}(\theta)^{-1} M_{12}(\theta)^T & I \end{bmatrix}^T \begin{bmatrix} M_{11}(\theta) & M_{12}(\theta) \\ M_{12}(\theta)^T & \tilde{M}_{22}(\theta) \end{bmatrix} \\
&\qquad \begin{bmatrix} I & 0 \\ -\tilde{M}_{22}(\theta)^{-1} M_{12}(\theta)^T & I \end{bmatrix} \begin{bmatrix} z_0 \\ z(\theta) \end{bmatrix} d\theta \\
&= \int_{-\tau}^{0} \begin{bmatrix} z_0 \\ z(\theta) \end{bmatrix}^T \begin{bmatrix} M_{11}(\theta) - M_{12}(\theta) \tilde{M}_{22}^{-1}(\theta) M_{12}(\theta)^T & 0 \\ 0 & \tilde{M}_{22}(\theta) \end{bmatrix} \begin{bmatrix} z_0 \\ z(\theta) \end{bmatrix} d\theta \\
&= z_0^T \left(\int_{-\tau}^{0} M_{11}(\theta) - M_{12}(\theta) \tilde{M}_{22}^{-1}(\theta) M_{12}(\theta)^T d\theta \right) z_0 + \int_{-\tau}^{0} z(\theta)^T \tilde{M}_{22}(\theta) z(\theta) d\theta \\
&= z_0^T T_0 z_0 + \int_{-\tau}^{0} z(\theta)^T \tilde{M}_{22}(\theta) z(\theta) d\theta \geq \epsilon' \|x\|_2^2
\end{aligned}$$

We now show that this implies that $T_0 \geq 0$. Suppose there exists some y such that $y^T T_0 y < 0$. Then there exists some z_0 such that $z_0 T_0 z_0 = -1$. Now let $z_0 = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T$,

$c_1 = z_1 + \tilde{M}_{22}(0)^{-1}M_{12}(0)^T z_0$, $c_2 = z_2 + \tilde{M}_{22}(-\tau)^{-1}M_{12}(-\tau)^T z_0$, $\alpha > 2/\tau$ and

$$z(\theta) = \begin{cases} c_1(1+\alpha\theta) & \theta \in [-1/\alpha, 0] \\ c_2(1-(\theta+\tau)\alpha) & \theta \in [-\tau, -\tau+1/\alpha] \\ 0 & \text{otherwise} \end{cases}.$$

Then z is continuous, $z(0) = c_1 = z_1 + \tilde{M}_{22}(0)^{-1}M_{12}(0)^T z_0$, $z(-\tau) = c_2 = z_2 + \tilde{M}_{22}(-\tau)^{-1}M_{12}(-\tau)^T z_0$, and $\|z(\theta)\|^2 \le c = \max\{\|c_1\|^2, \|c_2\|^2\}$ for all $\theta \in [-\tau, 0]$. Recall that $\tilde{M}_{22}(\theta) \le (\epsilon_2 + \epsilon')I = \epsilon_3 I$. Let $\alpha > 2\epsilon_3 c$ and then we have

$$\begin{aligned} z_0^T T_0 z_0 + \int_{-\tau}^{0} z(\theta)^T M_{22}(\theta) z(\theta) d\theta &\le -1 + \epsilon_3 \int_{-1/\alpha}^{0} \|z(\theta)\|^2 \\ &\le -1 + \epsilon_3 c/\alpha < -\frac{1}{2}. \end{aligned}$$

This contradicts the previous statement. Thus we have by contradiction that $T_0 \ge 0$. Now by using the invertibility of $\tilde{M}_{22} = M_{22}(\theta) - \epsilon' I \ge \epsilon' I$ and the Schur complement transformation, we have that statement 2 is equivalent to the following.

$$\begin{aligned} &\tilde{M}_{22}(\theta) \ge 0 \\ &M_{11}(\theta) + T(\theta) - M_{12}(\theta)\tilde{M}_{22}(\theta)^{-1}M_{12}(\theta)^T \ge 0 \end{aligned}$$

The first condition has already been proven. Finally, we have the following.

$$M_{11}(\theta) + T(\theta) - M_{12}(\theta)\tilde{M}_{22}(\theta)^{-1}M_{12}(\theta)^T = T_0 \ge 0$$

Thus we have shown that statement 2 is true. ■

Lemma 125. *Suppose M is a matrix valued function which is continuous except possibly at points $\{\tau_i\}_{i=1}^{K-1}$ where $\tau_i \in [-\tau_K, 0]$. Then the following are equivalent.*

1. *The following holds for all* $x \in \mathcal{C}_{\tau_K}$*.*

$$\int_{-\tau_K}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} \geq 0$$

2. *The following holds for all* x *where* x *is continuous except possibly at points* $\{-\tau_i\}_{i=1}^{K-1}$*.*

$$\int_{-\tau_K}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} \geq 0$$

Proof. Clearly statement 2 implies statement 1. Now suppose statement 2 is false. Then there exists some piecewise continuous x , $\epsilon > 0$ such that

$$V_1 = \int_{-\tau_K}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} M(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta \leq -\epsilon.$$

Now let

$$\hat{x}(\theta) = \begin{cases} x(-\tau_i - \frac{1}{\beta}) + (x(-\tau_i + \frac{1}{\beta}) - x(-\tau_i - \frac{1}{\beta}))(\beta(\theta + \tau_i + \frac{1}{\beta})/2) & \theta \in [-\tau_i - \frac{1}{\beta}, -\tau_i + \frac{1}{\beta}], \\ & i = 1, \ldots, K-1 \\ x(\theta) & \text{otherwise.} \end{cases}$$

Then $\hat{x}$ is continuous. Since x is piecewise continuous on $[-\tau_K, 0]$, it is bounded on this interval. Now let $c = \max_{\theta \in [-\tau_K, 0]} \|x(\theta)\|^2$. Then $\|\hat{x}(\theta)\|^2 \leq c$ for all $\theta \in [-\tau_k, 0]$. Now since M is piecewise continuous on $[-\tau_k, 0]$, there exists some $\epsilon' > 0$ such that

$M(\theta) < \epsilon' I$ for $\theta \in [-\tau_k, 0]$. Therefore we have the following.

$$\begin{aligned}
&\int_{-\tau}^{0} \begin{bmatrix} \hat{x}(0) \\ \hat{x}(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} \hat{x}(0) \\ \hat{x}(\theta) \end{bmatrix} d\theta \\
&= V_1 + \sum_{i=1}^{K-1} \int_{-\tau_i-1/\beta}^{-\tau_i+1/\beta} \left(\begin{bmatrix} \hat{x}(0) \\ \hat{x}(\theta) \end{bmatrix} M(\theta) \begin{bmatrix} \hat{x}(0) \\ \hat{x}(\theta) \end{bmatrix} - \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} M(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} \right) d\theta \\
&\le V_1 + \sum_{i=1}^{K-1} \int_{-\tau_i-1/\beta}^{-\tau_i+1/\beta} \epsilon' \left(\left(\|\hat{x}(\theta)\|^2 + \|x(0)\|^2 \right) + \left(\|x(\theta)\|^2 + \|x(0)\|^2 \right) \right) d\theta \\
&\le V_1 + 2\epsilon'(K-1)(4c)/\beta \\
&\le -\epsilon + 8\epsilon'(K-1)c/\beta \\
&\le -\epsilon/2
\end{aligned}$$

