Using Polynomial Semi-Separable Kernels to Construct Infinite-Dimensional Lyapunov Functions

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***Abstract*— In this paper, we introduce the class of semi-separable kernel functions for use in constructing Lyapunov functions for distributed-parameter systems such as delay-differential and partial-differential equations. We then consider the subset of semi-separable kernel functions defined by polynomials. We show that the set of such kernels which define positive integral operators can be parameterized by positive semidefinite matrices. We also show that, unlike for the class of separable kernels, semi-separable kernels defined by polynomials correspond to integral operators which map to dense subspaces of L_2. This means that for a system, the existence of a Lyapunov function defined by a Gaussian-type kernel function implies the existence of a Lyapunov function defined by a polynomial semi-separable kernel function.**

I. Introduction

The area of time-delay systems has long been an active area of research. Recently there has been much work on the construction of Lyapunov functions for linear time-delay systems. As a light sampling of the work in this area, consult the works in []. Some fundamental results in Lyapunov theory for delayed systems are given in [5]. A broad overview of research in time-delay systems can be found in, e.g., [3] or [6].

This paper is the third by the authors in a series of 3 CDC papers addressing computational questions in Lyapunov theory for time-delay systems. These papers all propose new ways of testing the positivity of Lyapunov structures of the following form.

$$V(x) = \int_{-\tau_k}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} M(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta + \int_{-\tau_k}^{0}\int_{-\tau_k}^{0} x(\theta) N(\theta,\omega) x(\omega) d\theta d\omega$$

In [8], we considered the first half of the Lyapunov function and gave an exact characterization of positivity using pointwise inequality conditions. We then demonstrated how to enforce these positivity conditions using a sum-of-squares methodology. In the paper [9], we considered the second half of the function. In this paper, a necessary and sufficient condition was given for positivity under the assumption of a polynomial function N. The two papers were connected by a joint positivity condition.

In this paper, we address problems created by the assumption of a polynomial N.

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While it is known that the function M in the first part of the function may be assumed to be polynomial (See [7]), very little is known about the class of functions to which N belongs. In a series of papers, of which [4] is representative, examples of these functions are derived given desired forms of the derivative. These functions include exponential terms and although polynomials can approximate exponentials in the L_∞ norm, is is not clear whether a quadratic form defined by such an approximation will approximate the original quadratic form. In fact, when using the algorithms defined in papers [8], [9], we have found that polynomials N of degree higher than 1 provide no improvement in accuracy (See the numeric examples at the end of the paper). There are a number of explanations for this result, and some of them are discussed in section II-B. Perhaps the simplest argument, however, is that a quadratic form defined by a polynomial, N, will only be positive on a finite-dimensional subspace of $\mathcal{C}_\infty$. For these forms, there will always exist a dense subspace of continuous functions $x \neq 0$ such that

$$\int_{-h}^{0}\int_{-h}^{0} x(\theta) N(\theta,\omega) x(\omega) d\theta d\omega = 0.$$

In this paper, we address the problem of polynomial kernels by considering a class of functions known as *semi-separable* functions. If a function, N, is semi-separable, then it can be represented as

$$N(t,s) = \begin{cases} N_1(t)N_2(s) & s < t \\ N_3(t)N_4(s) & s \geq t. \end{cases}$$

Semi-separable functions defined by polynomials can define quadratic forms which are positive on dense subspaces of $\mathcal{C}_\infty$. Furthermore, numerical tests indicate that an increase in the degree of the polynomial will always result in an increase in the accuracy of the condition. It is interesting to note that the quadratic forms defined by semi-separable kernels have a structure similar to a quadratic form originally considered by Repin in [10].

The main result of this paper is a method, based on sums-of-squares, for parameterizing semi-separable kernel functions using semidefinite programming. The paper is organized as follows. In the first section, we parameterize positive semi-separable kernel functions using a sum-of-squares approach. In section 2, we motivate semi-separable kernels by proving properties of their behavior. In Section 3, we derive the derivative of a semi-separable kernel function for a single and for multiple delays. Finally, in Section 4, we illustrate the use of semi-separable kernels with numerical experiments.

