The Weierstrass Approximation Theorem on Linear Varieties: Polynomial Lyapunov Functionals for Delayed Systems

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1 Introduction

In 1885, Weierstrass first published a result [16] showing that real-valued polynomials can be used to approximate any continuous function on a compact interval to arbitrary accuracy with respect to the supremum norm. Various generalizations of the Weierstrass approximation theorem have focused on generalized mappings [15] and alternate topologies [4]. More recently, the Weierstrass theorem has found applications in numerical computation due to the ease with which polynomial functions are parameterized and evaluated.

In this paper, we reexamine the Weierstrass theorem from the relatively new perspective of polynomial optimization. These problems consider optimization over $\mathcal{C}(X)$, the Banach space of continuous functions on X where $X \subset \mathbb{R}^n$ is compact. The structure of the problem is often a special case of

$$\begin{aligned} &\max Af : \\ &b + Bf = 0 \\ &c + Cf \succ 0, \end{aligned} \tag{1}$$

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where $A : \mathcal{C}(X) \to \mathbb{R}$, $B : \mathcal{C}(X) \to \mathbb{R}^{n\times m}$, and $C : \mathcal{C}(X) \to \mathcal{C}(X)$ are bounded linear operators, and c and b are fixed. For symmetric, matrix-valued functions on X, $\succ 0$ denotes positive definite on X. The main result of this paper is that if there exists a continuous solution to the optimization problem 1, then there exists a polynomial solution.

The analysis of time-delay systems has led to important results in the areas of communication networks [1, 7] and biological systems [5, 17], among others. See [6, 3] for an overview of applications of systems with delay. In this paper, the Weierstrass approximation result is applied to the synthesis of Lyapunov functionals for time-delay systems. In particular, consider a Lyapunov functional of the form

$$\int_{-h}^{0} \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} M(s) \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} ds.$$

If one considers a linear systems with discrete delays, then it has been shown in [11] that the Lyapunov functional proves stability if and only if there exist functions T and Q such that

$$M(s) + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \text{for all } s \in [-h, 0], \quad \int_{-h}^{0} T(s) ds = 0,$$
$$-(AM)(s) + \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \text{for all } s \in [-h, 0], \text{ and } \quad \int_{-h}^{0} Q(s) ds = 0,$$

where A is defined by the dynamics. For example, if

$$\dot{x}(t) = Ax(t-h),$$

then A is defined as

$$(AM)(s) = \begin{bmatrix} A_0^T M_{11} + M_{11} A_0 & M_{11} A_1 & 0 \\ A_1^T M_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_0^T M_{12}(s) \\ 0 & 0 & A_1^T M_{12}(s) \\ M_{21}(s) A_0 & M_{21}(s) A_1 & 0 \end{bmatrix}$$
$$+ \frac{1}{h} \begin{bmatrix} M_{12}(0) + M_{21}(0) + M_{22}(0) & -M_{12}(-h) & 0 \\ -M_{21}(-h) & -M_{22}(-h) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\dot{M}_{12}(s) \\ 0 & 0 & 0 \\ -\dot{M}_{21}(s) & 0 & -\dot{M}_{22}(s) \end{bmatrix}.$$

In this paper, we prove that the existence of continuous functions M, T, and Q, which prove exponential stability of the system implies the existence of polynomial functions B, C, and D, which also prove exponential stability of the system.

Once M, T, and Q are assumed to be polynomial, then we can apply recent advances in sum-of-squares optimization techniques [9] and results in semialgebraic geometry [14, 13, 12] which make it possible to numerically solve polynomial optimization problems in an asymptotic manner using semidefinite programming. See [11] for details on this technique and [8] for the extension to nonlinear systems.

2 Notation and Background

Most notation is standard. $\mathbb{R}$ is the real numbers. $\mathbb{R}^{n\times m}$ is the real matrices of dimension n by m. $\mathcal{C}[X,Y]$ is the Banach space of functions $f : X \to Y$ with norm

$$\|f\|_\infty = \sup_{s\in X} \|f(s)\|_Y.$$

For $Y = \mathbb{R}^{n\times m}$, denote

$$\|\cdot\|_Y = \bar{\sigma}(F(s)),$$

where $\bar{\sigma}(F)$ denotes the maximum singular value norm.