Which holds for $\beta \geq \frac{16\epsilon'(K-1)c}{\epsilon}$. Thus statement 2 is false implies statement 1 is false. Therefore statement 1 implies statement 2. ∎

Lemma 126. *Suppose $S_i : \mathbb{R} \mapsto \mathbb{S}^{2n}$, $i = 1 \ldots K$ are continuous symmetric matrix valued functions with domains $[-\tau_i, -\tau_{i-1}]$ where $\tau_K > \tau_i > \tau_{i-1} > \tau_0 = 0$ for $i = 2 \ldots K-1$. Then the following are equivalent.*

1. *There exists an $\epsilon > 0$ such that the following holds for all $x \in \mathcal{C}_{\tau_K}$.*

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} S_i(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta \geq \epsilon \|x\|_2^2$$

2. *There exists an $\epsilon' > 0$ and continuous symmetric matrix valued functions, $T_i : \mathbb{R} \mapsto \mathbb{S}^n$, such that*

$$S_i(\theta) + \begin{bmatrix} T_i(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \geq 0 \qquad \text{for } \theta \in [-\tau_i, -\tau_{i-1}] \qquad i = 1 \ldots K$$

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} T_i(\theta) = 0.$$

Proof. $(2 \Rightarrow 1)$ Suppose there exist continuous symmetric matrix valued functions, $T_i : \mathbb{R} \mapsto \mathbb{S}^n$, such that

$$S_i(\theta) + \begin{bmatrix} T_i(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \geq 0 \qquad i = 1 \ldots K$$
$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} T_i(\theta) = 0.$$

Then

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} S_i(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta - \epsilon' \|x\|_2^2$$
$$= \sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} S_i(\theta) + \begin{bmatrix} T_i(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta \geq 0.$$

$(1 \Rightarrow 2)$ Suppose that statement 1 holds for some S_i. Write S_i as

$$S_i(\theta) = \begin{bmatrix} M_{11i}(\theta) & M_{12i}(\theta) \\ M_{12i}(\theta)^T & M_{22i}(\theta) \end{bmatrix}.$$

We first prove that $M_{22i}(\theta) \geq \epsilon I$ for all $\theta \in [-\tau_i, -\tau_{i-1}]$, $i = 1, \ldots, K$. By statement 1, we have that

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} \begin{bmatrix} M_{11i}(\theta) & M_{12i}(\theta) \\ M_{12i}(\theta)^T & M_{22i}(\theta) - \epsilon I \end{bmatrix} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta \geq 0.$$

Now suppose that $M_{22i}(\theta) - \epsilon I$ is not positive semidefinite for all $\theta \in [-\tau_i, -\tau_{i-1}]$. Then there exists some $x_0 \in \mathbb{R}^n$ and $\theta_1 \in [-\tau_i, -\tau_{i-1}]$ such that $x_0^T(M_{22i}(\theta_1) - \epsilon I)x_0 < 0$. By continuity of M_{22i}, if $\theta_1 = -\tau_i$ or $\theta_1 = -\tau_{i-1}$, then there exists some $\theta_1' \in (-\tau_i, -\tau_{i-1})$ such that $x_0^T(M_{22i}(\theta_1') - \epsilon I)x_0 < 0$. Thus assume $\theta_1 \in (-\tau_i, -\tau_{i-1})$. Now, since M_{22i} is continuous, there exists some x_1 and $\delta > 0$ where $\theta_1 + \delta < -\tau_{i-1}$, $\theta_1 - \delta > -\tau_i$ and such that $x_1^T(M_{22i}(\theta) - \epsilon I)x_1 \leq -1$ for $\theta \in [\theta_1 - \delta, \theta_1 + \delta]$. Then for

$\beta > \max\{1/(-\tau_i - \theta_1 - \delta), 1/(\tau_i + \theta_1 - \delta)\}$, let

$$x(\theta) = \begin{cases} \beta(\theta - (\theta_1 - \delta - 1/\beta))x_1 & \theta \in [\theta_1 - \delta - 1/\beta, \theta_1 - \delta] \\ x_1 & \theta \in [\theta_1 - \delta, \theta_1 + \delta] \\ (1 - \beta(\theta - (\theta_1 + \delta)))x_1 & \theta \in [\theta_1 + \delta, \theta_1 + \delta + 1/\beta] \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in \mathcal{C}_{\tau_K}$, $x(0) = 0$ and $\|x(\theta)\|^2 \leq \|x_1\|^2$ for $\theta \in [-\tau_K, 0]$. Now, since every M_{22i} is continuous, they are bounded on $[-\tau_i, -\tau_{i-1}]$. Therefore, there exists some $\epsilon_2 > 0$ such that $M_{22i}(\theta) - \epsilon I \leq \epsilon_2 I$ for $\theta \in [-\tau_i, -\tau_{i-1}]$, $i = 1, \ldots, K$. Then let $\beta \geq 2\epsilon_2 \|x_1\|^2/\delta$ and we have the following.

$$\sum_{j=1}^{K} \int_{-\tau_j}^{-\tau_{j-1}} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T S_j(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta - \epsilon\|x\|^2 = \sum_{j=1}^{K} \int_{-\tau_j}^{-\tau_{j-1}} x(\theta)^T (M_{22j}(\theta) - \epsilon I) x(\theta) d\theta$$
$$= \int_{\theta_1-\delta}^{\theta_1+\delta} x_1^T (M_{22i}(\theta) - \epsilon I) x_1 d\theta + \int_{\theta_1-\delta-1/\beta}^{\theta_1-\delta} x(\theta)^T (M_{22i}(\theta) - \epsilon I) x(\theta) d\theta$$
$$+ \int_{\theta_1+\delta}^{\theta_1+\delta+1/\beta} x(\theta)^T (M_{22i}(\theta) - \epsilon I) x(\theta) d\theta$$
$$\leq -2\delta + \epsilon_2 \int_{\theta_1-\delta-1/\beta}^{\theta_1-\delta} \|x(\theta)\|^2 d\theta + \epsilon_2 \int_{\theta_1+\delta}^{\theta_1+\delta+1/\beta} \|x(\theta)\|^2 d\theta$$
$$\leq -2\delta + 2\epsilon_2 \|x_1\|^2/\beta \leq -\delta$$

