# Analysis of Nonlinear Time Delay Systems Using the Sum of Squares Decomposition

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**Abstract**

In this paper we present a new methodology for analyzing the stability of equilibria of systems described by nonlinear Delay Differential Equations (DDEs) with multiple incommensurate delays. The methodology we propose can be used to construct algorithmically Lyapunov-Krasovskii functionals both for testing delay-independent and delay-dependent stability. It is based on the sum of squares decomposition of multivariate polynomials and semidefinite programming (Linear Matrix Inequalities). By considering parameterized Lyapunov-Krasovskii functionals, robust stability analysis under parametric uncertainty can be performed in a unified way. We illustrate the methodology using examples from population dynamics and congestion control for the Internet.

**Index Terms**

Nonlinear time-delay systems, Lyapunov's direct method, stability analysis, robustness analysis, sum of squares decomposition, semidefinite programming.

## I. Introduction

Delay differential equations are important for modeling systems that involve transport and propagation of data, such as networked systems [1], but also for modeling maturation and growth in population dynamics [2], [3]. In the simplest case, these systems take the form

$$\dot{x}(t) = f(x(t), x(t-\tau)),$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ may be nonlinear. It is known that the presence of delays may induce undesirable effects such as instabilities and performance degradation; and that ignoring them in the modeling process may result in inadequate designs and incorrect analysis conclusions. Therefore the stability, robust stability and control of such systems have received a lot of interest in the past few years [4], [5]. More recently, significant research has focused on the analysis of network congestion control schemes for the Internet [1], which can be adequately modeled using Delay Differential Equations (DDEs).

DDEs fall in the category of Functional Differential Equations (FDEs), which differ from Ordinary Differential Equations (ODEs) because the system state belongs to an infinite dimensional space. This makes the problem of assessing the stability properties of steady state solutions more difficult, as more complicated analysis tools are required. Given local existence and uniqueness of solutions, stability properties can be assessed using appropriate Lyapunov functions, just as in the case of ODEs. These are functions of state that satisfy certain positivity conditions, and which are non-increasing along the system evolution. However, while for the case of ODEs these are functions, in the case of FDEs they are *functionals* owing to the fact that the state belongs in a function space itself.

Stability of time-delay systems can be *delay-independent* if it is retained irrespective of the size of the (finite) delay or *delay-dependent* if stability is valid dor some values of the delay, but not for others. In the case of linear DDEs, the form of these functionals that is necessary and sufficient for delay-dependent and strong delay-independent stability is known [6]–[8]. However, the delay-dependent conditions are difficult to construct algorithmically. Under restrictions on their structure, convex optimization has been used to construct them through the solution of a set of Linear Matrix Inequalities (LMIs). Inevitably these functionals yielded conservative results on the delay interval guaranteeing stability, see for example the attempts in [9] and [10]. The reason behind the difficulty in constructing the 'complete' (necessary and sufficient) structure is that this problem is equivalent to parameterizing the set of positive operators on an infinite-dimensional space. One approach is to use Lyapunov functionals with piecewise-linear kernels, which is known as the 'Discretized Lyapunov Functional Approach'; these functionals are constructed by solving a set of LMIs whose size depends on the discretization level [11], and as the discretization level is decreased, delay values closer to the boundary of stability can be tested. More recently a new approach was developed in [12] for constructing the complete Lyapunov-Krasovskii functional for linear DDEs, which is based on an explicit parameterization of positive operators and uses the Sum of Squares decomposition and semidefinite programming as the computational tool.

As far as nonlinear time delay systems are concerned, the only methodologies currently available for stability analysis centre on the construction of simple Lyapunov certificates for systems of low dimension, mainly through a judicious choice

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for a candidate Lyapunov function [2]. This is the case even for systems described by ODEs, where constructing Lyapunov functions is usually based on system structure and its properties (Volterra-type systems, gradient systems etc.). Recently, however, a computational methodology based on the sum of squares decomposition has been proposed [13], [14].

In this paper we present a methodology for constructing Lyapunov-Krasovskii functionals for nonlinear time delay systems with multiple incommensurate delays, using the Sum of Squares decomposition of multivariate polynomials and semidefinite programming. We construct appropriate Lyapunov-Krasovskii functionals both for delay-independent and delay-dependent stability. The Lyapunov functionals we construct have polynomial kernels, and in the case in which the DDE we consider is linear, they reduce to the complete (i.e., necessary and sufficient) Lyapunov functional structures.

The paper is organized as follows. In section II we present the main ingredients we use to construct these Lyapunov functionals. In section III we show how the methodology can be used for the robust stability analysis of nonlinear delayed systems (delay-independent and delay-dependent) as well as how systems with single and multiple, incommensurate delays can be treated. In section IV we illustrate our results with examples from population dynamics and network congestion control for the Internet. We conclude the paper in Section V. Background information on time-delay systems and the algorithmic methodology proposed can be found in the Appendix.

### A. Notation

Let $\mathbb{R}$ denote the reals, $\mathbb{R}^n$ the $n-$dimensional Euclidean space and $\mathcal{S}^n \subset \mathbb{R}^{n\times n}$ the set of symmetric real $n \times n$ matrices. For $x \in \mathbb{R}^n$, we denote by $\mathbb{R}[x]$ the ring of polynomials in $x$ with real coefficients and $Z_d[x]$ the vector of monomials in $x$ of degree $d$ or less. Further, denote $C([-\tau, 0], \mathbb{R}^n)$ the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. The norm on $C$ is defined as $\|\phi\| = \sup_{-\tau\leq\theta\leq 0} |\phi(\theta)|$. Similarly, $C^\gamma$ denotes the set defined by $C^\gamma = \{\phi \in C | \|\phi\| < \gamma\}$ with $\gamma > 0$. Suppose $\sigma \in \mathbb{R}$, $\rho \geq 0$ and $x \in C([\sigma - \tau, \sigma + \rho], \mathbb{R}^n)$; then for any $t \in [\sigma, \sigma + \rho]$, define $x_t \in C$ by $x_t(\theta) = x(t + \theta), \theta \in [-\tau, 0]$. In the multiple delay case, we will consider delays $\tau_i \in \mathbb{R}_+$, $i = 0, 1, \ldots, K$ with $0 = \tau_0 < \tau_1 \leq \ldots \leq \tau_K = \tau$. In this case, we define $H = \{-\tau_0, \ldots, -\tau_k\}$ and $H^c = [-\tau, 0]\backslash H$. We will also denote the intervals $I_i = [-\tau_i, -\tau_{i-1}]$.

## II. The Proposed Methodology

In this section we describe the approach used to construct Lyapunov Krasovskii functionals for nonlinear time-delay systems. Background theory on stability and Lyapunov theory for DDEs can be found in the Appendix. In order to introduce the methodology used in the rest of the paper, in this section we will consider global, delay-dependent stability of the steady state of a time-delay system with a single delay. In later sections we will concentrate on more general systems with multiple delays, different stability properties (local stability, robust stability, etc.) as well as different Lyapunov structures for delay-independent or delay-dependent stability.

Consider a time-delay system with a single delay of the form

$$\dot{x}(t) = f(x(t), x(t-\tau))$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $f(0, 0) = 0$ is such that a unique solution to the above delay-differential equation exists from an appropriate initial condition close to 0. One type of Lyapunov functional that can be used to verify the delay-dependent stability of the zero steady-state takes the form

$$V(\phi) = \int_{-\tau}^{0} v_1(\phi(0), \phi(\theta), \theta)d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} v_2(\phi(\xi), \phi(\theta), \theta, \xi)\, d\theta\, d\xi. \tag{1}$$

This functional has a derivative in the form

$$\dot{V}(\phi) = \int_{-\tau}^{0} v_3(\phi(0), \phi(-\tau), \phi(\theta), \theta)d\theta - \int_{-\tau}^{0}\int_{-\tau}^{0} \left(\frac{\partial v_2}{\partial\theta} + \frac{\partial v_2}{\partial\xi}\right) d\theta\, d\xi, \tag{2}$$

where the map $v_1, v_2$ to $v_3$ will be presented in later sections.

In [15], it was shown how to characterize the positivity of the first term in the Lyapunov functional, i.e.,

$$V_1 = \int_{-\tau}^{0} v_1(\phi(0), \phi(\theta), \theta)d\theta.$$

*Theorem 1:* Suppose $f : [-\tau, 0] \times \mathbb{R}^n \to \mathbb{R}$ is continuous on $H^c \times \mathbb{R}^n$ and suppose there exists a bounded function $z : [-\tau, 0] \to \mathbb{R}$, continuous on $H^c$, such that for all $t \in [-\tau, 0]$

$$f(t, z(t)) = \inf_x f(t, x).$$

Further suppose that for each bounded set $X \subset \mathbb{R}^n$ the set

$$\{f(t, x) | x \in X, t \in [-\tau, 0]\}$$

is bounded. Then the following are equivalent:

(i) For all $y \in C[-\tau, 0]$

$$\int_{-\tau}^{0} f(t, y(t))dt \geq 0.$$

(ii) There exists $g : [-\tau, 0] \to \mathbb{R}$ which is piecewise continuous and satisfies

$$f(t, z) + g(t) \geq 0 \qquad \text{for all } t, z$$
$$\int_{-\tau}^{0} g(t)dt = 0.$$

For example, quadratic functions $f$ which are strictly positive in a certain sense satisfy the conditions of Theorem 1 (see [15]); in particular, the Schur complement was shown to yield the continuous minimizing function $z(t)$. In the following lemma, we generalize the result from quadratic functions to functions which are lower bounded and obtain a global minimum. We start by giving a quantitative definition of global minimum.

*Definition 2:* We say that a function $f(y)$ has a unique global minimum, if there exists a $y^*$ such that for any $\epsilon > 0$, there exists a $\beta > 0$ such that

$$f(x) \geq f(y^*) + \beta$$

for all $x$ such that $\|x - y\| \geq \epsilon$.

*Lemma 3:* Suppose there exists $\alpha > 0$ such that

$$\alpha \|y\|^2 \leq f(t, y),$$

where $f(t, y)$ is continuous in $y$ and piecewise continuous in $t$. Suppose $f$ has a unique global minimum for every $t \in [-\tau, 0]$. Then there exists a piecewise continuous function $z$, such that $\min_y f(t, y) = f(t, z(t))$ for all $t \in [-\tau, 0]$.