II. Polynomial Matrices and Kernels

We will use a generalization of the framework as defined in[9]. First define the intervals

$$H_i = \begin{cases} [-h_1, 0] & \text{if } i = 1 \\ [-h_i, -h_{i-1}) & \text{if } i = 2, \dots, k. \end{cases}$$

A matrix-valued function $M : [-h, 0] \to \mathbb{S}^n$ is called a *piecewise polynomial matrix* if for each $i = 1, \dots, k$ the function M restricted to the interval H_i is a polynomial matrix. We parameterize such piecewise polynomial matrices as follows. Define the vector of indicator functions $g : [-h, 0] \to \mathbb{R}^k$ by

$$g_i(t) = \begin{cases} 1 & \text{if } t \in H_i \\ 0 & \text{otherwise} \end{cases}$$

for all $i = 1, \dots, k$ and all $t \in [-h, 0]$. Let $z_d(t)$ be the vector of monomials in variable t of degree d or less and also define the function $Z_{n,d} : [-h, 0] \to \mathbb{R}^{nk(d+1)\times n}$ by

$$Z_{n,d}(t) = g(t) \otimes I_n \otimes z(t).$$

M is a piecewise matrix polynomial if and only if there exist matrices $Q_i \in \mathbb{S}^{n(d+1)}$ for $i = 1, \dots, k$ such that

$$M(t) = Z_{n,d}(t)^T \operatorname{diag}(Q_1, \dots, Q_k) Z_{n,d}(t). \tag{1}$$

The function M is pointwise positive semidefinite, i.e.,

$$M(t) \succeq 0 \qquad \text{for all } t \in [-h, 0]$$

if there exists positive semidefinite matrices Q_i satisfying (1). We refer to such functions as *piecewise sum of squares matrices*, and define the set of such functions

$$\Sigma_{n,d} = \left\{ Z_{n,d}^T(t) Q Z_{n,d}(t) \mid \right. \\ \left. Q = \operatorname{diag}(Q_1, \dots, Q_k),\ Q_i \in \mathbb{S}^{n(d+1)}, Q_i \succeq 0 \right\}.$$

If we are given a function $M : [-h, 0] \to \mathbb{S}^n$ which is piecewise polynomial and want to know whether it is piecewise sum of squares, then this is computationally checkable using semidefinite programming. Naturally, the number of variables involved in this task scales as $kn^2(d+1)^2$ when the degree of M is $2d$.

A. Piecewise Polynomial Kernels

We consider functions N of two variables s, t which we will use as a kernel in the quadratic form

$$\int_{-h}^0 \int_{-h}^0 \phi(s)^T N(s,t)\, \phi(t)\, ds\, dt. \tag{2}$$

A polynomial in two variables is referred to as a *binary* polynomial. A function $N : [-h, 0] \times [-h, 0] \to \mathbb{S}^n$ is called a *binary piecewise polynomial matrix* if for each $i, j \in \{1, \dots, k\}$ the function N restricted to the set $H_i \times H_j$ is a binary polynomial matrix. It is straightforward to show that N is a symmetric binary piecewise polynomial matrix if and only if there exists a matrix $Q \in \mathbb{S}^{nk(d+1)}$ such that

$$N(s,t) = Z_{n,d}^T(s) Q Z_{n,d}(t),$$

where d is the degree of N. The following result appeared in [9].

Theorem 1: Suppose N is a symmetric binary piecewise polynomial matrix of degree $2d$. Then

$$\int_{-h}^0 \int_{-h}^0 \phi(s)^T N(s,t) \phi(t)\, ds\, dt \geq 0 \tag{3}$$

for all $\phi \in C([-h, 0], \mathbb{R}^n)$ if and only if there exists $Q \in \mathbb{S}^{nk(d+1)}$ such that

$$N(s,t) = Z_{n,d}^T(s) Q Z_{n,d}(t) \\ Q \succeq 0,$$

For convenience, we define the set of symmetric binary piecewise polynomial matrices which define positive quadratic forms by

$$\Gamma_{n,d} = \left\{ Z_{n,d}^T(s) Q Z_{n,d}(t) \mid Q \in \mathbb{S}^{nk(d+1)}, Q \succeq 0 \right\}.$$

If we are given a binary piecewise polynomial matrix $N : [-h, 0] \times [-h, 0] \to \mathbb{S}^n$ of degree $2d$ and want to know whether it defines a positive quadratic form, then this is checkable using semidefinite programming. The number of variables scales as $(nk)^2(d+1)^2$.