Therefore, by contradiction, we have that $M_{22i}(\theta) \geq \epsilon I$ for all $\theta \in [-\tau_i, -\tau_{i-1}]$ and $i = 1, \ldots, K$. Now we define $\epsilon' = \epsilon/2$ and $\tilde{M}_{22i}(\theta) = M_{22i}(\theta) - \epsilon' I \geq \epsilon' I$. We now show that $\tilde{M}_{22i}(\theta)^{-1}$ is continuous. We first note that $\tilde{M}_{22i}(\theta)^{-1}$ is bounded, since

$$\begin{aligned} \tilde{M}_{22i}(\theta) &\geq \epsilon' I \\ &\Rightarrow I \geq \epsilon' \tilde{M}_{22i}(\theta)^{-1} \\ &\Rightarrow \tilde{M}_{22i}(\theta)^{-1} \leq \frac{1}{\epsilon'} I. \end{aligned}$$

Now since $\tilde{M}_{22i}$ is continuous, for any $\beta > 0$, there exists a $\delta > 0$ such that

$$|\theta_1 - \theta_2| \leq \delta \Rightarrow \|\tilde{M}_{22i}(\theta_1) - \tilde{M}_{22i}(\theta_2)\| \leq \beta.$$

Therefore, for any $\beta' > 0$, let δ be defined as above for $\beta = \beta'\epsilon'^2$. Then $|\theta_1 - \theta_2| \leq \delta$ implies the following.

$$\begin{aligned}
&\|\tilde{M}_{22i}(\theta_1)^{-1} - \tilde{M}_{22i}(\theta_2)^{-1}\| \\
&= \|\tilde{M}_{22i}(\theta_1)^{-1}\tilde{M}_{22i}(\theta_2)\tilde{M}_{22i}(\theta_2)^{-1} - \tilde{M}_{22i}(\theta_1)^{-1}\tilde{M}_{22i}(\theta_1)\tilde{M}_{22i}(\theta_2)^{-1}\| \\
&= \|\tilde{M}_{22i}(\theta_1)^{-1}\left(\tilde{M}_{22i}(\theta_2) - \tilde{M}_{22i}(\theta_1)\right)\tilde{M}_{22i}(\theta_2)^{-1}\| \\
&\leq \|\tilde{M}_{22i}(\theta_1)^{-1}\|\|\tilde{M}_{22i}(\theta_2) - \tilde{M}_{22i}(\theta_1)\|\|\tilde{M}_{22i}(\theta_2)^{-1}\| \\
&\leq \frac{1}{\epsilon'^2}\|\tilde{M}_{22i}(\theta_2) - \tilde{M}_{22i}(\theta_1)\| \leq \beta'
\end{aligned}$$

Therefore $\tilde{M}_{22i}(\theta)^{-1}$ is continuous.

We now prove statement 2 by construction. Let

$$T_i(\theta) = T_0 - (M_{11i}(\theta) - M_{12i}(\theta)\tilde{M}_{22i}^{-1}(\theta)M_{12i}(\theta)^T),$$

where

$$T_0 = \frac{1}{\tau_K}\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}(M_{11i}(\theta) - M_{12i}(\theta)\tilde{M}_{22i}^{-1}(\theta)M_{12i}(\theta)^T)d\theta.$$

Then we have that T_i is continuous and

$$\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}T_i(\theta)d\theta = \tau_K T_0 - \tau_K T_0 = 0.$$

We now show that $T_0 \geq 0$. For any constant vector z_0, let z be a continuous function except possibly at points $\{-\tau_i\}_{i=1}^{K-1}$ and such that $z(0) = (I + \tilde{M}_{221}(0)^{-1}M_{121}(0)^T)z_0$, let $x(\theta) = z(\theta) - \tilde{M}_{22i}(\theta)^{-1}M_{12i}(\theta)^T z_0$ for $\theta \in [-\tau_i, -\tau_{i-1}]$. Then x is continuous except possibly at points $\{-\tau_i\}_{i=1}^{K-1}$, $x(0) = z_0$ and by statement 1, and Lemma 125 we have the following.

$$\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}\begin{bmatrix}x(0)\\x(\theta)\end{bmatrix}^T\begin{bmatrix}M_{11i}(\theta) & M_{12i}(\theta)\\M_{12i}(\theta)^T & \tilde{M}_{22i}(\theta)\end{bmatrix}\begin{bmatrix}x(0)\\x(\theta)\end{bmatrix}d\theta$$

$$=\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}\begin{bmatrix}z_0\\z(\theta)\end{bmatrix}^T\begin{bmatrix}I & 0\\-\tilde{M}_{22i}(\theta)^{-1}M_{12i}(\theta)^T & I\end{bmatrix}^T\begin{bmatrix}M_{11i}(\theta) & M_{12i}(\theta)\\M_{12i}(\theta)^T & \tilde{M}_{22i}(\theta)\end{bmatrix}$$

$$\begin{bmatrix}I & 0\\-\tilde{M}_{22i}(\theta)^{-1}M_{12i}(\theta)^T & I\end{bmatrix}\begin{bmatrix}z_0\\z(\theta)\end{bmatrix}d\theta$$

$$=\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}\begin{bmatrix}z_0\\z(\theta)\end{bmatrix}^T\begin{bmatrix}M_{11i}(\theta)-M_{12i}(\theta)\tilde{M}_{22i}^{-1}(\theta)M_{12i}(\theta)^T & 0\\0 & \tilde{M}_{22i}(\theta)\end{bmatrix}\begin{bmatrix}z_0\\z(\theta)\end{bmatrix}d\theta$$

$$=z_0^T\left(\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}M_{11i}(\theta)-M_{12i}(\theta)\tilde{M}_{22i}^{-1}(\theta)M_{12i}(\theta)^Td\theta\right)z_0+\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}z(\theta)^T\tilde{M}_{22i}(\theta)z(\theta)d\theta$$

$$=z_0^TT_0z_0+\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}z(\theta)^T\tilde{M}_{22i}(\theta)z(\theta)d\theta\geq\epsilon'\|x\|_2^2$$

We now show that this implies that $T_0 \geq 0$. Suppose there exists some y such that $y^TT_0y < 0$. Then there exists some z_0 such that $z_0T_0z_0 = -1$. Now let $\alpha > 1/\tau_1$ and

$$z(\theta)=\begin{cases}(I+\tilde{M}_{221}(0)^{-1}M_{121}(0)^T)z_0(1+\alpha\theta) & \theta\in[-1/\alpha,0]\\0 & \text{otherwise}\end{cases}.$$

Then z is continuous, $z(0) = (I + \tilde{M}_{221}(0)^{-1}M_{121}(0)^T)z_0$ and $\|z(\theta)\|^2 \leq \|z(0)\|^2$ for all $\theta \in [-\tau, 0]$. Recall that $\tilde{M}_{22i}(\theta) \leq (\epsilon_2 + \epsilon')I = \epsilon_3 I$. Let $\alpha > 2\epsilon_3\|z(0)^2\|$ and then we have

$$\begin{aligned}z_0^TT_0z_0+\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}z(\theta)^T\tilde{M}_{22i}(\theta)z(\theta)d\theta &\leq -1+\epsilon_3\int_{-1/\alpha}^{0}\|z(\theta)\|^2\\&\leq -1+\epsilon_3\|z(0)\|^2/\alpha < -\frac{1}{2}.\end{aligned}$$