*Proof:* Let $z(t) = \arg\min_{y \in \mathbb{R}^n} f(t, y)$. We now demonstrate that $z$ is piecewise continuous. Suppose $f(t, y)$ is continuous on a closed interval $t \in I$. We first show that $z(t)$ is bounded on $I$. Choose an arbitrary $z_0 \in \mathbb{R}^n$ and let

$$B = \sup_{t \in I} f(t, z_0).$$

$B$ is finite because $f$ is continuous in $t$ on the compact interval $I$. Now choose $r > \sqrt{B/\alpha}$. Then if $\|z(t_1)\| \geq r$ for some $t_1 \in I$, we have that

$$f(t_1, z(t_1)) \geq \alpha \|z(t_1)\|^2 > B,$$

while

$$f(t_1, z_0) \leq B,$$

which contradicts the definition of $z(t_1)$. Thus we have that $z(t) \leq r$ for all $t \in I$.

Now suppose we are given some arbitrary $\epsilon > 0$ and a point $t$ in the interior of $I$. To show that $z$ is continuous at $t$, we must find a $\gamma > 0$ such that $|t - s| \leq \gamma$ implies $\|z(t) - z(s)\| \leq \epsilon$. Since $z(t)$ is the unique global minimizer of $f$ at $t$, there exists a $\beta > 0$ such that

$$f(t, z(t)) \leq f(t, y) - \beta \qquad \text{for any } y \text{ such that } \|y - z(t)\| \geq \epsilon$$

By continuity of $f$ at $t$, there exists a $\delta > 0$ such that for any $s \in I$ with $|t - s| \leq \delta$, we have

$$\|f(t, y) - f(s, y)\| \leq \frac{\beta}{4}$$

for all $\|y\| \leq r$.

Now choose $\gamma < \delta$ such that $s \in I$ for all $|s - t| \leq \gamma$. Suppose there exists an $s$, with $|t - s| \leq \gamma$ and $|z(t) - z(s)| > \epsilon$. Then

$$\begin{aligned} f(s, z(s)) &\geq f(t, z(s)) - \beta/4 \\ &\geq f(t, z(t)) + \beta - \beta/4 \\ &\geq f(s, z(t)) + \beta - \beta/2 = f(s, z(t)) + \beta/2. \end{aligned}$$

This contradicts the definition of $z(s)$ as being the minimizer of $f$ at $s$. Therefore, $\|z(t) - z(s)\| \leq \epsilon$ for all $|s - t| \leq \gamma$, as desired and $z$ is continuous at $t$. Concluding, since $f(t, y)$ is piecewise continuous in $t$, if $f$ is continuous at a point $t$, then it is continuous in an open neighborhood of $t$. Since we have shown that this implies $z$ is also continuous at point $t$, we have that $z$ is continuous at every point for which $f(t, z)$ is continuous. Since $f$ is piecewise continuous, this proves that $z$ is piecewise continuous. ■

We now state conditions on $v_1$ in (1) for positivity of the first term in the Lyapunov functional.

*Theorem 4:* Consider a continuous function $v(x, y(\theta), \theta)$ that satisfies the conditions of Lemma 3. Then

$$\int_{-\tau}^{0} v(x, y(\theta), \theta) d\theta \geq 0 \text{ for all } x \in \mathbb{R}^m \text{ and } y \in C,$$

if and only if there exists a continuous function $t(x, \theta)$ such that for all $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$,

$$v(x, z, \theta) + t(x, \theta) \geq 0 \text{ for all } \theta \in [-\tau, 0], \tag{3}$$

$$\int_{-\tau}^{0} t(x, \theta) d\theta = 0. \tag{4}$$

*Proof:* Suppose that (2) holds. Integrating condition (3) on $[-\tau, 0]$ we get:

$$\begin{aligned} & \int_{-\tau}^{0} v(x, z, \theta) d\theta + \int_{-\tau}^{0} t(x, \theta) d\theta \geq 0 \\ \Rightarrow \quad & \int_{-\tau}^{0} v(x, z, \theta) d\theta \geq 0, \end{aligned}$$

since condition (4) holds. Therefore (1) holds. The converse is also true, combining the results in Theorem 1 and Lemma 3. ■

Returning now to the original Lyapunov-functional (1), we will show how to test non-negativity of the second term, i.e.,

$$V_2 = \int_{-\tau}^{0} \int_{-\tau}^{0} v_2(\phi(\xi), \phi(\theta), \theta, \xi) d\theta d\xi. \tag{5}$$

Again, we generalize from the quadratic case. One simple condition is that the function $v_2(y, z, \theta, \xi) \geq 0$ for $\theta, \xi \in [-\tau, 0]$ and for all $y, z \in \mathbb{R}^n$ but this is many times too restrictive. An alternative condition is given by the following proposition:

*Proposition 5:* Given a continuous function $v(y, z, \theta, \xi)$ suppose there exists a continuous vector-valued function $s : (y, \theta) \mapsto \mathbb{R}^d$ for some $d$ so that:

$$v(y, z, \theta, \xi) = (s(y, \theta))^T s(z, \xi). \tag{6}$$

Then we have:

$$\int_{-\tau}^{0} \int_{-\tau}^{0} v(\phi(\xi), \phi(\theta), \theta, \xi) d\theta \, d\xi \geq 0.$$

*Proof:* Suppose that there exists a decomposition of the form (6) for $v$. Then we have:

$$\begin{aligned} & \int_{-\tau}^{0} \int_{-\tau}^{0} v(\phi(\xi), \phi(\theta), \theta, \xi) d\theta d\xi \\ = \quad & \left( \int_{-\tau}^{0} s(\phi(\theta), \theta) d\theta \right)^T \left( \int_{-\tau}^{0} s(\phi(\xi), \xi) d\xi \right) \geq 0. \end{aligned}$$

Therefore the result follows. ■

Using Theorem (4) and Proposition (5) we can reformulate the problem of testing positivity of an integral form to testing certain conditions on its kernel. However, even if we are given a function $v_1$, it may be hard to ensure that conditions (3–4) are satisfied. Moreover, a more interesting and at least as hard question to answer is how to *construct* $v_1$ to ensure that these conditions are satisfied. In this paper we make the following assumption that will allow the algorithmic construction of these functions.

*Assumption 6:* We assume that $v_1$ and $v_2$ are *polynomials* in their arguments, and in particular that for some positive integer $d$,

$$\begin{aligned} v_1(\hat{x}_0, \hat{y}, \theta) &= Z_d^T[\hat{x}_0, \hat{y}] P(\theta) Z_d[\hat{x}_0, \hat{y}], \\ v_2(\hat{y}, \hat{z}, \theta, \xi) &= Z_d^T[\hat{y}, \theta] R Z_d[\hat{z}, \xi]. \end{aligned}$$

where $P(\theta)$ is a polynomial matrix and $R$ is a matrix of appropriate dimensions. We also assume that the vector field $f \in \mathbb{R}[\hat{x}_0, \hat{x}_1]$ as well as that $t(\hat{x}_0, \theta) \in \mathbb{R}[\hat{x}_0, \theta]$.

*Remark 7:* The above assumption may seem restrictive, especially the restriction that the vector field be polynomial. However, it is true that in many cases polynomial vector fields arise in modeling of physical processes (e.g., using Volterra-type equations). In some other cases, one can employ a non-polynomial transformation to render a non-polynomial vector field polynomial. See [16] for more details.

Under Assumption 6, all conditions in Theorem (4) and Proposition (5) are polynomial non-negativity conditions, which are in general difficult to test. In fact, it is known that testing non-negativity of polynomials of high degree (more than 4) is

NP-hard. A sufficient condition for polynomial non-negativity which is worst-case polynomial-time verifiable is the existence of a sum of squares (SOS) decomposition. This is the approach we propose to follow in order to construct polynomials $v_i$ to satisfy the conditions in Theorem (4) and Proposition (5). More details on the sum of squares decomposition can be found in [13] — see Appendix II for some background remarks. Here, we denote by $\Sigma$ the sum of squares cone and by $\Sigma_d$ the subset of $\Sigma$ of polynomials of degree $d$ of less. We now present two important results that we will be using in the sequel.

First note that if $a \in \mathbb{R}[x]$ is a sum of squares, then it is globally nonnegative. In order to ensure that it is positive *definite* and radially unbounded, we can construct a polynomial 'shaping' function $\varphi(x)$ which can be used to impose positive definiteness.

*Proposition 8:* Given a polynomial $a(x)$ of degree $2d$, let

$$\varphi(x) = \sum_{i=1}^{n}\sum_{j=1}^{d} \epsilon_{ij} x_i^{2j}, \quad \sum_{j=1}^{d} \epsilon_{ij} \geq \gamma \quad \forall\ i = 1, \ldots, n, \tag{7}$$

with $\gamma$ a positive number, and $\epsilon_{ij} \geq 0$ for all $i$ and $j$. Then the condition

$$a(x) - \varphi(x) \in \Sigma$$

guarantees the positive definiteness of $a(x)$. Moreover, $a(x)$ is radially unbounded.

*Proof:* The function $\varphi(x)$ as is positive definite if $\epsilon_{ij}$'s satisfy the above. Then $a(x) - \varphi(x)$ being SOS implies that $a(x) \geq \varphi(x)$, and therefore $a(x)$ is positive definite. Moreover it is radially unbounded since $\varphi$ is radially unbounded – it is the positive sum of simple monomials (i.e., in only one variable) squared. ■

The second remark we need to make is related to testing non-negativity of a polynomial over a bounded domain instead of globally. We have already come across Condition (3) in Proposition 4 where polynomial non-negativity is required only for $\theta \in [-\tau, 0]$. We will also see similar conditions in the sequel: when studying local stability the non-negativity conditions will only be required to hold in some region of the state-space; when studying robust stability, these conditions will be required to hold for parameters inside a parameter set; etc. These conditional satisfiability conditions can be tested using a generalization to the S-procedure, which is based on Putinar's representation [17] in Real Algebraic Geometry.

Given $p \in \mathbb{R}[x]$, suppose we want to ensure that $p(x) > 0$ on the set

$$D = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \ldots, N_1\}$$

Then one can search for Lagrange-type multipliers $\lambda_i \in \Sigma_k$ so that:

$$p(x) + \sum_{i=1}^{N_1} \lambda_i(x) g_i(x) \in \Sigma$$

Searching for $\lambda_i(x)$ of a fixed degree $k$ so that the above expression is SOS is a semidefinite programme. Note that if $p$ and $g_i$ are quadratic forms and $\lambda_i$ are constants, the above test is indeed the S-procedure, which can fail if $i \geq 2$. However, it has been shown in [17] that if $D$ is compact and another mild condition holds on the $g_i(x)$, then there is a $k$ for which the above test will succeed – it is indeed a necessary and sufficient condition. Other tests can also be formulated, which may have different properties. See e.g. [18].

We now show how to use these ideas to provide an algorithmic methodology for assessing the stability of nonlinear time-delay systems.