B. Piecewise Polynomial Semi-Separable Kernels

A function $N : [-h, 0] \times [-h, 0] \to \mathbb{S}^n$ is called a *piecewise polynomial semi-separable matrix* if the function $N(s,t)$ restricted to the $s \leq t$ or $s \geq t$ is a binary piecewise polynomial matrix. N is a piecewise polynomial semi-separable matrix if and only if there exist matrices $Q_1, Q_1 \in \mathbb{S}^{nk(d+1)}$ such that

$$N(s,t) = \begin{cases} Z_{n,d}^T(s) Q_1 Z_{n,d}(t) & s \leq t \\ Z_{n,d}^T(s) Q_2 Z_{n,d}(t) & s > t \end{cases},$$

A piecewise-polynomial semi-separable matrix defines a positive quadratic form if it has a "sum-of-squares" representation.

Theorem 2: Suppose

$$Q(s) \geq 0$$

Let Z be the standard vector of monomial bases. Now define N as follows

$$R(t,s,\omega) = \begin{bmatrix} R_{11}(t,s,\omega) & R_{12}(t,s,\omega) \\ R_{12}(t,s,\omega)^T & R_{22}(t,s,\omega) \end{bmatrix} \\ = Z_{2n,d}(t)^T Q(s) Z_{2n,d}(\omega)$$

Let

$$N(\omega,t) = \begin{cases} N_1(\omega,t) & \omega \leq t \\ N_2(\omega,t) & \omega > t, \end{cases}$$

where

$$N_1(\omega,t) = \int_{-h}^{\omega} R_{11}(t,s,\omega)\,ds + \int_{\omega}^{t} R_{21}(t,s,\omega)\,ds + \int_{t}^{0} R_{22}(t,s,\omega)\,ds,$$

$$N_2(\omega,t) = \int_{-h}^{t} R_{11}(t,s,\omega)\,ds + \int_{t}^{\omega} R_{12}(t,s,\omega)\,ds + \int_{\omega}^{0} R_{22}(t,s,\omega)\,ds.$$

Then for any $x \in \mathcal{C}([-h,0],\mathbb{R}^n)$

$$\int_{-h}^{0}\int_{-h}^{0} x(s)^T N(s,t)x(t)dsdt \geq 0$$

Proof: By positivity, there exists a $P(s)$ so that $Q(s) = P(s)^T P(s)$. Now equipartition

$$P(s)Z_{2n,d}(\omega) = \begin{bmatrix} K_1(s,\omega) & K_2(s,\omega) \end{bmatrix}.$$

Then

$$\begin{aligned} R(t,s,\omega) &= \left(P(s)Z_{2n,d}(t)\right)^T P(s)Z_{2n,d}(\omega) \\ &= \begin{bmatrix} K_1(s,t)^T \\ K_2(s,t)^T \end{bmatrix} \begin{bmatrix} K_1(s,\omega) & K_2(s,\omega) \end{bmatrix} \\ &= \begin{bmatrix} K_1(s,t)^T K_1(s,\omega) & K_1(s,t)^T K_2(s,\omega) \\ K_2(s,t)^T K_1(s,\omega) & K_2(s,t)^T K_2(s,\omega) \end{bmatrix} \\ &= \begin{bmatrix} R_{11}(t,s,\omega) & R_{12}(t,s,\omega) \\ R_{12}(t,s,\omega)^T & R_{22}(t,s,\omega) \end{bmatrix} \end{aligned}$$

Now let

$$K(s,t) = \begin{cases} K_1(s,t) & s \leq t \\ K_2(s,t) & s > t. \end{cases}$$

We define the integral operator A by $y = Ax$ if

$$y(s) = \int_{-h}^{0} K(s,t)x(t)\,dt.$$

Then

$$\begin{aligned} \langle y,y\rangle &= \int_{-h}^{0}\int_{-h}^{0}\int_{-h}^{0} x(\omega)^T K(s,\omega)^T K(s,t)x(t)\,dt\,ds\,d\omega \\ &= \int_{-h}^{0}\int_{-h}^{0} x(\omega)^T \left(\int_{-h}^{0} K(s,\omega)^T K(s,t)\,ds\right) x(t)\,dt\,d\omega. \end{aligned}$$

Now, for $\omega \leq t$

$$\begin{aligned} \int_{-h}^{0} K(s,\omega)^T K(s,t)\,ds &= \int_{-h}^{\omega} K_1(s,\omega)^T K_1(s,t)\,ds \\ &+ \int_{\omega}^{t} K_2(s,\omega)^T K_1(s,t)\,ds + \int_{t}^{0} K_2(s,\omega)^T K_2(s,t)\,ds, \\ &= N_1(\omega,t) \end{aligned}$$