But this is in contradiction to the previous relation. Thus we have by contradiction that $T_0 \geq 0$. Now by using the invertibility of $\tilde{M}_{22i} = M_{22i}(\theta)-\epsilon' I \geq \epsilon' I$ and the Schur complement transformation, we have that statement 2 is equivalent to the following for $i = 1, \ldots, K$.

$$\begin{aligned} &\tilde{M}_{22i}(\theta) \geq 0 && \theta \in [-\tau_i, -\tau_{i-1}] \\ &M_{11i}(\theta) + T_i(\theta) - M_{12i}(\theta)\tilde{M}_{22i}(\theta)^{-1}M_{12i}(\theta)^T \geq 0 && \theta \in [-\tau_i, -\tau_{i-1}] \end{aligned}$$

We have already shown the first statement. Finally, we have the following for $i = 1 \ldots K$.

$$M_{11i}(\theta) + T_i(\theta) - M_{12i}(\theta)\tilde{M}_{22i}(\theta)^{-1}M_{12i}(\theta)^T = T_0 \geq 0$$

Thus we have shown that statement 2 is true. ∎

Lemma 127. *Suppose $\{M_i\}_{i=1}^K$ are continuous symmetric matrix valued functions with domains $[-\tau_i, -\tau_{i-1}]$ and $\tau_K > \tau_i > \tau_{i-1} > \tau_0 = 0$ for $i = 2 \ldots K-1$. Then the following are equivalent.*

1. *The following holds for all $x \in \mathcal{C}_{\tau_K}$.*

$$\sum_{i=1}^K \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(-\tau_0) \\ \vdots \\ x(-\tau_K) \\ x(\theta) \end{bmatrix}^T M_i(\theta) \begin{bmatrix} x(-\tau_0) \\ \vdots \\ x(-\tau_K) \\ x(\theta) \end{bmatrix} d\theta \geq 0$$

2. *The following holds for all x where x is continuous except possibly at points $\{-\tau_i\}_{i=1}^{K-1}$.*

$$\sum_{i=1}^K \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(-\tau_0)^- \\ x(-\tau_1)^+ \\ \vdots \\ x(-\tau_K)^+ \\ x(\theta) \end{bmatrix}^T M_i(\theta) \begin{bmatrix} x(-\tau_0)^- \\ x(-\tau_1)^+ \\ \vdots \\ x(-\tau_K)^+ \\ x(\theta) \end{bmatrix} d\theta \geq 0$$

Where $x(\tau)^-$ denotes the left-handed limit and $x(\tau)^+$ denotes the right-handed limit.

Proof. Clearly statement 2 implies statement 1. Now suppose statement 2 is false. Then there exists some piecewise continuous x , and $\epsilon > 0$ such that

$$V_1 = \sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(-\tau_0)^- \\ x(-\tau_1)^+ \\ \vdots \\ x(-\tau_K)^+ \\ x(\theta) \end{bmatrix}^T M_i(\theta) \begin{bmatrix} x(-\tau_0)^- \\ x(-\tau_1)^+ \\ \vdots \\ x(-\tau_K)^+ \\ x(\theta) \end{bmatrix} d\theta \leq -\epsilon.$$

Now let $\beta > \max_{i=1,\ldots,K}\{2/(\tau_i - \tau_{i-1})\}$ and

$$\hat{x}(\theta) = \begin{cases} x(-\tau_i - \frac{1}{\beta}) + (x(-\tau_i)^+ - x(-\tau_i - \frac{1}{\beta}))\beta(\theta + \tau_i + \frac{1}{\beta}) & \begin{array}{c}\theta \in [-\tau_i - \frac{1}{\beta}, -\tau_i], \\ i = 1, \ldots, K-1\end{array} \\ x(\theta) & \text{otherwise.} \end{cases}$$

Then $\hat{x}$ is continuous on $[-\tau_K, 0]$, $\hat{x}(0) = x(0)^-$ and $\hat{x}(-\tau_i) = x(-\tau_i)^+$ for $i = 1, \ldots, K$. Since x is piecewise continuous on $[-\tau_K, 0]$, it is bounded on this interval. Now let $c = \max_{\theta\in[-\tau_K,0]} \|x(\theta)\|^2$. Then $\|\hat{x}(\theta)\|^2 \leq c$ for all $\theta \in [-\tau_k, 0]$. Now since M is piecewise continuous on $[-\tau_k, 0]$, there exists some $\epsilon' > 0$ such that $M(\theta) < \epsilon' I$

for $\theta \in [-\tau_k, 0]$. Therefore we have the following.

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} \hat{x}(-\tau_0) \\ \vdots \\ \hat{x}(-\tau_K) \\ \hat{x}(\theta) \end{bmatrix}^T M_i(\theta) \begin{bmatrix} \hat{x}(-\tau_0) \\ \vdots \\ \hat{x}(-\tau_K) \\ \hat{x}(\theta) \end{bmatrix} d\theta$$

$$= \sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(-\tau_0)^- \\ x(-\tau_1)^+ \\ \vdots \\ x(-\tau_K)^+ \\ \hat{x}(\theta) \end{bmatrix}^T M_i(\theta) \begin{bmatrix} x(-\tau_0)^- \\ x(-\tau_1)^+ \\ \vdots \\ x(-\tau_K)^+ \\ \hat{x}(\theta) \end{bmatrix} d\theta$$

$$= V_1 + \sum_{i=1}^{K-1} \int_{-\tau_i-1/\beta}^{-\tau_i} \left(\begin{bmatrix} x(-\tau_0)^- \\ x(-\tau_1)^+ \\ \vdots \\ x(-\tau_K)^+ \\ \hat{x}(\theta) \end{bmatrix}^T M_{i+1}(\theta) \begin{bmatrix} x(-\tau_0)^- \\ x(-\tau_1)^+ \\ \vdots \\ x(-\tau_K)^+ \\ \hat{x}(\theta) \end{bmatrix} - \begin{bmatrix} x(-\tau_0)^- \\ x(-\tau_1)^+ \\ \vdots \\ x(-\tau_K)^+ \\ x(\theta) \end{bmatrix}^T M_{i+1}(\theta) \begin{bmatrix} x(-\tau_0)^- \\ x(-\tau_1)^+ \\ \vdots \\ x(-\tau_K)^+ \\ x(\theta) \end{bmatrix} \right) d\theta$$

$$\le V_1 + \sum_{i=1}^{K-1} \int_{-\tau_i-1/\beta}^{-\tau_i} \epsilon' \left(\left(\|\hat{x}(\theta)\|^2 + \sum_{j=0}^{K} \|\hat{x}(-\tau_j)\|^2 \right) + \left(\|x(\theta)\|^2 + \sum_{j=0}^{K} \|\hat{x}(-\tau_j)\|^2 \right) \right) d\theta$$

$$\le V_1 + \epsilon'(K-1)(2c + 2(K+1)c)/\beta$$

$$\le -\epsilon + 2\epsilon'(K-1)(K+2)/\beta$$

$$\le -\epsilon/2$$

Which holds for $\beta \geq \frac{4\epsilon'(K-1)(K+2)}{\epsilon}$. Thus statement 2 is false implies statement 1 is false. Therefore statement 1 implies statement 2. ■