## III. Nonlinear Time Delay Systems

In this paper we propose a methodology, based on the sum of squares decomposition of multivariate polynomials, for the algorithmic construction of Lyapunov-Krasovskii functionals for nonlinear time-delay systems. We will consider robust stability analysis, local stability analysis, and systems with multiple, non-commensurate delays. Both delay-independent and delay-dependent conditions will be proposed.

It is worth mentioning that testing the stability of the steady-state of a time delay system is a difficult task even if the vector field is linear; for example, establishing whether a linear system with multiple delays is stable independent of the delay or stable for all $\tau_i \in [0, \overline{\tau}_i)$ with $\overline{\tau}_i \in \mathbb{R}_+$, $i = 1, \ldots, K$ given, is known to be NP-Hard [19]. Although obtaining an exact answer to these questions would probably be computationally intractable, sufficient stability tests that are algorithmically verifiable can still be formulated which may answer the stability question for some problem instances. In this case it is convenient to provide a nested family of such tests, each of which is at least as powerful as the previous one, but comes with an increased computational cost. This is the approach we follow here.

### A. Delay-independent stability

We say that a steady-state of a time-delay system is delay-independent stable if it is stable for all fixed values of the delay. For example, delay-independent stability conditions are used in controller synthesis when the size of the delay is unknown [20].

For general nonlinear time-delay systems, the appropriate structure of a Lyapunov functional that will prove stability is not known. For this reason, we seek to generalize the structures used for linear time-delay systems, where a necessary and sufficient Lyapunov structure has been established. Given a linear time-delay system of the form $\dot{x} = A_0 x(t) + A_1 x(t-\tau)$, the property of "*strong delay-independent stability*" is equivalent to

$$\det(sI_n - A_0 - zA_1) \neq 0$$

for all $s = j\omega$, $\tau \geq 0$ and $z$ in the closed unit disk [7]. This property is stronger than the standard notion of delay-independent stability, for which only $\det(sI_n - A_0 - e^{-s\tau}A_1) \neq 0$ for all $s = j\omega$ and $\tau \geq 0$ is required. The property is, however, robust with respect to perturbations in $A_0$ and $A_1$. In [7] the class of Lyapunov functionals necessary and sufficient for strong delay-independent stability of linear systems has been characterized. The system $\dot{x} = A_0 x(t) + A_1 x(t-\tau)$ is strongly delay-independent stable if and only if it possesses, for a certain $L \in \mathbb{N}$, a Lyapunov functional of the form

$$V(\phi) = v_0(\phi(0), \phi(-\tau), \ldots, \phi(-L\tau)) + \int_{-\tau}^{0} v_1(\phi(\theta), \phi(\theta-\tau), \ldots, \phi(\theta - L\tau))\, d\theta \tag{8}$$

where $v_0$ and $v_1$ are *quadratic* polynomials.

Returning to the nonlinear case, we consider for simplicity a system with a single discrete delay given by:

$$\dot{x}(t) = f(x(t), x(t-\tau)), \tag{9}$$

where we assume that $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is polynomial in its arguments and that 0 is the only steady-state of the system. We will return to multiple, incommensurate delays and multiple equilibria later in the paper.

In the sequel, we will be introducing a number of results which give equilibrium stability conditions for the above system using sum of squares programming. To state these results succinctly, we must introduce a number of symbolic independent variables. Because these variables are derived from the state, it is useful to use notation which reflects this relationship. In particular, we use the variable $\hat{x}_{kl}^{(i)}$ to reference $x_i(t - l\tau_k)$ where $k = 0, \ldots, K$ and $l = 1, \ldots, L$. We will use $\hat{x}_{kl}$ to denote the vector of $\hat{x}_{kl}^{(i)}$ and let $\hat{x}_{k1} = \hat{x}_k$ and $\hat{x}_{1l} = \hat{x}_l$ when $L = 1$ or $K = 1$, respectively. In a similar fashion, $\hat{y}_{kl}^{(i)}$ will be used for $x_i(t + \theta - l\tau_k)$ an $\hat{z}_{kl}^{(i)}$ for $x_i(t + \xi - l\tau_k)$. Finally, we will use

$$\hat{X}_L = [\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_L], \quad \hat{Y}_L = [\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_L].$$

Note that the relationships given above are only provided to aide understanding. All such variables are, in fact, independent. We have the following conditions for delay-independent stability:

*Proposition 9:* Consider the system described by Equation (9). For a positive integer $L$, suppose there exist polynomials $v_0, v_1 : \mathbb{R}^{n(L+1)} \to \mathbb{R}$, a positive definite, radially unbounded polynomial $\varphi : \mathbb{R}^{n(L+1)} \to \mathbb{R}$ and a non-negative polynomial $\psi : \mathbb{R}^{n(L+1)} \to \mathbb{R}$ such that the following hold for all $\hat{X}_L, \hat{Y}_L \in \mathbb{R}^{n(L+1)}$, and $\hat{x}_{L+1} \in \mathbb{R}^n$:

1) $v_0(\hat{X}_L) - \varphi(\hat{X}_L) \geq 0$,
2) $v_1(\hat{Y}_L) \geq 0$,
3) $\sum_{m=0}^{L} \nabla_{\hat{x}_m} v_0(\hat{X}_L)^T f(\hat{x}_m, \hat{x}_{m+1}) + v_1(\hat{x}_0, \ldots, \hat{x}_L) - v_1(\hat{x}_1, \ldots, \hat{x}_{L+1}) + \psi(\hat{X}_L) \leq 0$,

where

$$\hat{X}_L = [\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_L], \quad \hat{Y}_L = [\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_L].$$

Then the 0 steady-state is globally delay-independent stable. If moreover $\psi$ is positive definite, then the 0 steady-state is globally asymptotically delay-independent stable.

*Proof:* Consider the functional

$$V(\phi) = v_0(\phi(0), \phi(-\tau), \ldots, \phi(-L\tau)) + \int_{-\tau}^{0} v_1(\phi(\theta), \phi(\theta-\tau), \ldots, \phi(\theta - L\tau))d\theta.$$

The first two constraints impose that $V(\phi)$ is non-negative and radially unbounded. The derivative of $V$ along the trajectories of system (9) is:

$$\begin{aligned} \dot{V}(\phi) = & \sum_{m=0}^{L} \nabla_{\phi(-m\tau)} v_0(\phi(0), \phi(-\tau), \ldots, \phi(-L\tau))^T f(\phi(-m\tau), \phi(-(m+1)\tau)) \\ & + v_1(\phi(0), \phi(-\tau), \ldots, \phi(-L\tau)) - v_1(\phi(-\tau), \phi(-2\tau), \ldots, \phi(-(L+1)\tau)) \end{aligned}$$

Under the third condition the above derivative is non-positive. Therefore if all three conditions are satisfied, then by Theorem 22, the steady-state of the system given by (9) is globally stable; since the delay size does not appear explicitly in the above conditions, then the zero steady-state is globally stable independent of delay. Moreover if $\psi > 0$, then the zero steady-state is globally asymptotically stable independent of delay. ∎

The functional $V$ is the functional given by Equation (8), but here we used it to analyze stability of nonlinear systems. In particular, if the system (9) is linear, we recover the conditions given in [7].

Compared to the Lyapunov functional given by (1) that was investigated in the previous section, only the first term of that expression is used, and the kernel $v_1(\phi(0), \phi(\theta), \theta)$ is only a function of $\phi(0)$ and $\phi(\theta)$ with no cross-terms. These restrictions ensure that the delay, $\tau$, does not appear explicitly either in the Lyapunov positivity or the derivative non-positivity conditions.

Making use of Assumption 6, the conditions in the above proposition can be tested algorithmically using the sum of squares decomposition and SOSTOOLS [21]. The functions $\varphi$ and $\psi$ can be constructed as per Equation (7). For illustration, the following is a simple example.

*Example 10:* Consider the system:

$$\dot{x}_1(t) = -x_1(t) + x_2^2(t-\tau), \quad \dot{x}_2(t) = -x_2(t)$$

This system is delay-independent stable, a property which can be proven by constructing a Lyapunov functional $V(z_0)$ with $v_0$ and $v_1$ polynomials of bounded degree. When $v_0$ and $v_1$ are constrained to be second order polynomials, no certificate is found. However, when we search for fourth order polynomials, a certificate of stability is obtained. The functional and its derivative become

$$\begin{aligned} V(x_t) &= x_2^2(t) + \frac{3}{4}x_1^2(t) + (0.5x_1(t) + x_2^2(t))^2 + \int_{-\tau}^{0} x_2^4(t+\theta)d\theta. \\ -\dot{V}(x_t) &= (x_1(t) + x_2^2(t) - x_2^2(t-\tau))^2 + 2x_2^2(t) + \\ &\quad + x_2^2(t)x_2^2(t-\tau) + 2(x_2^2(t) + \frac{1}{4}x_1(t))^2 + \frac{14}{16}x_1(t)^2. \end{aligned}$$

To get this certificate, only a handful of SOSTOOLS commands are required.

In the next three subsections, we will concentrate on local, robust and multiple-delay delay-independent stability analysis.