Similarly, for $\omega > t$,

$$\begin{aligned} \int_{-h}^{0} K(s,\omega)^T K(s,t)\,ds &= \int_{-h}^{t} K_1(s,\omega)^T K_1(s,t)\,ds \\ &+ \int_{t}^{\omega} K_1(s,\omega)^T K_2(s,t)\,ds + \int_{\omega}^{0} K_2(s,\omega)^T K_2(s,t)\,ds \\ &= N_2(\omega,t). \end{aligned}$$

Therefore,

$$\langle y,y\rangle = \int_{-h}^{0}\int_{-h}^{0} x(\omega)^T N(s,t)x(t)\,dt\,d\omega \geq 0$$

■

For convenience, we define the set of piecewise polynomial semi-separable matrices which define positive quadratic forms by

$$\begin{aligned} &\Xi_{n,d,r} = \\ &\big\{(N_1,\, N_2) \mid \\ &\quad N_1(\omega,t) = \int_{-h}^{\omega} R_{11}(t,s,\omega)\,ds + \int_{\omega}^{t} R_{21}(t,s,\omega)\,ds \\ &\qquad + \int_{t}^{0} R_{22}(t,s,\omega)\,ds, \\ &\quad N_2(\omega,t) = \int_{-h}^{t} R_{11}(t,s,\omega)\,ds + \int_{t}^{\omega} R_{12}(t,s,\omega)\,ds \\ &\qquad + \int_{\omega}^{0} R_{22}(t,s,\omega)\,ds, \\ &\quad R_{11} \mapsto \mathbb{R}^{n\times n}, \\ &\quad R(t,s,\omega) = Z_{2n,d}(t)^T Q(s) Z_{2n,d}(\omega) \\ &\quad Q(s) = Z_{2nk(d+1),r}(s)^T P Z_{2nk(d+1),r}(s) \\ &\quad P = \operatorname{diag}(P_1,\cdots,P_k) \\ &\quad P_i \in \mathbb{S}^{4nk(d+1)(r+1)},\, P_i \succeq 0\,\big\}. \end{aligned}$$

Here r is the degree of the SOS representation and d is the degree of the kernel matrix. In practice, the complexity can be reduced by separating the kernel into a piecewise polynomial component and k continuous semi-separable components. See Section IV-A for details on the separation. We will not directly address the associated complexity reduction in this paper.

III. Properties of Separable and Semi-Separable Kernels

To motivate the synthesis of positive semi-separable kernels, we will use this section to examine some of the properties of these functions. The motivation given is in terms of certain operator-theoretic concepts. A discussion of the known properties of operators defined by semi-separable kernels can be found in [2].

It is well-known that a stable dynamical system which defines a strongly continuous semigroup on a Hilbert space X will have a Lyapunov function of the form

$$\langle x, Ax\rangle,$$

where $A : X \to X$ is a positive operator (See, e.g. [1]). For time-delay systems, $x \in \mathbb{R}^n \times \mathcal{C}([-h,0],\mathbb{R}^n)$ equipped with the L_2-inner product. For linear time-delay systems, it is known that A may be assumed to have the form

$$(Ax)(\theta) = M(\theta)\begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} + \int_{-h}^{0} \begin{bmatrix} 0 & 0 \\ 0 & N(\theta,\omega) \end{bmatrix} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}.$$

Thus we can assume that A consists of the combination of a multiplier and integral operator defined by matrix-valued functions M and N, respectively. Most results on constructing Lyapunov-Krasovskii functionals are attempts to parameterize classes of positive linear operators by using positive semidefinite matrices to construct the functions M and N. For example, the "piecewise-linear" method of [3] is used to construct functions M and N which are linear on certain subintervals of $[-h, 0]$. We note briefly that the method in [3] also constructs semi-separable kernels, although, naturally, the separable components are piecewise-linear.

Properties of a kernel function are most easily expressed by properties of the operator it defines. Consider the operator A_k defined by the function $k \in L_2$.

$$(A_k x)(t) := \int_{-h}^{0} k(t,s)x(s)ds$$

The following properties are listed here without proof.