Lemma 128. *Suppose* $S_i : \mathbb{R} \mapsto \mathbb{S}^{n(K+2)}$ *are continuous matrix valued functions with domains* $[-\tau_i, -\tau_{i-1}]$ *for* $i = 1, \ldots, K$ *where* $\tau_K > \tau_i > \tau_{i-1} > \tau_0 = 0$ *for* $i = 2, \ldots, K-1$. *Then the following are equivalent.*

1. *There exists an* $\epsilon > 0$ *such that the following holds for all* $x \in \mathcal{C}_{\tau_K}$.

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(-\tau_0) \\ \vdots \\ x(-\tau_K) \\ x(\theta) \end{bmatrix}^T S_i(\theta) \begin{bmatrix} x(-\tau_0) \\ \vdots \\ x(-\tau_K) \\ x(\theta) \end{bmatrix} d\theta \geq \epsilon \|x\|_2^2$$

2. *There exists an* $\epsilon' > 0$ *and continuous matrix valued functions* $T_i : \mathbb{R} \mapsto \mathbb{S}^{n(K+1)}$ *such that*

$$S_i(\theta) + \begin{bmatrix} T_i(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \geq 0 \qquad \theta \in [-\tau_i, -\tau_{i-1}],\ i = 1 \ldots K$$
$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} T_i(\theta) = 0.$$

Proof. ($2 \Rightarrow 1$) Suppose there exist continuous symmetric matrix valued functions, $T_i : \mathbb{R} \mapsto \mathbb{S}^{n(K+1)}$, such that

$$S_i(\theta) + \begin{bmatrix} T_i(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \geq 0 \qquad i = 1 \ldots K$$
$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} T_i(\theta) = 0.$$

Then

$$\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(-\tau_0)\\ \vdots \\ x(-\tau_K)\\ x(\theta)\end{bmatrix}^T S_i(\theta) \begin{bmatrix} x(-\tau_0)\\ \vdots \\ x(-\tau_K)\\ x(\theta)\end{bmatrix} d\theta - \epsilon\|x\|_2^2$$

$$= \sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(-\tau_0)\\ \vdots \\ x(-\tau_K)\\ x(\theta)\end{bmatrix}^T \left(S_i(\theta) - \begin{bmatrix} T_i(\theta) & 0 \\ 0 & -\epsilon' I\end{bmatrix}\right) \begin{bmatrix} x(-\tau_0)\\ \vdots \\ x(-\tau_K)\\ x(\theta)\end{bmatrix} d\theta \geq 0.$$

($1 \Rightarrow 2$) Suppose that statement 1 holds for some S_i. Write S_i as

$$S_i(\theta) = \begin{bmatrix} M_{11i}(\theta) & M_{12i}(\theta) \\ M_{12i}(\theta)^T & M_{22i}(\theta)\end{bmatrix},$$

where $M_{22i} : \mathbb{R} \mapsto \mathbb{S}^n$. We first prove that $M_{22i}(\theta) \geq \epsilon I$ for all $\theta \in [-\tau_i, -\tau_{i-1}]$, $i = 1, \ldots, K$. By statement 1, we have that

$$\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(-\tau_0)\\ \vdots \\ x(-\tau_K)\\ x(\theta)\end{bmatrix}^T \begin{bmatrix} M_{11i}(\theta) & M_{12i}(\theta) \\ M_{12i}(\theta)^T & M_{22i}(\theta) - \epsilon I\end{bmatrix} \begin{bmatrix} x(-\tau_0)\\ \vdots \\ x(-\tau_K)\\ x(\theta)\end{bmatrix} d\theta \geq 0.$$

Now suppose that $M_{22i}(\theta) - \epsilon I$ is not positive semidefinite for all $\theta \in [-\tau_i, -\tau_{i-1}]$. Then there exists some $x_0 \in \mathbb{R}^n$ and $\theta_1 \in [-\tau_i, -\tau_{i-1}]$ such that $x_0^T(M_{22i}(\theta_1) - \epsilon I)x_0 < 0$. By continuity of M_{22i}, if $\theta_1 = -\tau_i$ or $\theta_1 = -\tau_{i-1}$, then there exists some $\theta_1' \in (-\tau_i, -\tau_{i-1})$ such that $x_0^T(M_{22i}(\theta_1') - \epsilon I)x_0 < 0$. Thus assume $\theta_1 \in (-\tau_i, -\tau_{i-1})$. Now, since M_{22i} is continuous, there exists some x_1 and $\delta > 0$ where $\theta_1 + \delta < -\tau_{i-1}$, $\theta_1 - \delta > -\tau_i$ and such that $x_1^T(M_{22i}(\theta) - \epsilon I)x_1 \leq -1$ for $\theta \in [\theta_1 - \delta, \theta_1 + \delta]$. Then for

$\beta > \max\{1/(-\tau_i - \theta_1 - \delta), 1/(\tau_i + \theta_1 - \delta)\}$, let

$$x(\theta) = \begin{cases} \beta(\theta - (\theta_1 - \delta - 1/\beta))x_1 & \theta \in [\theta_1 - \delta - 1/\beta, \theta_1 - \delta] \\ x_1 & \theta \in [\theta_1 - \delta, \theta_1 + \delta] \\ (1 - \beta(\theta - (\theta_1 + \delta)))x_1 & \theta \in [\theta_1 + \delta, \theta_1 + \delta + 1/\beta] \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in \mathcal{C}_\tau$, $x(-\tau_i) = 0$ for $i = 0, \ldots, K$ and $\|x(\theta)\|^2 \leq \|x_1\|^2$ for all $\theta \in [-\tau_K, 0]$. Now, since every M_{22i} is continuous, each are bounded on $[-\tau_i, -\tau_{i-1}]$. Therefore, there exists some $\epsilon_2 > 0$ such that $M_{22i}(\theta) - \epsilon I \leq \epsilon_2 I$ for $\theta \in [-\tau_i, -\tau_{i-1}]$, $i = 1, \ldots, K$. Then let $\beta \geq 2\epsilon_2\|x_1\|^2/\delta$ and we have the following.