*1) Local Stability:* Nonlinear systems may have more than one equilibria, or the stability properties of a steady-state may not be global. In order to obtain a local result, we have to restrict our attention to a region around the equilibrium of interest — this is the region $\Omega$ in Theorem 22 in the Appendix. In the sequel, we will be using the following form for the region $\Omega$, but other descriptions can also be accommodated:

$$\Omega = \left\{ x_t \in C : \|x_t\| = \sup_{-\tau \le \theta \le 0} |x(t+\theta)|_\infty \le \gamma \right\}$$

Here $|\cdot|_\infty := \max_i |x_i|$ is the $\infty$-norm on $\mathbb{R}^n$. Related to this is the semi-algebraic set $\{y \in \mathbb{R} : h(y) \le 0\}$, where

$$h(y) := (y - \gamma)(y + \gamma).$$

By using a variant of the $S$-procedure, we can formulate conditions for stability that impose validity of the corresponding Lyapunov conditions when $x_t \in \Omega$. In particular, we have the following result:

*Proposition 11:* Let 0 be a steady-state of system (9) and given $\gamma$, let $h$ be as defined above. For an integer $L \ge 0$, let there exist functions $v_0, v_1 : \mathbb{R}^{n(L+1)} \to \mathbb{R}$, a positive definite function $\varphi : \mathbb{R}^{n(L+1)} \to \mathbb{R}$, non-negative functions $\psi, p_{ij} : \mathbb{R}^{n(L+1)} \to \mathbb{R}$, $i = 1, \dots, n$, $j = 0, \dots, L$ and non-negative functions $q_{ij} : \mathbb{R}^{n(L+2)} \to \mathbb{R}$ for $i = 1, \dots, n$, $j = 0, \dots, L+1$ such that the following hold for all $\hat{X}_L, \hat{Y}_L \in \mathbb{R}^{n(L+1)}$, and $\hat{x}_{L+1} \in \mathbb{R}^n$:

1) $v_0(\hat{X}_L) - \varphi(\hat{X}_L) + \sum_{j=0}^{L} \sum_{i=1}^{n} p_{ij}(\hat{X}_L) h(\hat{x}_j^{(i)}) \ge 0$,
2) $v_1(\hat{Y}_L) \ge 0$,
3) $\sum_{m=0}^{L} \nabla_{\hat{x}_m} v_0(\hat{X}_m)^T f(\hat{x}_m, \hat{x}_{m+1}) + v_1(\hat{x}_0, \dots, \hat{x}_m) - v_1(\hat{x}_1, \dots, \hat{x}_{L+1}) + \psi(\hat{X}_L)$
$- \sum_{j=0}^{L+1} \sum_{i=1}^{n} q_{ij}(\hat{X}_{L+1}) h(\hat{x}_j^{(i)}) \le 0$,

where

$$\hat{X}_L = [\hat{x}_0, \hat{x}_1, \dots, \hat{x}_L], \quad \hat{X}_{L+1} = [\hat{x}_0, \hat{x}_1, \dots, \hat{x}_{L+1}], \quad \hat{Y}_L = [\hat{y}_0, \hat{y}_1, \dots, \hat{y}_L]$$

and $\hat{x}_j^{(i)}$ is the $i^{\text{th}}$ component of $\hat{x}_j$. Then 0 is (locally) delay-independent stable. If moreover $\psi(\hat{X}_L) > 0$, then 0 is (locally) delay-independent asymptotically stable.

*Proof:* Consider the functional

$$V(\phi) = v_0(\phi(0), \phi(-\tau), \dots, \phi(-L\tau)) + \int_{-\tau}^{0} v_1(\phi(\theta), \phi(\theta - \tau), \dots, \phi(\theta - L\tau))d\theta.$$

When $h(\phi_i(-j\tau)) \leq 0$ and $p_{ij}(\phi(0), \phi(-\tau), \ldots, \phi(-L\tau)) \geq 0$ for $i = 1, \ldots, n$ and $j = 0, \ldots, L$, we have from the first two conditions:

$$\begin{aligned} V(\phi) =& v_0(\phi(0), \phi(-\tau), \ldots, \phi(-L\tau)) + \int_{-\tau}^{0} v_1(\phi(\theta), \phi(\theta-\tau), \ldots, \phi(\theta - L\tau))d\theta \\ \geq& -\sum_{j=0}^{L}\sum_{i=1}^{n} p_{ij}(\phi(0), \phi(-\tau), \ldots, \phi(-L\tau))h(\phi_i(-j\tau)) + \varphi(\phi(0), \phi(-\tau), \ldots, \phi(-L\tau)) \\ >& 0 \end{aligned}$$

and so the first Lyapunov condition is satisfied, i.e., $V > 0$ on $\Omega$. The same is true for the derivative condition, given constraint (3) above, and so the zero steady-state of system (9) is locally delay-independent stable. If $\psi > 0$, then

$$-\frac{dV}{dt} > 0 \text{ on } \Omega$$

and so asymptotic stability independent of delay is concluded. ■

The conditions in the above proposition can be tested algorithmically if we assume a polynomial structure of $v_0(\hat{X}_L)$, $v_1(\hat{Y}_L)$, $p_{ij}(\hat{X}_L)$ for $i = 1, \ldots, n$, $j = 0, \ldots, L$ and $q_{ij}(\hat{X}_{L+1})$ for $i = 1, \ldots, n$, $j = 0, \ldots, L+1$; construct $\varphi(\hat{X}_L)$ and possibly $\psi(\hat{X}_L)$ using Equation (7); and replace non-negativity conditions by the existence of a sum of squares decomposition for them. SOSTOOLS can then be used to construct these polynomial functions algorithmically.

At this point we should note that having obtained a Lyapunov function that is valid locally and proves asymptotic stability, the domain of attraction of the steady-state can be estimated as the maximal level set of $V$ that is contained in $\Omega$. This can also be formulated as a SOS program, but it is beyond the scope of this paper.

*2) Robust Stability:* Another important issue is robust stability under parametric uncertainty, which can be treated in a unified way as we will see in the sequel. Consider a time-delay system of the form (9) with an uncertain parameter $p$:

$$\dot{\tilde{x}}(t) = \tilde{f}(\tilde{x}(t), \tilde{x}(t-\tau), p), \tag{10}$$

where $p \in P$, where $P$ is a semi-algebraic set defined by

$$P = \{p \in \mathbb{R}^m \,|\, q_i(p) \leq 0,\ i = 1, \ldots, N\} \tag{11}$$

where $q_i \in \mathbb{R}[p]$. Define a new variable $x(t) := \tilde{x}(t) - \tilde{x}^*$, where $\tilde{x}^*$ is a steady-state of (10), which satisfies $\tilde{f}(\tilde{x}^*, \tilde{x}^*, p) = 0$, and may change as the parameters $p \in P$ vary. Then we have:

$$\dot{x}(t) = \tilde{f}(x(t) + \tilde{x}^*, x(t-\tau) + \tilde{x}^*, p) = f(x(t), x(t-\tau), p, \tilde{x}^*) \tag{12}$$
$$0 = \tilde{f}(\tilde{x}^*, \tilde{x}^*, p) \tag{13}$$

This system has a steady-state at the origin. We assume for simplicity that there is a single steady-state for any given values of the parameters. Then the stability of this steady-state can be tested by constructing a *Parameter-Dependent* Lyapunov functional. See the remark at the end of this section for systems with multiple equilibria. The robust stability properties of the above system can be tested using the following proposition. This constructs a Lyapunov functional which is only parameterized by $p$, even though the Lyapunov functional could be parameterized also by $\tilde{x}^*$.

*Proposition 12:* Consider the system given by (12), where $p \in P$ as defined by (11). For a positive integer $L$, suppose that there exist functions $v_0, v_1 : \mathbb{R}^{n(L+1)} \times \mathbb{R}^m \to \mathbb{R}$, a positive definite radially unbounded function $\varphi : \mathbb{R}^{n(L+1)} \to \mathbb{R}$ and a non-negative function $\psi : \mathbb{R}^{n(L+1)} \to \mathbb{R}$ such that the following conditions hold for all $\hat{X}_L, \hat{Y}_L \in \mathbb{R}^{n(L+1)}$, $\hat{x}_{L+1} \in \mathbb{R}^n$ and $p \in P$:

1) $v_0(\hat{X}_L, p) - \varphi(\hat{X}_L) \geq 0$,
2) $v_1(\hat{Y}_L, p) \geq 0$,
3) $\sum_{m=0}^{L} \nabla_{\hat{x}_m} v_0(\hat{X}_L, p)^T f(\hat{x}_m, \hat{x}_{m+1}, p, \tilde{x}^*) + v_1(\hat{x}_0, \ldots, \hat{x}_L, p) - v_1(\hat{x}_1, \ldots, \hat{x}_{L+1}, p) + \psi(\hat{X}_L) \leq 0$, when (13) is satisfied,

where

$$\hat{X}_L = [\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_L], \quad \hat{Y}_L = [\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_L].$$

Then the steady-state 0 of the system given by (12–13) is robustly globally delay-independent stable for all $p \in P$. Moreover, if $\psi(\hat{x}_L) > 0$, 0 is delay-independent robustly globally asymptotically stable for all $p \in P$.

The proof is based on the fact that the following functional

$$V(\phi) = v_0(\phi(0), \phi(-\tau), \ldots, \phi(-L\tau), p) + \int_{-\tau}^{0} v_1(\phi(\theta), \phi(\theta-\tau), \ldots, \phi(\theta - L\tau), p)d\theta \tag{14}$$

is a Lyapunov-Krasovskii functional, and is omitted for brevity. The constraints defining the set $P$ can be adjoined in the three conditions above using non-negative multipliers. Polynomial multipliers can also be used to adjoin condition (13) to the third constraint. If we assume that all functions involved above are polynomial, then these multipliers can be chosen to be sum of squares and polynomial respectively, allowing their construction to be done using sum of squares programming.

Here we illustrate the construction of the sum of squares program for $L = 0$. This program ensures robust asymptotic stability independent of delays.

Find polynomials $v_0(\hat{x}_0, p)$ and $v_1(\hat{y}_0, p)$
and $\varphi(\hat{x}_0)$ and $\psi(\hat{x}_0)$ constructed as per Equation 7
and sum of squares $\sigma_{1i}(\hat{x}_0, p), \sigma_{2i}(\hat{y}_0, p)$ and $\sigma_{3i}(\hat{x}_0, \hat{x}_1, p, \tilde{x}^*), \quad i = 1, \ldots, N$
and polynomials $r_i(\hat{x}_0, \hat{x}_1, p, \tilde{x}^*), \quad i = 1, \ldots, n$
Such that:

$$v_0(\hat{x}_0, p) - \varphi(\hat{x}_0) + \sum_{i=1}^{N} \sigma_{1i}(\hat{x}_0, p) q_i(p) \text{ is a SOS} \tag{15}$$

$$v_1(\hat{y}_0, p) + \sum_{i=1}^{N} \sigma_{2i}(\hat{y}_0, p) q_i(p) \text{ is a SOS} \tag{16}$$

$$\begin{aligned} &- \nabla_{\hat{x}_0} v_0(\hat{x}_0, p)^T f(\hat{x}_0, \hat{x}_1, p, \tilde{x}^*) - v_1(\hat{x}_0, p) + v_1(\hat{x}_1, p) - \psi(\hat{x}_0) \\ &\qquad + \sum_{i=1}^{N} \sigma_{3i}(\hat{x}_0, \hat{x}_1, p, \tilde{x}^*) q_i(p) + \sum_{i=1}^{n} r_i(\hat{x}_0, \hat{x}_1, p, \tilde{x}^*) \tilde{f}_i(\tilde{x}^*, \tilde{x}^*, p) \text{ is a SOS} \end{aligned} \tag{17}$$

*Remark 13:* Many times there are multiple steady-states in (10) that move as $p$ is allowed to vary in $P$ and the result we seek in that case is a local result. In this case, the parameter set $P$ should be extended to include the 'motion' of the steady-state $\tilde{x}^*$ and the region $\Omega$ has to be sufficiently small so that no other equilibria cross into $\Omega$ as the parameters change within $P$. An example of this case can be found in Section IV.