- if k is separable, it is semi-separable.
- If k is separable, then A_k has a finite number of non-zero singular values, denoted $\sigma_i(A_k)$.
- If $k \in L_2[-2, 0]$, then A_k is a compact, Hilbert-Schmidt operator and so

  $$\sum_{i}^{\infty} \sigma_i(A_k)^2 < \infty$$

  We assume both separable and semi-separable kernels are in $L_2[-h, 0]$.
- The class of semi-separable operators is not of the "trace class", i.e. there exist semi-separable functions, k, such that

  $$\sum_{i}^{\infty} \sigma_i(A_k) = \infty.$$

  A simple example of a kernel for a non-"trace class" operator is $k_1(t)k_2(s) = 0$ and $k_3(t)k_4(s) = 1$.
- Any non-negative compact Hermetian operator A has a compact Hermitian square root B such that $A = B^* B$

One possible explanation for the effectiveness of semi-separable kernel methods is that positive operators defined by polynomial semi-separable kernels will be non-zero on a dense subset of L_2. The question of what type of kernel functions are necessary in the definition of the Lyapunov-Krasovskii functions has no satisfactory answer and further work will be required before making a more definitive statement.

IV. The Derivative of a Function with Semi-Separable Kernel

In this section, we consider the derivative of the functional defined by a semi-separable kernel on the vector field defined by a linear time-delay system of the following form.

$$\dot{x}(t) = \sum_{i=0}^{k} A_i x(t - h_i) \tag{4}$$

Solutions of this type of system are well-defined, and we have the following result.

Proposition 3: Suppose that N_1 and N_2 are continuous, differentiable functions. Let

$$V(x) = \int_{-h}^{0}\int_{-h}^{0} x(s)N(s,t)x(t)\,ds\,dt$$

where

$$N(t,s) = \begin{cases} N_1(t,s) & s < t \\ N_2(t,s) & s \geq t. \end{cases}$$

Then $\dot{V}(x)$ along trajectories of Equation 4 is given by

$$\dot{V}(x) = \int_{-h} 0 \begin{bmatrix} x(t) \\ x(t-h) \\ x(\theta) \end{bmatrix}^T D(\theta) \begin{bmatrix} x(t) \\ x(t-h) \\ x(\theta) \end{bmatrix} - \int_{-h}^{0}\int_{-h}^{0} x(\theta)^T E(\theta,\omega) x(\omega)\,d\omega\,d\theta$$

Where

$$D(\theta) = \begin{bmatrix} 0 & 0 & N_1(0,\theta) \\ 0 & 0 & -N_2(-h,\theta) \\ N_2(\theta,0) & -N_1(\theta,-h) & 0 \end{bmatrix}$$

and

$$E(\theta,\omega) = \begin{cases} \frac{\partial}{\partial\omega}N_1(\theta,\omega) + \frac{\partial}{\partial\omega}N_1(\theta,\omega) & \theta < \omega \\ \frac{\partial}{\partial\omega}N_2(\theta,\omega) + \frac{\partial}{\partial\omega}N_2(\theta,\omega) & \theta \geq \omega. \end{cases}$$

Proof: Suppose x is a trajectory of Equation 4.

$$\begin{aligned} V(t) &= \int_{-h}^{0}\int_{-h}^{0} x(t+\theta)^T N(\theta,\omega) x(t+\omega)\,d\theta\,d\omega \\ &= \int_{-h}^{0}\int_{-h}^{\theta} x(t+\theta)^T N_1(\theta,\omega) x(t+\omega)\,d\theta\,d\omega \\ &+ \int_{-h}^{0}\int_{\theta}^{0} x(t+\theta)^T N_2(\theta,\omega) x(t+\omega)\,d\theta\,d\omega \\ &= V_1(t) + V_2(t) \end{aligned}$$

Now we will examine these parts individually.

$$\begin{aligned} \dot{V}_1(t) &= \int_{-h}^{0}\int_{-h}^{\theta} \dot{x}(t+\theta)^T N_1(\theta,\omega) x(t+\omega)\,d\theta\,d\omega \\ &+ \int_{-h}^{0}\int_{-h}^{\theta} x(t+\theta)^T N_1(\theta,\omega) \dot{x}(t+\omega)\,d\theta\,d\omega \end{aligned}$$

By noting that $\frac{\partial}{\partial t}x(t+\theta) = \frac{\partial}{\partial\theta}x(t+\theta)$ and using integration by parts on the first term of V_1, we obtain

$$\begin{aligned} &\int_{-h}^{0} \dot{x}(t+\theta)^T \int_{-h}^{\theta} N_1(\theta,\omega) x(t+\omega)\,d\omega\,d\theta \\ &= x(t)^T \int_{-h}^{0} N_1(0,\omega) x(t+\omega)\,d\omega \\ &- \int_{-h}^{0} x(t+\theta)^T N_1(\theta,\theta) x(t+\theta) d\theta \\ &- \int_{-h}^{0} x(t+\theta)^T \int_{-h}^{\theta} \left(\frac{\partial}{\partial\theta} N_1(\theta,\omega) x(t+\omega)\,d\omega \right) d\theta. \end{aligned}$$