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(-\tau_0) \\ \vdots \\ x(-\tau_K) \\ x(\theta) \end{bmatrix}^T S_i(\theta) \begin{bmatrix} x(-\tau_0) \\ \vdots \\ x(-\tau_K) \\ x(\theta) \end{bmatrix} d\theta - \epsilon\|x\|^2$$
$$= \sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} x(\theta)^T (M_{22i}(\theta) - \epsilon I) x(\theta) d\theta$$
$$= \int_{\theta_1-\delta}^{\theta_1+\delta} x_1^T (M_{22i}(\theta) - \epsilon I) x_1 d\theta + \int_{\theta_1-\delta-1/\beta}^{\theta_1-\delta} x(\theta)^T (M_{22i}(\theta) - \epsilon I) x(\theta) d\theta$$
$$+ \int_{\theta_1+\delta}^{\theta_1+\delta+1/\beta} x(\theta)^T (M_{22i}(\theta) - \epsilon I) x(\theta) d\theta$$
$$\leq -2\delta + \epsilon_2 \int_{\theta_1-\delta-1/\beta}^{\theta_1-\delta} \|x(\theta)\|^2 d\theta + \epsilon_2 \int_{\theta_1+\delta}^{\theta_1+\delta+1/\beta} \|x(\theta)\|^2 d\theta$$
$$\leq -2\delta + 2\epsilon_2\|x_1\|^2/\beta \leq -\delta$$

Therefore, by contradiction, we have that $M_{22i}(\theta) \geq \epsilon I$ for all $\theta \in [-\tau_i, -\tau_{i-1}]$, $i = 1, \ldots, K$. Now we define $\epsilon' = \epsilon/2$ and $\tilde{M}_{22i}(\theta) = M_{22i}(\theta) - \epsilon' I \geq \epsilon' I$. We now show that $\tilde{M}_{22i}(\theta)^{-1}$ is continuous. We first note that $\tilde{M}_{22i}(\theta)^{-1}$ is bounded, since

$$\begin{aligned}\tilde{M}_{22i}(\theta) &\geq \epsilon' I \\ &\Rightarrow I \geq \epsilon' \tilde{M}_{22i}(\theta)^{-1} \\ &\Rightarrow \tilde{M}_{22i}(\theta)^{-1} \leq \frac{1}{\epsilon'} I.\end{aligned}$$

Now since $\tilde{M}_{22i}$ is continuous, for any $\beta > 0$, there exists a $\delta > 0$ such that

$$|\theta_1 - \theta_2| \leq \delta \Rightarrow \|\tilde{M}_{22i}(\theta_1) - \tilde{M}_{22i}(\theta_2)\| \leq \beta.$$

Therefore, for any $\beta' > 0$, let δ be defined as above for $\beta = \beta'\epsilon'^2$. Then $|\theta_1 - \theta_2| \leq \delta$ implies the following.

$$\begin{aligned}&\|\tilde{M}_{22i}(\theta_1)^{-1} - \tilde{M}_{22i}(\theta_2)^{-1}\| \\ &= \|\tilde{M}_{22i}(\theta_1)^{-1}\tilde{M}_{22i}(\theta_2)\tilde{M}_{22i}(\theta_2)^{-1} - \tilde{M}_{22i}(\theta_1)^{-1}\tilde{M}_{22i}(\theta_1)\tilde{M}_{22i}(\theta_2)^{-1}\| \\ &= \|\tilde{M}_{22i}(\theta_1)^{-1}\left(\tilde{M}_{22i}(\theta_2) - \tilde{M}_{22i}(\theta_1)\right)\tilde{M}_{22i}(\theta_2)^{-1}\| \\ &\leq \|\tilde{M}_{22i}(\theta_1)^{-1}\|\|\tilde{M}_{22i}(\theta_2) - \tilde{M}_{22i}(\theta_1)\|\|\tilde{M}_{22i}(\theta_2)^{-1}\| \\ &\leq \frac{1}{\epsilon'^2}\|\tilde{M}_{22i}(\theta_2) - \tilde{M}_{22i}(\theta_1)\| \leq \beta'\end{aligned}$$

Therefore $\tilde{M}_{22i}(\theta)^{-1}$ is continuous.

We now prove statement 2 by construction. Let

$$T_i(\theta) = T_0 - (M_{11i}(\theta) - M_{12i}(\theta)\tilde{M}_{22i}^{-1}(\theta)M_{12i}(\theta)^T)$$

Where

$$T_0 = \frac{1}{\tau_K}\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}(M_{11i}(\theta) - M_{12i}(\theta)\tilde{M}_{22i}^{-1}(\theta)M_{12i}(\theta)^T)d\theta$$

Then we have that T_i is continuous and

$$\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}} T_i(\theta)d\theta = \tau_K T_0 - \tau_K T_0 = 0.$$

Now for any constant vector $z_c = \begin{bmatrix} z_0 & \cdots z_K \end{bmatrix}^T$, suppose that z is a continuous function except possibly at points $\{-\tau_i\}_{i=1}^{K-1}$ such that

$$\begin{aligned} z(-\tau_i)^+ &= z_i + \tilde{M}_{22i}(-\tau_i)^{-1} M_{12i}(-\tau_i)^T z_c \qquad i = 1, \ldots, K, \\ z(-\tau_0)^- &= z_0 + \tilde{M}_{221}(-\tau_0)^{-1} M_{121}(-\tau_0)^T z_c. \end{aligned}$$

Then, let $x(\theta) = z(\theta) - \tilde{M}_{22i}(\theta)^{-1} M_{12i}(\theta)^T z_c$ for $\theta \in [-\tau_i, -\tau_{i-1}]$. Then x is continuous except possibly at points $\{-\tau_i\}_{i=1}^{K-1}$, $x(-\tau_0)^- = z(-\tau_0)^- - \tilde{M}_{221}(-\tau_0)^{-1} M_{121}(-\tau_0)^T z_c = z_0$, $x(-\tau_i)^+ = z(-\tau_i)^+ - \tilde{M}_{22i}(-\tau_i)^{-1} M_{12i}(-\tau_i)^T z_c = z_i$ for $i = 1, \ldots, K$ and by statement 1 and Lemma 127, we have the following.

$$\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(-\tau_0)^- \\ x(-\tau_1)^+ \\ \vdots \\ x(-\tau_K)^+ \\ x(\theta) \end{bmatrix}^T \begin{bmatrix} M_{11i}(\theta) & M_{12i}(\theta) \\ M_{12i}(\theta)^T & \tilde{M}_{22i}(\theta) \end{bmatrix} \begin{bmatrix} x(-\tau_0)^- \\ x(-\tau_1)^+ \\ \vdots \\ x(-\tau_K)^+ \\ x(\theta) \end{bmatrix} d\theta$$

$$= \sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} z_c \\ z(\theta) \end{bmatrix}^T \begin{bmatrix} I & 0 \\ -\tilde{M}_{22i}(\theta)^{-1}M_{12i}(\theta)^T & I \end{bmatrix}^T \begin{bmatrix} M_{11i}(\theta) & M_{12i}(\theta) \\ M_{12i}(\theta)^T & \tilde{M}_{22i}(\theta) \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ -\tilde{M}_{22i}(\theta)^{-1}M_{12i}(\theta)^T & I \end{bmatrix} \begin{bmatrix} z_c \\ z(\theta) \end{bmatrix} d\theta$$

$$= \sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} z_c \\ z(\theta) \end{bmatrix}^T \begin{bmatrix} M_{11i}(\theta) - M_{12i}(\theta)\tilde{M}_{22i}^{-1}(\theta)M_{12i}(\theta)^T & 0 \\ 0 & \tilde{M}_{22i}(\theta) \end{bmatrix} \begin{bmatrix} z_c \\ z(\theta) \end{bmatrix} d\theta$$

$$= z_c^T(\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}} M_{11i}(\theta) - M_{12i}(\theta)\tilde{M}_{22i}^{-1}(\theta)M_{12i}(\theta)^T d\theta)z_c$$

$$+\sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}} z(\theta)^T\tilde{M}_{22i}(\theta)z(\theta)d\theta$$

$$= z_c^T T_0 z_c + \sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}} z(\theta)^T\tilde{M}_{22i}(\theta)z(\theta)d\theta \geq \epsilon'\|x\|_2^2$$