*3) Multiple Delays:* Here we consider a system of the form:

$$\dot{x}(t) = f(x(t), x(t - \tau_1), \ldots, x(t - \tau_K)) \triangleq f(\hat{x}_0, \ldots, \hat{x}_K). \tag{18}$$

As mentioned already, we assume that $0 \le \tau_1 \le \ldots \le \tau_K = \tau$, i.e., the delays are put in non-decreasing order, with $\tau_0 = 0$, and $f$ is polynomial in its arguments. For simplicity, here we only develop the case in which there is a single steady-state and we seek to show global stability. Moreover, we set $L = 1$. Simple sufficient delay-independent stability conditions for global stability can be found in the following proposition:

*Proposition 14:* Consider the system given by (18) with 0 a steady state. Suppose there exist functions $v_0, v_{1i} : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, K$, a positive definite radially unbounded function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and a non-negative function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that for all $\hat{x}_0, \ldots, \hat{x}_K, \hat{y}_0, \ldots, \hat{y}_K \in \mathbb{R}^n$:

1) $v_0(\hat{x}_0) - \varphi(\hat{x}_0) \ge 0$,
2) $v_{1i}(\hat{y}_0) \ge 0$ for all $i = 1, \ldots, K$,
3) $\nabla_{\hat{x}_0} v_0(\hat{x}_0)^T f(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K) + \sum_{i=1}^{K} (v_{1i}(\hat{x}_0) - v_{1i}(\hat{x}_i)) + \psi(\hat{x}_0) \le 0$.

Then the steady-state is globally stable. If $\psi(\hat{x}_0) > 0$, then the steady-state is globally asymptotically stable.
The proof is omitted for brevity, and uses the following as a Lyapunov-Krasovskii functional:

$$V = v_0(\phi(0)) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} v_{1i}(\phi(\theta)) d\theta.$$

### *B. Delay-dependent stability*

When the stability properties of the steady-state change as the delay size, seen as a static parameter, changes, the stability is termed *delay-dependent*. In this case, a different type of Lyapunov functional has to be used to allow for the delay size to appear explicitly in the Lyapunov conditions.

For a linear time-delay system of the form $\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{K} A_i x(t - \tau_i)$ where $A_0, A_1, \ldots, A_K \in \mathbb{R}^{n \times n}$ are fixed matrices, the complete Lyapunov structure for delay-dependent stability takes the form:

$$V(\phi) \quad = \quad \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix} d\theta + \int_{-\tau}^{0} \int_{-\tau}^{0} \phi(\theta)^T R(\theta, \xi) \phi(\xi) d\theta d\xi \tag{19}$$

where $M : \mathbb{R} \to \mathcal{S}^{2n}$ and $R : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ are piecewise continuous functions with possible discontinuities at $\xi, \theta = \tau_i$ for $i = 1, \ldots, K-1$. Here $\tau = \max_i \tau_i$. As mentioned already, the conditions for positivity of $V$ and non-negativity of the derivative condition are infinite dimensional and difficult to test. Here we will use the sum of squares decomposition to construct them. The structure of the Lyapunov-Krasovskii functional we will be considering will take the form (1) which we repeat here for convenience:

$$V(\phi) = \int_{-\tau}^{0} v_1(\phi(0), \phi(\theta), \theta) d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} v_2(\phi(\xi), \phi(\theta), \theta, \xi) d\theta d\xi$$

Note that this functional reduces to the complete Lyapunov functional (19) used for linear time-delay systems when $v_1$ and $v_2$ are quadratic in $\phi$. We have the following result, again referring to the system given by (9), i.e., with one delay. The multiple delay case will be considered next.

*Proposition 15:* Let 0 be a steady-state for the system given by (9) with a polynomial vector field. Given $Z_d[\hat{y}, \theta]$, let $Q \succeq 0$ be of appropriate dimensions and define

$$v_2(\hat{y}, \hat{z}, \theta, \xi) = Z_d[\hat{y}, \theta]^T Q Z_d[\hat{z}, \xi]. \tag{20}$$

Suppose also that there exists a function $v_1 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, a radially unbounded positive definite function $\varphi : \mathbb{R}^n \to \mathbb{R}$, as well as a non-negative function $\psi : \mathbb{R}^n \to \mathbb{R}$, and functions $t_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $t_2 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, all polynomials, that satisfy the following conditions for all $\hat{x}_0, \ldots, \hat{x}_K, \hat{y}_0, \ldots, \hat{y}_K \in \mathbb{R}^n$:

(1) $v_1(\hat{x}_0, \hat{y}, \theta) + t_1(\hat{x}_0, \theta) - \varphi(\hat{x}_0) \geq 0$, for $\theta \in [-\tau, 0]$,

(2) $\nabla_{\hat{x}_0} v_1(\hat{x}_0, \hat{y}, \theta)^T f(\hat{x}_0, \hat{x}_1) - \dfrac{\partial v_1(\hat{x}_0, \hat{y}, \theta)}{\partial \theta} + \dfrac{1}{\tau} v_1(\hat{x}_0, \hat{x}_0, 0) - \dfrac{1}{\tau} v_1(\hat{x}_0, \hat{x}_1, -\tau) + 2v_2(\hat{y}, \hat{x}_0, \theta, 0)$
$-2v_2(\hat{y}, \hat{x}_1, \theta, -\tau) + t_2(\hat{x}_0, \hat{x}_1, \theta) + \psi(\hat{x}_0) \leq 0$ for $\theta \in [-\tau, 0]$,

(3) $\displaystyle\int_{-\tau}^{0} t_1(\hat{x}_0, \theta)\, d\theta = 0$,

(4) $\displaystyle\int_{-\tau}^{0} t_2(\hat{x}_0, \hat{x}_1, \theta)\, d\theta = 0$.

(5) There exists a $R \succeq 0$ so that

$$\frac{\partial v_2}{\partial \theta}(\hat{y}, \hat{z}, \theta, \xi) + \frac{\partial v_2}{\partial \xi}(\hat{y}, \hat{z}, \theta, \xi) = Z_d[\hat{y}, \theta]^T R Z_d[\hat{z}, \xi].$$

Then the steady-state 0 of the system given by (34) is *globally stable* for delay size $\tau$. Moreover, if $\psi(\hat{x}_0) > 0$, then 0 is *globally asymptotically stable* for delay size $\tau$.

*Proof:* Consider the following functional:

$$V(\phi) \quad = \quad \int_{-\tau}^{0} v_1(\phi(0), \phi(\theta), \theta) d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} v_2(\phi(\xi), \phi(\theta), \theta, \xi) d\theta d\xi \tag{21}$$

where $v_2$ is given by (20). The first term is positive definite owing to conditions (1) and (3) in the above proposition, and the fact that $\varphi(\phi(0)) > 0$. The second term is positive semidefinite by construction, as $Q \succeq 0$. Therefore the first Lyapunov condition is satisfied.

The time derivative of $V(\phi)$ is:

$$\dot{V}(\phi) = \int_{-\tau}^{0} \begin{pmatrix} \nabla_{\phi(0)} v_1(\phi(0), \phi(\theta), \theta)^T f(\phi(0), \phi(-\tau)) - \frac{\partial v_1}{\partial \theta}(\phi(0), \phi(\theta), \theta) \\ +\frac{1}{\tau} v_1(\phi(0), \phi(0), 0) - \frac{1}{\tau} v_1(\phi(0), \phi(-\tau), -\tau) \\ +2v_2(\phi(\theta), \phi(0), \theta, 0) - 2v_2(\phi(\theta), \phi(-\tau), \theta, -\tau) \end{pmatrix} d\theta$$
$$- \int_{-\tau}^{0}\int_{-\tau}^{0} \left( \frac{\partial v_2}{\partial \theta}(\phi(\xi), \phi(\theta), \theta, \xi) + \frac{\partial v_2}{\partial \xi}(\phi(\xi), \phi(\theta), \theta, \xi) \right) d\theta d\xi.$$

This derivative condition has a form similar to (2), which was discussed in Section II. Conditions (2) and (4) in the Proposition guarantee that the first term of $\dot{V}$ is non-positive for $\theta \in [-\tau, 0]$. Condition (5) guarantees that the second term in $\dot{V}$ is non-positive. So (21) is a Lyapunov-Krasovskii functional, and the zero steady-state is *stable*. Since there is no constraint on the state-space, and $\varphi$ is radially unbounded, the result holds globally. Moreover if $\psi > 0$, then the steady-state is globally asymptotically stable for delay $\tau$. ■

In order to algorithmically construct the Lyapunov-Krasovskii functional for the nonlinear system, we can use the above proposition in a similar way as described in the delay-independent case. To impose the conditions $\theta \in [-\tau, 0]$, we use a process similar to the S-procedure. Different Lyapunov-Krasovskii structures can also be used and similar conditions can be imposed.

An important issue which is unique to the case of delay-dependent stability, is to prove that the stability properties hold for a *delay interval* rather than for a specific value of the delay. The conditions in Proposition 15 are not affine in $\tau$. However, one can consider $\tau$ as a static parameter, itself being allowed to vary within the interval $\tau \in [0, \overline{\tau}]$. One can then view this as a robustness problem and construct a (possibly parameterized) Lyapunov-Krasovskii functional that guarantees delay-dependent stability for the whole delay interval.

Similar arguments allow the construction of Lyapunov-Krasovskii functionals for *local* delay-dependent stability and robust delay-dependent stability under parameter variations. Here, we present a simple example to show how Lyapunov-Krasovskii functionals can be constructed using SOSTOOLS to verify delay-dependent stability of the equilibrium of a simple delay differential equation.

*Example 16:* Consider the delayed logistic (Hutchinson's) equation

$$\frac{dx}{dt} = \alpha x(t-1)[1+x(t)]$$

which has been used extensively in the past to model single species growth with delay in population dynamics. By looking at the properties of the solution and not resorting to Lyapunov arguments, it was shown that the stability condition is $\alpha < \frac{37}{24}$ for $x_t > -1$ [2].