Similarly for the second term,

$$\int_{-h}^{0} x(t+\theta)^T \int_{-h}^{\theta} N_1(\theta,\omega)\dot{x}(t+\omega)\, d\theta\, d\omega$$
$$= \int_{-h}^{0} x(t+\theta)^T \left(N_1(\theta,\theta)x(t+\theta) - N_1(\theta,-h)x(t-h)\right)\, d\theta$$
$$+ \int_{-h}^{0} x(t+\theta)^T \int_{-h}^{\theta} \frac{\partial}{\partial\omega} N_1(\theta,\omega)x(t+\omega)\, d\theta\, d\omega.$$

Collecting terms, we have

$$\dot{V}_1(x) = -x(t+\theta)^T \int_{-h}^{0} N_1(\omega,-h)x(t-h)\, d\theta$$
$$+ \int_{-h}^{0} x(t)^T N_1(0,\omega)x(t+\omega) d\omega$$
$$- \int_{-h}^{0}\int_{-h}^{\theta} x(t+\theta)^T \Big(\frac{\partial}{\partial\omega} N_1(\theta,\omega)$$
$$+ \frac{\partial}{\partial\omega} N_2(\theta,\omega)\Big) x(t+\omega) d\omega d\theta$$

A Similar analysis of V_2 yields the following results.

$$\dot{V}_2(t) = \int_{-h}^{0} \dot{x}(t+\theta)^T \int_{\theta}^{0} N_2(\theta,\omega)x(t+\omega)\, d\theta\, d\omega$$
$$+ \int_{-h}^{0} x(t+\theta)^T \int_{\theta}^{0} N_2(\theta,\omega)\dot{x}(t+\omega)\, d\theta\, d\omega,$$

where

$$\int_{-h}^{0} \dot{x}(t+\theta)^T \int_{\theta}^{0} N_2(\theta,\omega)x(t+\omega)\, d\theta\, d\omega$$
$$= 0 - x(t-h)^T \int_{-h}^{0} N_2(-h,\omega)x(t+\omega)\, d\omega$$
$$- \int_{-h}^{0} x(t+\theta)^T \Big(0 - N_2(\theta,\theta)x(t+\theta)\Big)\, d\theta$$
$$- \int_{-h}^{0}\int_{\theta}^{0} x(t+\theta)^T \frac{\partial}{\partial\omega} N_2(\theta,\omega)x(t+\omega) d\omega d\theta$$

and

$$\int_{-h}^{0} x(t+\theta)^T \int_{\theta}^{0} N_2(\theta,\omega)\dot{x}(t+\omega)\, d\theta\, d\omega$$
$$= \int_{-h}^{0} x(t+\theta)^T \left(N_2(\theta,0)x(t) - N_2(\theta,\theta)x(t+\theta)\right)\, d\theta$$
$$- \int_{-h}^{0} x(t+\theta)^T \int_{\theta}^{0} \frac{\partial}{\partial\omega} N_2(\theta,\omega)x(t+\omega)\, d\theta\, d\omega$$

Collecting terms, we get the following.

$$\dot{V}_2(x) = -x(t-h)^T \int_{-h}^{0} N_2(-h,\omega)x(t+\omega)\, d\omega$$
$$+ \int_{-h}^{0} x(t+\theta)^T N_2(\theta,0)x(t) d\theta$$
$$- \int_{-h}^{0}\int_{\theta}^{0} x(t+\theta)^T \Big(\frac{\partial}{\partial\omega} N_2(\theta,\omega)$$
$$+ \frac{\partial}{\partial\omega} N_2(\theta,\omega)\Big) x(t+\omega) d\omega d\theta$$

By combining $\dot{V}_1$ and $\dot{V}_2$, we obtain the desired result. ■

A. Multiple Delays

For the case of multiple delays, the functions may be discontinuous at points h_i. The derivatives are therefore more complicated. For taking the derivative, it is convenient to decomposed the kernels into separable and semi-separable parts as follows

$$V(t) = V_1(t) + V_2(t).$$

The separable part is defined by a piecewise-polynomial kernel, Q.

$$V_1(t) = \int_{-h}^{0}\int_{-h}^{0} x(t+\theta)^T Q(\theta,\omega)x(t-\omega)\, d\theta\, d\omega$$