We now show that this implies that $T_0 \geq 0$. Suppose there exists some y such that $y^T T_0 y < 0$. Then there exists some z_c such that $z_c T_0 z_c = -1$. Now let $\alpha > 2/\max_i\|\tau_i - \tau_{i-1}\|$ and

$$z(\theta) = \begin{cases} (z_0 + \tilde{M}_{221}(0)^{-1}M_{121}(0)^T z_c)(1+\alpha\theta) & \theta \in [-1/\alpha, 0] \\ (z_i + \tilde{M}_{22i}(-\tau_i)^{-1}M_{12i}(-\tau_i)^T z_c)(1+\alpha\theta) & \theta \in [-\tau_i, -\tau_i + 1/\alpha] \quad i = 1\ldots K \\ 0 & \text{otherwise.} \end{cases}$$

Then z is continuous except at points $\{-\tau_i\}_{i=1}^{K-1}$, $z(0)^- = z_0 + \tilde{M}_{221}(0)^{-1}M_{121}(0)^T z_c$,

$z(-\tau_i)^+ = z_i + \tilde{M}_{22i}(-\tau_i)^{-1}M_{12i}(-\tau_i)^T z_c$ and $\|z(\theta)\|^2 \le c$ for all $\theta \in [-\tau, 0]$ where

$$c = \max\left\{\max_{i=0\ldots K}\{\|z_i + \tilde{M}_{22i}(-\tau_i)^{-1}M_{12i}(-\tau_i)^T z_c\|^2\}, \|z_0 + \tilde{M}_{221}(0)^{-1}M_{121}(0)^T z_c\|\right\}.$$

Recall that $\tilde{M}_{22i}(\theta) \le (\epsilon_2 + \epsilon')I = \epsilon_3 I$. Let $\alpha > 2\epsilon_3 c(K+1)$ and then we have

$$\begin{aligned}
& z_0^T T_0 z_0 + \sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}} z(\theta)^T \tilde{M}_{22i}(\theta) z(\theta) d\theta \\
&= z_0^T T_0 z_0 + \int_{-1/\alpha}^{0} z(\theta)^T \tilde{M}_{221}(\theta) z(\theta) d\theta + \sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_i+1/\alpha} z(\theta)^T \tilde{M}_{22i}(\theta) z(\theta) d\theta \\
&\le -1 + \epsilon_3 \int_{-1/\alpha}^{0} \|z(\theta)\|^2 + \sum_{i=1}^{K} \epsilon_3 \int_{-\tau_i}^{-\tau_i+1/\alpha} \|z(\theta)\|^2 d\theta \\
&\le -1 + \epsilon_3 c(K+1)/\alpha < -\frac{1}{2}.
\end{aligned}$$

But this contradicts the previous relation. Thus we have by contradiction that $T_0 \ge 0$. Now by using the invertibility of $\tilde{M}_{22i} = M_{22i}(\theta) - \epsilon' I \ge \epsilon' I$ and the Schur complement transformation, we have that statement 2 is equivalent to the following for $i = 1, \ldots, K$.

$$\begin{aligned}
&\tilde{M}_{22i}(\theta) \ge 0 && \theta \in [-\tau_i, -\tau_{i-1}] \\
&M_{11i}(\theta) + T_i(\theta) - M_{12i}(\theta)\tilde{M}_{22i}(\theta)^{-1}M_{12i}(\theta)^T \ge 0 && \theta \in [-\tau_i, -\tau_{i-1}]
\end{aligned}$$

We have already proven the first statement. Finally, we have the following.

$$M_{11i}(\theta) + T_i(\theta) - M_{12i}(\theta)\tilde{M}_{22i}(\theta)^{-1}M_{12i}(\theta)^T = T_0 \ge 0$$

Thus we have shown that statement 2 is true. ∎

Lemma 129. *Suppose $M : \mathbb{R}^2 \mapsto \mathbb{R}^{n\times n}$ is a matrix valued function which is continuous except possibly at points $\{\tau_i\}_{i=1}^K$ where $\tau_i \le \tau_{i-1}$ for $i = 1, \ldots, K$ and $\tau_0 = 0$. Then the following are equivalent.*

1. *The following holds for all* $x \in \mathcal{C}_{\tau_K}$.

$$\int_{-\tau_K}^{0} \int_{-\tau_K}^{0} x(\theta)^T M(\theta, \omega) x(\theta) d\theta d\omega \geq 0$$

2. *The following holds for all* x *where* x *is continuous except possibly at points* $\{-\tau_i\}_{i=1}^{K-1}$.

$$\int_{-\tau_K}^{0} \int_{-\tau_K}^{0} x(\theta)^T M(\theta, \omega) x(\theta) d\theta d\omega \geq 0$$

Proof. Clearly statement 2 implies statement 1. Now suppose statement 2 is false. Then there exists some piecewise continuous x , $\epsilon > 0$ such that

$$V_1 = \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} x(\theta)^T M(\theta, \omega) x(\omega) d\theta d\omega \leq -\epsilon.$$

Now let

$$\hat{x}(\theta) = \begin{cases} x(-\tau_i - \frac{1}{\beta}) + \beta(x(-\tau_i + \frac{1}{\beta}) - x(-\tau_i - \frac{1}{\beta}))(\theta + \tau_i + \frac{1}{\beta})/2 & \theta \in [-\tau_i - \frac{1}{\beta}, -\tau_i + \frac{1}{\beta}] \\ & i = 1, \dots, K-1 \\ x(\theta) & \text{otherwise.} \end{cases}$$

Then $\hat{x}$ is continuous. Since x is piecewise continuous on $[-\tau_K, 0]$ it is bounded on this interval. Now let $c = \max_{\theta \in [-\tau_K, 0]} \|x(\theta)\|^2$. Then $\|\hat{x}(\theta)\|^2 \leq c$ for all $\theta \in [-\tau_K, 0]$. Now since M is piecewise continuous on $[-\tau_K, 0]$, it is bounded on this interval, i.e. there exists some $\epsilon' > 0$ such that $y^T M(\theta, \omega) z < \epsilon' \max\{\|y\|^2, \|z\|^2\}$ for $\theta, \omega \in [-\tau_K, 0]$.