The proposition above can be used to construct algorithmically a Lyapunov-Krasovskii functional of the form

$$V(\phi) = \int_{-1}^{0} v_1(\phi(0), \phi(\theta), \theta) d\theta + \int_{-1}^{0}\int_{-1}^{0} v_2(\phi(\theta), \phi(\xi)) d\theta d\xi$$

to verify the (local) asymptotic stability of the zero equilibrium. For example, for $\alpha = 1.2$ and for $|x_t| < 0.2$ the following polynomials have been constructed:

$$\begin{aligned}
v_1(\hat{x}_0, \hat{y}, \theta) &= (3.88 + \theta + 0.73\theta^2)\hat{x}_0^2 - (6.15 + 0.45\theta + 1.5\theta^2)\hat{x}_0\hat{y} \\
&\quad + (4.5 + 2.5\theta + 0.13\theta^2)\hat{y}^2 \\
v_2(\hat{y}, \hat{z}) &= \begin{bmatrix} \hat{y} \\ \hat{y}^2 \end{bmatrix}^T \begin{bmatrix} 2.2 & 0.1 \\ 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} \hat{z} \\ \hat{z}^2 \end{bmatrix} \\
&= 2.2052(0.9986\hat{y} + 0.0524\hat{y}^2)(0.9986\hat{z} + 0.0524\hat{z}^2) \\
&\quad + 0.2948(-0.0524\hat{y} + 0.9986\hat{y}^2)(-0.0524\hat{z} + 0.9986\hat{z}^2)
\end{aligned}$$

It can be readily seen that $\int_{-1}^{0}\int_{-1}^{0} v_2(\phi(\theta), \phi(\xi)) d\theta d\xi \geq 0$. In order to verify that $\int_{-1}^{0} v_1(\phi(0), \phi(\theta), \theta) d\theta > 0$, we constructed multipliers as shown in the proposition above. In particular, for $q(\theta) = \theta(\theta+1) \leq 0$ we obtained the multiplier

$$\sigma(\hat{x}_0, \hat{y}) = 2\hat{y}^2 - 2.5\hat{x}_0\hat{y} + \hat{x}_0^2 \geq 0.$$

Moreover, a function $t_1$ was constructed, of the form

$$t_1(\hat{x}_0, \theta) = \left(-1.42 - 0.4\theta + 3.66\theta^2\right)\hat{x}_0^2$$

for which $\int_{-1}^{0} t_1(\hat{x}_0, \theta) d\theta = 0$. Given these functions, we can verify that

$$\begin{aligned}
v_1 + q\sigma + t_1 - 0.06\hat{x}_0^2 &= \begin{bmatrix} \hat{x}_0 \\ \hat{y} \end{bmatrix}^T \begin{bmatrix} 2.4 + 1.6\theta + 5.39\theta^2 & -(3.075 + 1.475\theta + 2\theta^2) \\ -(3.075 + 1.475\theta + 2\theta^2) & 4.5 + 4.5\theta + 2.13\theta^2 \end{bmatrix} \begin{bmatrix} \hat{x}_0 \\ \hat{y} \end{bmatrix} \\
&\geq 0
\end{aligned}$$

since the matrix in the middle is positive semidefinite for all $\theta$ (a plot of its eigenvalues could verify that). Integrating the left hand side on $[-1, 0]$ we satisfy

$$\int_{-1}^{0} v_1(\phi(0), \phi(\theta), \theta) d\theta \geq 0.06\phi(0)^2$$

as required. The derivative condition is also satisfied (details are omitted for brevity). Therefore, the above functional shows local asymptotic stability of the zero equilibrium for $\alpha = 1.2$.

We now turn our attention to the problem of investigating stability for nonlinear time-delay systems with multiple delays.

*1) The Multiple-Delay Case:* The presence of multiple, often incommensurate delays in a functional differential equation causes significant complications in analysis. Necessary and sufficient conditions for stability in this case are very difficult to test — in fact, the problem of deciding stability is NP-hard [19]. This of course does not mean that for certain delay values the stability property cannot be verified by considering certificates of a particular structure: this is the approach we follow in this section, in which we propose tests for delay-dependent stability.

The Lyapunov functions we will be using in this case have piecewise continuous kernels, and take the form (1). Sufficient conditions for delay-dependent stability can then be written as follows:

*Proposition 17:* Consider the system given by (18) with $0$ a steady state and $f$ a polynomial vector field. Given a vector of monomials $Z_d[\hat{y},\theta]$ consider $Q \succeq 0$ and let $Q_{ij}$ be the $i$,$j$th block of $Q$ and

$$v_{2ij}(\hat{y},\hat{z},\theta,\xi) = Z_d^T[\hat{y},\theta] Q_{ij} Z_d[\hat{z},\xi], \text{ for } i = 1,\ldots,K, \quad j = 1,\ldots,K.$$

Furthermore, suppose there exist polynomials $v_{1i}(\hat{x}_0,\hat{y},\theta)$, $t_{1i}(\hat{x}_0,\theta)$ and $t_{2i}(\hat{x}_0,\hat{x}_1,\ldots,\hat{x}_K,\theta)$ for $i = 1,\ldots,K$, a positive definite radially unbounded polynomial function $\varphi(\hat{x}_0) > 0$ and a non-negative polynomial function $\psi(\hat{x}_0)$ such that for all $\hat{y},\hat{x}_0,\ldots,\hat{x}_K \in \mathbb{R}^n$:

(1) $v_{1i}(\hat{x}_0,\hat{y},\theta) + t_{1i}(\hat{x}_0,\theta) - \varphi(\hat{x}_0) \geq 0$, for all $\theta \in [-\tau_i,-\tau_{i-1}]$ for each $i = 1,\ldots,K$.

(2) $$\nabla_{\hat{x}_0} v_{1i}(\hat{x}_0,\hat{y},\theta)^T f(\hat{x}_0,\hat{x}_1,\ldots,\hat{x}_K) - \frac{\partial v_{1i}}{\partial \theta}(\hat{x}_0,\hat{y},\theta) + \frac{1}{\tau_i - \tau_{i-1}} v_{1i}(\hat{x}_0,\hat{x}_{i-1},-\tau_{i-1})$$
$$-\frac{1}{\tau_i - \tau_{i-1}} v_{1i}(\hat{x}_0,\hat{x}_i,-\tau_i) + 2\sum_{j=1}^{K} v_{2ij}(\hat{x}_{j-1},\hat{y},-\tau_{j-1},\theta)$$
$$-2\sum_{j=1}^{K} v_{2ij}(\hat{x}_j,\hat{y},-\tau_j,\theta) + t_{2i}(\hat{x}_0,\hat{x}_1,\ldots,\hat{x}_K,\theta) + \psi(\hat{x}_0) \leq 0$$
for all $\theta \in [-\tau_i,-\tau_{i-1}]$ and for each $i = 1,\ldots,K$,

(3) $\displaystyle\sum_{i=1}^{K} \int_{I_i} t_{1i}(\hat{x}_0,\theta) d\theta = 0,$

(4) $\displaystyle\sum_{i=1}^{K} \int_{I_i} t_{2i}(\hat{x}_0,\hat{x}_1,\ldots,\hat{x}_K,\theta) d\theta = 0.$

(5) There exist $R \succeq 0$ and let $R_{ij}$, the $i$,$j$th block of $R$ satisfy
$$\frac{\partial v_{2ij}}{\partial \theta}(\hat{y},\hat{z},\theta,\xi) + \frac{\partial v_{2ij}}{\partial \xi}(\hat{y},\hat{z},\theta,\xi) = Z_d[\hat{y},\theta]^T R_{ij} Z_d[\hat{z},\xi], \quad i,j = 1,\ldots,K.$$

Then the steady-state is globally stable. If $\psi(\hat{x}_0) > 0$, then the steady-state is globally asymptotically stable.

*Proof:* The proof of this proposition is based on ensuring that the following functional is a Lyapunov-Krasovksii functional:

$$V(\phi) \;=\; \sum_{i=1}^{K} \int_{I_i} v_{1i}(\phi(0),\phi(\theta),\theta) d\theta + \sum_{i=1}^{K}\sum_{j=1}^{K} \int_{I_i}\int_{I_j} v_{2ij}(\phi(\theta),\phi(\xi),\theta,\xi) d\theta d\xi. \tag{22}$$

Conditions (1) and (3) in the proposition ensure that $V(\phi) > 0$, so the first Lyapunov condition is satisfied and moreover $V$ is radially unbounded.

The derivative of this functional along the trajectories of (18) is:

$$\dot{V}(\phi) = \sum_{i=1}^{K} \int_{I_i} \begin{pmatrix} \nabla_{\phi(0)} v_{1i}(\phi(0),\phi(\theta),\theta)^T f(\phi(0),\phi(-\tau_1),\ldots,\phi(-\tau_K)) - \frac{\partial v_{1i}}{\partial \theta}(\phi(0),\phi(\theta),\theta) \\ +\frac{1}{\tau_i - \tau_{i-1}} \left(v_{1i}(\phi(0),\phi(-\tau_{i-1}),-\tau_{i-1}) - v_{1i}(\phi(0),\phi(-\tau_i),-\tau_i)\right) \\ +2\sum_{j=1}^{K} v_{2ij}(\phi(-\tau_{j-1}),\phi(\theta),-\tau_{j-1},\theta) \\ -2\sum_{j=1}^{K} v_{2ij}(\phi(-\tau_j),\phi(\xi),-\tau_j,\theta) \end{pmatrix} d\theta$$
$$-\sum_{i=1}^{K}\sum_{j=i}^{K} \int_{I_i}\int_{I_j} \left( \frac{\partial v_{2ij}}{\partial \theta}(\phi(\theta),\phi(\xi),\theta,\xi) + \frac{\partial v_{2ij}}{\partial \xi}(\phi(\theta),\phi(\xi),\theta,\xi) \right) d\theta d\xi$$

The non-positivity of this time derivative is ensured by condition (2) when condition (4) is taken into account, as well as the decomposition given by (5). Therefore $V(\phi)$ is a Lyapunov-Krasovskii functional that proves global stability of the steady-state. If $\psi(\phi) > 0$, then $\dot{V}(\phi) < 0$ and the steady-state is globally asymptotically stable. ■

In Section IV, we will see an example of a system from network congestion control for the Internet, where the inhomogeneous communication delays are taken into account in the stability test.

## IV. Examples

In this section we present two more complicated examples, one from population dynamics, and one from network congestion control for the Internet.

*A. Robust stability analysis of a predator-prey model*

A simple model of predator-prey interactions is

$$\dot{x} = bx - k_1xy, \quad \dot{y} = k_2xy - \sigma y,$$

where $x$ and $y$ are the prey and predator populations, $b$ is the rate of increase of prey, $k_1$ and $k_2$ are the coefficients of the effect of predation on $x$ and $y$ and $\sigma$ is the death rate of $y$. Evidently, the cause of death of the prey is due to predation alone, and the growth of the predator population has as the only limitation the number of prey. These equations give rise to Lotka-Volterra predator-prey cycles, but the model is not biologically meaningful because it is *conservative* giving rise to a family of closed trajectories rather than a single limit cycle [22].