The semi-separable part is defined by piecewise-polynomial kernels, N_1 and N_2, as

$$V_2(t) = \sum_{i=1}^{k} \int_{-h_i}^{-h_{i-1}} \int_{-h_i}^{\theta} x(t+\theta)^T N_1(\theta,\omega)x(t-\omega)\, d\theta\, d\omega$$
$$+ \sum_{i=1}^{k} \int_{-h_i}^{-h_{i-1}} \int_{\theta}^{-h_{i-1}} x(t+\theta)^T N_2(\theta,\omega)x(t-\omega)\, d\theta\, d\omega.$$

Since the derivative of a separable kernel is already well-known, we can instead focus on the derivative of the semi-separable part. We have the following framework for the functional and its derivative.

$$\begin{aligned} Y = \Big\{ & N : [-h,0]\times[-h,0] \to \mathbb{S}^n \mid \\ & N(s,t) = N(t,s)^T && \text{for all } s,t\in[-h,0] \\ & N \text{ is } C^1 \text{ on } H_i\times H_j && \text{for all } i,j = 1,\dots,k \\ & && \text{and for } s\neq t \Big\} \end{aligned}$$

and for its derivative, define

$$\begin{aligned} Z_1 = \Big\{ & D : [-h,0] \to \mathbb{S}^{(k+2)n} \mid \\ & D_{ij}(t) \text{ is constant} && \text{for all } t\in[-h,0] \\ & && \text{for } i,j = 1,\dots,3 \\ & D \text{ is } C^0 \text{ on } H_i && \text{for all } i = 1,\dots,k \Big\} \\ Z_2 = \Big\{ & E : [-h,0]\times[-h,0] \to \mathbb{S}^n \mid \\ & E(s,t) = E(t,s)^T && \text{for all } s,t\in[-h,0] \\ & E \text{ is } C^0 \text{ on } H_i\times H_j && \text{for all } i,j = 1,\dots,k \\ & && \text{and for } s\neq t \Big\} \end{aligned}$$

Here $D \in Z_1$ is partitioned according to

$$D(t) = \begin{bmatrix} D_{1,1} & \dots & D_{1,k+1}(t) \\ \vdots & & \vdots \\ D_{k+1,1}(t) & \dots & D_{k+1,k+1}(t) \end{bmatrix} \quad (5)$$

where $D_{i,j} \in \mathbb{R}^{n\times n}$. Let $Z = Z_1 \times Z_2$. The derivative of a Lyapunov function can be defined as a linear map $Y \mapsto Z$. This is made explicit in the following definition.

Definition 4: Define the map $L : Y \to Z$ by $(D, E) = L(N)$ if for all $t, s \in [-h, 0]$ and $i = 1, \cdots, k$, we have

$$D_{i,k+1}(t) = \begin{cases} -\frac{1}{2}(N_1(\theta, -h_i) + N_2(-h_i, \theta)) & \theta \in H_i \\ \frac{1}{2}(N_1(-h_i, \theta) + N_2(\theta, -h_i)) & \theta \in H_{i+i} \end{cases}$$

where those values undefined by symmetry are zero and

$$E(\theta, \omega) = \begin{cases} \frac{\partial}{\partial \omega} N_1(\theta, \omega) + \frac{\partial}{\partial \omega} N_1(\theta, \omega) & \theta < \omega,\ \theta, \omega \in H_i \\ \frac{\partial}{\partial \omega} N_2(\theta, \omega) + \frac{\partial}{\partial \omega} N_2(\theta, \omega) & \theta \geq \omega,\ \theta, \omega \in H_i \\ 0 & \text{otherwise.} \end{cases}$$

Here D is partitioned as in (5).

Lemma 5: Suppose $N \in Y$ and V is given by **(??)**. Let $(D, E) = L(M, N)$. Then the Lie derivative of V on the vector field of (4) is given by

$$\dot{V}(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(-h_0) \\ \vdots \\ \phi(-h_k) \\ \phi(s) \end{bmatrix}^T D(s) \begin{bmatrix} \phi(-h_0) \\ \vdots \\ \phi(-h_k) \\ \phi(s) \end{bmatrix} ds + \int_{-h}^{0} \int_{-h}^{0} \phi(s)^T E(s, t) \phi(t)\, ds\, dt. \qquad (6)$$