Therefore we have the following.

$$\begin{aligned}
&\int_{-\tau_K}^{0}\int_{-\tau_K}^{0} \hat{x}(\theta)^T M(\theta,\omega)\hat{x}(\omega)d\theta d\omega \\
&= V_1 + \sum_{i,j=1}^{K-1}\int_{-\tau_i-1/\beta}^{-\tau_i+1/\beta}\int_{-\tau_j-1/\beta}^{-\tau_j+1/\beta}\left(\hat{x}(\theta)^T M(\theta,\omega)\hat{x}(\omega) - x(\theta)^T M(\theta,\omega)x(\omega)\right)d\theta d\omega \\
&\le V_1 + \sum_{i,j=1}^{K-1}\int_{-\tau_i-1/\beta}^{-\tau_i+1/\beta}\int_{-\tau_j-1/\beta}^{-\tau_j+1/\beta}(2\epsilon' c)d\theta d\omega \\
&= -\epsilon + 4\epsilon'(K-1)^2 c/\beta < -\epsilon/2
\end{aligned}$$

Which holds for $\beta \ge \frac{8\epsilon'(K-1)^2 c}{\epsilon}$. Thus statement 2 is false implies statement 1 is false. Therefore statement 1 implies statement 2. ∎

Lemma 130. *Let M be a matrix valued function $M : \mathbb{R}^2 \to \mathbb{S}^n$ which is discontinuous only at points $\theta, \omega = -\tau_i$ for $i = 1, \ldots, K-1$ where the τ_i are increasing and $\tau_0 = 0$. Then $M \in \tilde{H}_2^+$ if and only if there exists some continuous matrix valued function $R : \mathbb{R}^2 \to \mathbb{R}^{nK\times nK}$ such that $R \in H_2^+$ and the following holds where $I_i = [-\tau_i, -\tau_{i-1}]$, $\Delta_i = \tau_i - \tau_{i-1}$.*

$$\begin{aligned}
M(\theta,\omega) &= M_{ij}(\theta,\omega) \qquad \text{for all } \theta \in I_i, \quad \omega \in I_j \\
M_{ij}(\theta,\omega) &= R_{ij}\left(\frac{\tau_K}{\Delta_i}\theta + \tau_{i-1}\frac{\tau_K}{\Delta_i}, \frac{\tau_K}{\Delta_j}\omega + \tau_{j-1}\frac{\tau_K}{\Delta_j}\right) \\
R(\theta,\omega) &= \begin{bmatrix} R_{11}(\theta,\omega) & \ldots & R_{1K}(\theta,\omega) \\ \vdots & & \vdots \\ R_{K1}(\theta,\omega) & \ldots & R_{KK}(\theta,\omega) \end{bmatrix}
\end{aligned}$$

Proof. ($\Leftarrow$) Define $\theta_i(\theta) = \frac{\tau_K}{\Delta_i}\theta + \tau_{i-1}\frac{\tau_K}{\Delta_i}$ and $\omega_i(\omega) = \frac{\tau_K}{\Delta_i}\omega + \tau_{i-1}\frac{\tau_K}{\Delta_i}$. Then $\theta_i(-\tau_i) = -\tau_K$, $\theta_i(-\tau_{i-1}) = 0$, $\theta(\theta_i) = \frac{\Delta_i}{\tau_K}\theta_i - \tau_{i-1}$ and $\frac{d\theta}{d\theta_i} = \frac{\Delta_i}{\tau_K}$. The same relation holds between ω and ω_i. Because M is piecewise continuous, M_{ij} as defined above are continuous. Then for any $x \in \mathcal{C}_{\tau_K}$, let $x_i(\theta) = \frac{\Delta_i}{\tau_K}x\left(\frac{\Delta_i}{\tau_K}\theta - \tau_{i-1}\right)$. Then $x_i \in \mathcal{C}_{\tau_K}$ and we have the following.

$$\begin{aligned}
&\int_{-\tau_K}^{0}\int_{-\tau_K}^{0} x(\theta)^T M(\theta,\omega)x(\omega) \\
&= \sum_{i,j=1}^{K}\int_{I_i}\int_{I_j} x(\theta)^T M_{ij}(\theta,\omega)x(\omega)d\theta d\omega \\
&= \sum_{i,j=1}^{K}\int_{-\tau_K}^{0}\int_{-\tau_K}^{0} x_i(\theta_i)^T R_{ij}(\theta_i,\omega_j)x_j(\omega_j)d\theta_i d\omega_j \\
&= \int_{-\tau_K}^{0}\int_{-\tau_K}^{0} \begin{bmatrix} x_1(\theta) \\ \vdots \\ x_K(\theta)\end{bmatrix}^T R(\theta,\omega) \begin{bmatrix} x_1(\omega) \\ \vdots \\ x_K(\omega)\end{bmatrix} d\theta d\omega \geq 0
\end{aligned}$$

Thus $R \in H_2^+$ implies $M \in H_2^+$.

($\Rightarrow$) Now suppose $M \in H_2^+$. Then for any nK dimensional vector valued function in $\mathcal{C}_{\tau_K}$, $x(\theta) = \begin{bmatrix} x_1(\theta)^T & \cdots & x_K(\theta)^T \end{bmatrix}^T$, define $\tilde{x}(\theta) = \frac{\tau_K}{\Delta_i} x_i\left(\frac{\tau_K}{\Delta_i}\theta + \tau_{i-1}\frac{\tau_K}{\Delta_i}\right)$ for $\theta \in I_i$. Then $\tilde{x}$ is continuous except possibly at points $\{-\tau_i\}$ and we have the following.

$$\begin{aligned}
&\int_{-\tau_K}^{0}\int_{-\tau_K}^{0} x(\theta)^T R(\theta,\omega)x(\omega)d\theta d\omega \\
&= \sum_{i,j=1}^{K}\int_{-\tau_K}^{0}\int_{-\tau_K}^{0} x_i(\theta_i)^T R_{ij}(\theta_i,\omega_j)x_j(\omega_j)d\theta_i d\omega_j \\
&= \sum_{i,j=1}^{K}\int_{I_i}\int_{I_j} \tilde{x}(\theta)^T M_{ij}(\theta,\omega)\tilde{x}(\omega)d\theta d\omega \\
&= \int_{-\tau_K}^{0}\int_{-\tau_K}^{0} \tilde{x}(\theta)^T M(\theta,\omega)\tilde{x}(\omega)d\theta d\omega
\end{aligned}$$

where M is as defined above. Since M is bounded and positive on $x \in \mathcal{C}_{\tau_K}$, Lemma 129 shows that M is also positive on the space of functions which are continuous except possibly at points $\{-\tau_i\}_{i=1}^{K-1}$. Therefore, we have that $R \in H_2^+$. ∎

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