The above equations describe ideal populations that can react instantaneously to any change in the environment; in real populations this change comes with a delay that represents *maturation* of the predator population. A more realistic set of equations is [23]:

$$\begin{aligned}\dot{x}(t) &= x(t)[b - ax(t) - k_1y(t)],\\ \dot{y}(t) &= -\sigma y(t) + k_2x(t-\tau)y(t-\tau),\end{aligned}$$

where $-ax(t)^2$ limits the growth of the prey, and $\tau \geq 0$ is a constant capturing the average period between death of prey and birth of a subsequent number of predators.

*Assumption 18:* $a, b, k_1, k_2$ and $\sigma$ are positive.

The equilibria $(x^*, y^*)$ of the above system are:

$$\begin{aligned}(x^*, y^*) &= (0,0), \quad (x^*, y^*) = (b/a, 0),\\ (x^*, y^*) &= \left(\frac{\sigma}{k_2}, \frac{bk_2 - a\sigma}{k_1k_2}\right). \end{aligned} \tag{23}$$

We are only interested in the steady-state given by (23).

*Assumption 19:* $(bk_2 - a\sigma) > 0$.

Assumption 19 ensures that the steady-state (23) is in the positive quadrant. We now shift the coordinates to $(x_1, x_2) = (x - \frac{\sigma}{k_2}, y - \frac{bk_2 - a\sigma}{k_1k_2})$ to get:

$$\dot{x}_1(t) = \left[x_1(t) + \frac{\sigma}{k_2}\right][-ax_1(t) - k_1x_2(t)] \tag{24}$$

$$\dot{x}_2(t) = -\sigma x_2(t) + \sigma x_2(t-\tau) + \frac{bk_2 - a\sigma}{k_1}x_1(t-\tau) + k_2x_1(t-\tau)x_2(t-\tau) \tag{25}$$

We can linearize the above system about $(0,0)$ to get:

$$\dot{x}_1(t) = \frac{\sigma}{k_2}[-ax_1(t) - k_1x_2(t)] \tag{26}$$

$$\dot{x}_2(t) = -\sigma x_2(t) + \sigma x_2(t-\tau) + \frac{bk_2 - a\sigma}{k_1}x_1(t-\tau) \tag{27}$$

For the linearised system, we have the following result:

*Proposition 20:* Consider the system (26–27) under the assumptions (18,19). Then if $(bk_2 - 3a\sigma) < 0$ the zero steady-state is stable independent of the delay. If $(bk_2 - 3a\sigma) > 0$ the zero steady-state is stable if the delay satisfies $\tau < \tau^*$ and is unstable otherwise, where $\tau^*$ is given by:

$$\tau^* = \frac{1}{\omega}\ \text{atan}\ \left[\omega\frac{(a\sigma^2 - \omega k_2)k_2 - \sigma(2a\sigma + bk_2)(k_2 + a)}{k_2\sigma\omega^2(k_2 + a) + (2a\sigma - bk_2)(a\sigma^2 - \omega k_2)}\right]$$

where $\omega$ solves

$$\omega^4 + \frac{a^2\sigma^2}{k_2^2}\omega^2 + \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2) = 0.$$

*Proof:* In the absence of delay, and under the two assumptions, the system is *asymptotically stable*. Substituting $s = j\omega$ in the characteristic equation and separating real and imaginary parts we get:

$$\begin{aligned}-\omega^2 + \frac{a\sigma^2}{k_2} &= \sigma\omega\sin(\omega\tau) + \sigma\left(\frac{2a\sigma}{k_2} - b\right)\cos(\omega\tau)\\ \sigma\left[1 + \frac{a}{k_2}\right]\omega &= \sigma\omega\cos(\omega\tau) - \sigma\left(\frac{2a\sigma}{k_2} - b\right)\sin(\omega\tau)\end{aligned}$$

Squaring the two equations and adding we get:

$$\omega^4 + \frac{a^2\sigma^2}{k_2^2}\omega^2 + \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2) = 0. \tag{28}$$

Denoting $p_1 = \frac{a^2\sigma^2}{k_2^2}$ and $p_2 = \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2)$ the roots of this equation are:

$$\omega^2 = -\frac{p_1}{2} \pm \frac{\sqrt{p_1^2 - 4p_2}}{2}. \tag{29}$$

Under assumption 19, if $(bk_2 - 3a\sigma) < 0$ (i.e $p_2 > 0$), then there are no real solutions to (28). Since the steady-state is stable when the delay is zero, and there is no $\omega$ for which poles cross to the RHP, we conclude that (26–27) is delay-independent stable.

Under assumption 19 and $(bk_2 - 3a\sigma) > 0$ then $p_2 < 0$ and one of the two roots of (29) is positive and the other one is negative. Therefore the poles cross the imaginary axis at only one $\omega$ — there is no possibility for *stability reversal*. If $\omega$ is the solution to the above equation, then at $\tau = \tau^*$ given in the statement of the Proposition a Hopf bifurcation occurs; the system is stable for $\tau < \tau^*$ and unstable for $\tau > \tau^*$. ■

We now analyze the nonlinear description of the system (24–25) using the methodology that was developed in the previous sections. We choose as nominal values for the parameters $\sigma = 10$, $a = 1$, $k_1 = 1$, and $k_2 = 3$.

*1) Delay-independent stability analysis:* The system (24–25) has many equilibria, and so we need to define a region around the zero steady-state to obtain a stability condition (this is the region $\Omega$ in Theorem 22). We let:

$$|x_{1_t}| \leq \gamma_1 x^*, \quad |x_{2_t}| \leq \gamma_2 y^*, \tag{30}$$

where the steady-state $(x^*, y^*)$ is given by (23). We consider $b$ to be a parameter in the problem. From Proposition 20, the linear version of this system is delay-independent stable when $\frac{a\sigma}{k_2} < b < \frac{3a\sigma}{k_2}$. For the given values of $a, \sigma$ and $k_2$, the system is delay-independent stable for $10/3 < b < 10$. For the purpose of calculating $(x^*, y^*)$ we use a value of $b = 20/3$. The steady-state $(0, 0)$ of system (24–25) does not move as $b$ changes, however the other two equilibria cross through the region defined by (30). If we choose $\gamma_1 = \gamma_2 = 0.1$, then no other steady-state enters this region for $11/3 < b < 10$.

We consider the following Lyapunov structure:

$$V(x_t) = v_0(x_1(t), x_2(t), b) + \int_{-\tau}^{0} v_1(x_1(t+\theta), x_2(t+\theta), b)d\theta.$$

We use Proposition 12 to obtain parameter regions for which robust delay-independent stability of the origin can be proven. When $v_0$ is second order and $v_1$ is 4th order, we can construct $V(x_t)$ for $4.56 \leq b \leq 7.11$. When they are 4th and 6th order respectively, then this region becomes $3.67 \leq b \leq 9.95$, which is essentially the full interval.

*2) Delay-dependent stability analysis:* Now we will try to test values for $\tau$ for which stability is retained using the same parameters as before and fixed $b = 15$. Given these parameters $\tau^* = 0.0541$. The system has several equilibria and so we use the same constraints on $x_1$ and $x_2$ on the state-space given by (30) with $\gamma_1 = \gamma_2 = 0.1$.

We can construct the Lyapunov functional $V(x_t)$ given by (21) with $v_1$ 0th order with respect to $\theta$ and 2nd order with respect to the rest of the variables for $\tau = 0.04$. When $v_1$ is quartic with respect to all variables but $\theta$ (which is kept at 0 order) then we can construct this $V(x_t)$ for $\tau = 0.053$. Here instead of the term $V_2$ in (5) we use

$$V_2 = \int_{-\tau}^{0}\int_{t+\theta}^{t} v_2(\phi(\zeta))d\zeta d\theta.$$

Terms like these were used in [9] and are useful when $v_1$ is zeroth order in $\theta$. The corresponding SDP is bigger as the functional is more complicated, but we can see that larger values of the delay closer to the stability boundary can be tested.

### *B. Network Congestion Control for the Internet*

Internet congestion control [1] is a distributed algorithm to allocate available bandwidth to competing sources so as to avoid congestion collapse by ensuring that link capacities are not exceeded. The need for congestion control for the Internet emerged in the mid-1980s, when congestion collapse resulted in unreliable file transfer. In 1988 Jacobson [24] proposed an admittedly ingenious scheme for congestion control. The shortcomings of this scheme and its successors such as TCP Reno and Vegas have only recently become apparent: they are not scalable to arbitrary networks with very large capacities and multiple, non-commensurate time-delays. New designs of Active Queue Management (AQM) and/or Transmission Control Protocol (TCP) schemes have been proposed that provide scalable stability in the presence of heterogeneous delays, which can be verified at least for the linearization about a steady-state.

The simplest adequate modeling framework for network congestion control is in the form of nonlinear deterministic delay-differential equations [25]–[27]. Some work has been done on the analysis of such systems usually for the single-bottleneck link case, using either Lyapunov-Razumikhin functions or Lyapunov-Krasovskii functionals, IQC methods, passivity etc.

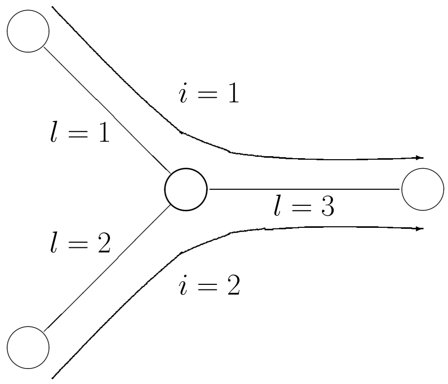


Fig. 1. A network topology under consideration.

In this section we analyze a simple network instantiation of what is known a 'primal' congestion control scheme, shown in Figure 1. It consists of $L = 3$ links, labeled $l = 1, 2, 3$ and $S = 2$ sources, $i = 1, 2$. For this network, we define an $L \times S$ routing matrix $R$ which is 1 if source $i$ uses link $l$ and 0 otherwise:

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

The architecture of network congestion control is shown in Figure 2. To each source $i$ we associate a transmission rate $x_i$. All sources whose flow passes through resource $l$ contribute to the *aggregate rate* $y_l$ for resource $l$, the rates being added with some forward time delay $\tau^f_{i,l}$. Hence we have:

$$y_l(t) = \sum_{i=1}^{S} R_{li} x_i(t - \tau^f_{i,l}) \triangleq r_f(x_i, \tau^f_{i,l}) \tag{31}$$

The resources $l$ react to the aggregate rate $y_l$ by setting congestion information $p_l$, the price at resource $l$. This is the Active Queue Management part of the picture. The prices of all the links that source $i$ uses are aggregated to form $q_i$, the *aggregate price* for source $i$, again through a delay $\tau^b_{i,l}$:

$$q_i(t) = \sum_{l=1}^{L} R_{li} p_l(t - \tau^b_{i,l}) \triangleq r_b(p_l, \tau^b_{i,l}) \tag{32}$$

The prices $q_i$ can then be used to set the rate of source $i$, $x_i$, which completes the loop. The forward and backward delays can be combined to yield the Round Trip Time (RTT):

$$\tau_i = \tau^f_{i,l} + \tau^b_{i,l}, \quad \forall\ l \tag{33}$$

The capacity of link $l$ is given by $c_l$. For a general network, the interconnection is shown in Figure 2. In this section we choose the control laws for TCP and AQM as follows:

$$\begin{aligned} p_l(t) &= \left(\frac{y_l(t)}{c_l}\right)^B \\ \dot{x}_i(t) &= 1 - x_i(t - \tau_i) q_i(t) \end{aligned}$$

Here $p_l(t)$ corresponds to the probability that the queue length exceeds $B$ in a $M/M/1$ queue with arrival rate $y_l(t)$ and capacity $c_l$, and the source law corresponds to a queue length with proportionally fair source dynamics.