Proof: The proof is now a straightforward extension of the single-delay case. In particular, if

$$V_i(x) = \int_{-h_i}^{-h_{i-1}} \int_{-h_i}^{\theta} x(t+\theta)^T N_1(\theta, \omega) x(t-\omega)\, d\theta\, d\omega, + \int_{-h_i}^{-h_{i-1}} \int_{\theta}^{-h_{i-1}} x(t+\theta)^T N_2(\theta, \omega) x(t-\omega)\, d\theta\, d\omega,$$

then

$$\dot{V}_i(x) = \int_{-h_i}^{-h_{i-1}} \begin{bmatrix} x(t-h_{i-1}) \\ x(t-h_i) \\ x(\theta) \end{bmatrix}^T D(\theta) \begin{bmatrix} x(t-h_{i-1}) \\ x(t-h_i) \\ x(\theta) \end{bmatrix} - \int_{-h_i}^{-h_{i-1}} \int_{-h_i}^{-h_{i-1}} x(\theta)^T E(\theta, \omega) x(\omega)\, d\omega\, d\theta,$$

where

$$D(\theta) = \begin{bmatrix} 0 & 0 & N_1(-h_{i-1}, \theta) \\ 0 & 0 & -N_2(-h_i, \theta) \\ N_2(\theta, -h_{i-1}) & -N_1(\theta, -h_i) & 0 \end{bmatrix}$$

for $\theta \in H_i$ and

$$E(\theta, \omega) = \begin{cases} \frac{\partial}{\partial \omega} N_1(\theta, \omega) + \frac{\partial}{\partial \omega} N_1(\theta, \omega) & \theta < \omega \\ & \theta, \omega \in H_i \\ \frac{\partial}{\partial \omega} N_2(\theta, \omega) + \frac{\partial}{\partial \omega} N_2(\theta, \omega) & \theta \geq \omega. \\ & \theta, \omega \in H_i \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, since $V_2 = \sum_{i=1}^{k} V_i$, we have the desired result. ∎

V. Numerical Investigations

In this section, we perform a number of numerical examples in order to determine the importance of semi-separable kernels for the stability of linear time-delay systems. To this end, we compare algorithms which include semi-separable kernels with ones which only include separable kernels. We begin with what is perhaps the best understood linear time-delay system.

$$\dot{x}(t) = x(t - \tau)$$

It is well-known that this system is stable for $\tau \leq \frac{\pi}{2}$. The following summarizes the results of our numerical experiments as applied to this problem.

Maximum Stable Delay			
degree bound	0	2	4
semi-separable kernel	1.417	1.564	1.570
separable kernel	1.417	1.532	1.532
true			1.5708

TABLE I

τ_{max} USING A FIXED DEGREE BOUND OF 4 ON THE FIRST PART OF THE FUNCTIONAL AND A VARIABLE BOUND ON THE SECOND PART

We now consider a randomly chosen example.

$$\dot{x}(t) = \begin{bmatrix} -1 & -1 \\ .1 & -.2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t - \tau)$$

Maximum Stable Delay			
# of bases	0	1	2
semi-separable kernel	1.586	1.693	1.694
separable kernel	1.586	1.690	1.690
Method of Gu et al.	1.6940		1.6941

TABLE II

τ_{max} USING A FIXED DEGREE BOUND OF 4 ON THE FIRST PART OF THE FUNCTIONAL AND A VARIABLE BOUND ON THE SECOND PART

In our numerical experiments, we chose a relatively high fixed degree for the multiplier and observed improvement in accuracy as the degree of the kernel increased. This is for practical reasons, as the degree of the multiplier must match the degree of the kernel. This because when forming the derivative for time-delay systems, the kernel appears in the multiplier term. Therefore to refute positivity of the derivative, the degree of the refutation must match that of the kernel. The result of this observation is that for many examples, using the multiplier alone was sufficient to obtain accuracy to 4 significant digits. However, for the examples given here, the effect of the kernel is relatively clear, if only at higher levels of accuracy.

The interesting feature of the results presented in this section is not the quantitative rate of increase in accuracy due to increasing polynomial degree, but rather the qualitative shape of the increase. While for a separable kernel, there is no increase in accuracy above polynomials of degree 2, for semi-separable kernels, there is a consistent increase in accuracy for increasing the degree at any level. This is a feature we have observed in all numerical examples.

VI. CONCLUSION

This paper leaves a number of important unanswered questions. Although numerical tests indicate that semi-separable polynomial kernels are far more effective that separable kernels, we have not, as of yet, been able to construct a proof that this class of functions is necessary or sufficient.

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