For the network shown in Figure 1, the closed loop dynamics become:

$$\begin{aligned} \dot{x}_1 &= 1 - x_1(t - \tau_1) \left[ \left(\frac{x_1(t-\tau_1)}{c_1}\right)^B + \left(\frac{x_1(t-\tau_1) + x_2(t - \tau^b_{1,3} - \tau^f_{2,3})}{c_3}\right)^B \right] \\ \dot{x}_2 &= 1 - x_2(t - \tau_2) \left[ \left(\frac{x_2(t-\tau_2)}{c_2}\right)^B + \left(\frac{x_2(t-\tau_2) + x_1(t - \tau^f_{1,3} - \tau^b_{2,3})}{c_3}\right)^B \right] \end{aligned}$$

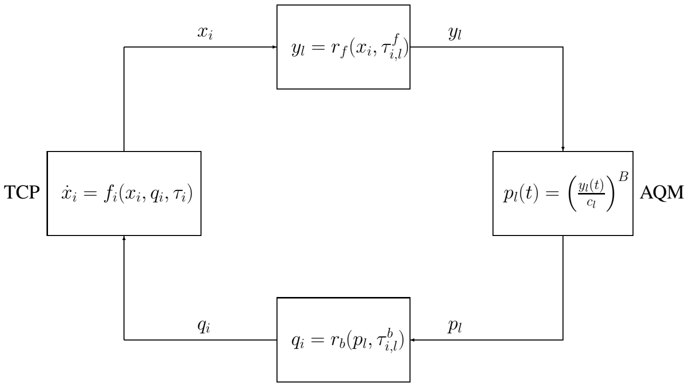


Fig. 2. The internet as an interconnection of sources and links through delays.

Let $B = 2$, $c_1 = c_2 = 3$, and $c_3 = 1$. Then the steady-state of this system is $(x_1^*, x_2^*) = (0.6242, 0.6242)$. We consider delay sizes such that $\tau^b_{1,3} = 63$ms, $\tau^f_{1,3} = 93$ms, $\tau^f_{2,3} = 49$ms, $\tau^b_{2,3} = 77$ms. The system equations about the new steady-state become:

$$\begin{aligned}
\dot{x}_1' &= 1 - [x_1'(t-0.154) + x_1^*] \\
&\quad \left[\left(\frac{x_1'(t-0.154)+x_1^*}{3}\right)^2 + (x_1'(t-0.154) + x_2'(t-0.14) + x_1^* + x_2^*)^2\right] \\
\dot{x}_2' &= 1 - [x_2'(t-0.126) + x_2^*] \\
&\quad \left[\left(\frac{x_2'(t-0.126)+x_2^*}{3}\right)^2 + (x_2'(t-0.126) + x_1'(t-0.14) + x_1^* + x_2^*)^2\right]
\end{aligned}$$

where $x_i(t) = x_i'(t) + x^*$. The linearization of this system about the steady-state is stable and a Lyapunov function of the form (22) can be constructed. The same Lyapunov function can be constructed in a region of the state-space satisfying $\|x_{1_t}\| \leq 0.8x_1^*$ and $\|x_{2_t}\| \leq 0.8x_2^*$, thus showing that the steady-state is nonlinearly stable. The Lyapunov functional has the following form: all $v_{1i}$ are 4th order, quadratic in $(\phi(0), \phi(\theta))$ and at most second order in $\theta$ and $Z_d$ to construct $v_{2ij}$ are third order: at most first order in $\phi(\theta)$ and second order in $\theta$.

## V. Concluding Remarks

In this paper we presented a methodology to construct Lyapunov-Krasovskii functionals for time delay systems based on the Sum of Squares decomposition. The construction is entirely algorithmic and is achieved through the solution of a set of Linear Matrix Inequalities (LMIs). Linear and nonlinear delay-independent and delay-dependent stability can now be treated using the same tools.

The above methods can be easily extended to systems with distributed delays. Still a judicious choice for the structure of the Lyapunov functional would be required. Functional differential equations of neutral type can also be treated in a unified way. These are topics of future research.

## Appendix I
## Stability of Time-delay systems

In this appendix we will present some facts about autonomous Delay Differential Equations and introduce the notion of Lyapunov stability. A more detailed account can be found in [28].

Assume $\Omega$ is a subset of $C$, $f : \Omega \to \mathbb{R}^n$ is a given function, and ‘ ˙ ’ represents the right-hand derivative. Then we call

$$\dot{x}(t) = f(x_t) \tag{34}$$

a Delay Differential Equation (DDE) on $\Omega$. Given $\phi \in \Omega$ and $\rho > 0$, a function $x(\phi) \in C([-\tau, \rho), \mathbb{R}^n)$ is said to be a solution to Equation (34) on $[-\tau, \rho)$ with initial condition $\phi$, if $x_t \in \Omega$ for $t \in [0, \rho]$, $x(t)$ satisfies (34) for $t \in [0, \rho)$ and $x_0(\phi) = \phi$. Such a solution exists and is unique if $f$ is continuous everywhere and Lipschitz continuous on every compact set in $\Omega$; see [28] for more details.

A point $x^* \in \Omega$ is called a steady-state (equilibrium) of (34) if $x(t) = x^*$ for all $t \geq 0$ with initial condition $x^*$. Without loss of generality we assume that 0 is a steady-state for the system; for if it is at $x^*$, one can employ a transformation $z(t) = x(t) - x^*$ so that the transformed system

$$\dot{z}(t) = f(z_t + x^*)$$

has the steady state $z^*$ at 0.

We now give the following stability definitions.

*Definition 21:* The trivial solution of (34) is called

1) *stable* if for any $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that $|x(t)| \leq \epsilon$ for any initial condition $\phi \in C^\delta$ and all $t \geq 0$; otherwise it is termed *unstable*.
2) *asymptotically stable* if it is stable and there is a $\gamma > 0$ such that $\lim_{t\to\infty} x(t) = 0$ for any initial condition $\phi \in C^\gamma$.

Just as in the case of nonlinear systems described by Ordinary Differential Equations (ODEs), a Lyapunov argument can be formulated for the stability analysis of DDEs.

Consider a continuous functional $V : C \to \mathbb{R}$ and define:

$$\dot{V}(\phi) = \lim \sup_{h\to 0^+} \frac{1}{h}[V(x_h(\phi)) - V(\phi)]$$

This is the upper Lie derivative of $V$ along a solution of (34). For autonomous systems we have the following theorem (Lyapunov-Krasovskii) [28]:

*Theorem 22:* Assume $f : \Omega \to \mathbb{R}^n$ is completely continuous[1] and the solutions of (34) depend continuously on initial data. Suppose that $V : \Omega \to \mathbb{R}$ is continuous and there exist nonnegative continuous functions $a(s)$ and $b(s)$ satisfying $a(0) = b(0) = 0$, $a(s)$ strictly increasing, such that:

$$V(\phi) \geq a(|\phi(0)|) \text{ on } \Omega$$
$$\dot{V}(\phi) \leq -b(|\phi(0)|) \text{ on } \Omega$$

Then the solution $x = 0$ of (34) is stable, and every solution is bounded. If in addition, $b(s) > 0$ for $s > 0$, the $x = 0$ is asymptotically stable.

The stability of steady-states of time-delay systems is classified as *delay-dependent* or *delay-independent*, based on the persistence of stability as the delay size, seen as a parameter, is increased. We say that a system is *delay-independent* stable, if the stability property is retained for all positive values of the delays in the system. On the other hand, we say that a system is *delay-dependent* stable if the stability is preserved for some values of delays and is lost for some others. The property of delay-independent stability is therefore stronger than that of delay-dependent stability.

## APPENDIX II
## THE SUM OF SQUARES DECOMPOSITION

Recall that $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_m]$ denotes the ring of polynomials in $x$ with real coefficients.

*Definition 23:* An element $p \in \mathbb{R}[x]$ is said to be a *Sum of Squares* (SOS) if there exist polynomials $f_i(x)$, $i = 1, \ldots, M$ such that

$$p(x) = \sum_{i=1}^{M} f_i^2(x).$$

The set of polynomials that admit a SOS decomposition is denoted by $\Sigma$; similarly, the subset of $\Sigma$ for which the degree of the polynomials is $2d$ is denoted by $\Sigma_d$. An equivalent characterization of SOS polynomials is given in the following proposition, the proof of which can be found in [13].

*Proposition 24:* A polynomial $p \in \Sigma_d$ is SOS if and only if there exists a positive semidefinite matrix $Q$ and a vector $Z(x)$ containing monomials in $x$ of degree $\leq d$ so that

$$p(x) = Z(x)^T Q Z(x).$$

Searching for such a $Q$ can be performed using semidefinite programming [13]. Moreover, if the monomials in the polynomial $p(x)$ have *unknown* coefficients then the search for feasible values of those coefficients such that $p(x)$ is a Sum of Squares is also an SDP, a fact that is important for the construction of Lyapunov functions and other S-procedure type multipliers. Integral constraints such as the ones that appear in Theorem 4 and Proposition 5 in the main text are still affine in the unknown coefficients and so they can be added as extra constraints in a Semidefinite Program. The construction of the SDP related to the SOS conditions can be performed efficiently using SOSTOOLS [21], a software that formulates general sum of squares programs as SDPs and calls semidefinite programming solvers to solve them. All examples in this paper were solved using SOSTOOLS.

[1]Recall that if $\Omega$ is a subset of a Banach space $C$ and $A : \Omega \to C$, then $A$ is *completely continuous* if $A$ is continuous and for any bounded set $B \subseteq \Omega$, the closure of $AB$ is compact.

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