The following is a statement of the Weierstrass approximation theorem.

Theorem 1. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and $X \subset \mathbb{R}^n$ is compact. Then there exists a sequence of polynomials which converges to f uniformly in X.*

3 Linear Varieties

The following proposition is an extension of the Weierstrass theorem to linear varieties of the Banach space $\mathcal{C}[X,\mathbb{R}^{n\times m}]$ and is the main technical result of the paper.

Theorem 2. *Let X be a compact subset of $\mathbb{R}^n$ and $L_i : \mathcal{C}(X,\mathbb{R}^{n\times o}) \to \mathbb{R}^{p\times q}$ be bounded linear operators. Then for any $f \in \mathcal{C}(X,\mathbb{R}^{n\times o})$ and $\delta > 0$, there exists polynomial r such that $\|f-r\|_\infty \le \delta$ and $L_i r = L_i f$ for $i = 1,\dots,k$.*

Proof. First we note that by increasing the number of constraints, we can assume that $L_i : \mathcal{C}(X,\mathbb{R}^{n\times o}) \to \mathbb{R}$. i.e. The L_i map to the real numbers.

Proceed by induction. Suppose that the proposition is true for $k = m-1$. If $L_m d = 0$ for all $d \in \mathcal{C}(X,\mathbb{R}^{n\times o})$, then let r be as given by the proposition for $m-1$. In this case $L_m r = L_m f = 0$. Otherwise, there exists some $g \in \mathcal{C}(X,\mathbb{R}^{n\times o})$ such that $L_m g = c > 0$. Let β be a uniform bound for operators L_i in $i = 1,\dots,k$. Assume without loss of generality that $\|g\|_\infty = \delta/4$. Let $\gamma = \min\{\frac{\delta}{4}, \frac{c}{\beta}\}$.

By assuming that the proposition is true for $k = m-1$, we assume there exists some polynomial p such that

$$L_i f = L_i p \text{ for } i = 1,\dots,m-1 \quad \text{and} \quad \|f+g-p\|_\infty \le \gamma.$$

Therefore

$$\|f-p\|_\infty \le \|f+g-p\|_\infty + \|g\|_\infty \le \frac{\delta}{4} + \frac{\delta}{4} \le \delta/2.$$

Furthermore,

$$L_m p = L_m f + L_m g - L_m\left(f+g-p\right) \ge L_m f + c - \beta\gamma \ge L_m f.$$

By similar logic, there also exists a polynomial b with $\|f-b\|_\infty \leq \delta/2$, $L_m b \leq L_m f$ and $L_i f = L_i b$ for $i = 1, \ldots, m-1$. Now since $L_m p \geq L_m f$ and $L_m b \leq L_m f$, there exists some $\lambda \in [0,1]$ such that $\lambda L_m p + (1-\lambda) L_m b = Lf$. Now let $r = \lambda p + (1-\lambda) b$. Then r is polynomial,

$$L_m r = \lambda L_m p + (1-\lambda) L_m b = L_m f,$$

$$L_i r = \lambda L_i p + (1-\lambda) L_i b = \lambda L_i f + (1-\lambda) L_i f = L_i f \qquad \text{for } i = 1, \ldots, m-1,$$

and

$$\begin{aligned} \|f-r\|_\infty &= \|\lambda(f-p) + (1-\lambda)(f-b)\|_\infty \leq \lambda \|f-p\|_\infty + (1-\lambda)\|f-b\|_\infty \\ &\leq \delta/2 + \delta/2 = \delta. \end{aligned}$$

Therefore, if the proposition is true for $k = m-1$, it is also true for $k = m$. Assume without loss of generality that $L_1 = 0$. Then the proposition is true for $k = 1$ by the Weierstrass Approximation Theorem. □ We now proceed to develop a number of

extensions to the main result.

4 Derivatives

The first extension deals with the unbounded linear operator of differentiation.

Proposition 3. *Let $L_i, K_i : \mathcal{C}([a,b], \mathbb{R}^{n\times m}) \to \mathbb{R}^{p\times q}$ be bounded linear operators. Then for any $f \in \mathcal{C}^1([a,b], \mathbb{R}^{n\times m})$ and $\delta > 0$, there exists a polynomial r such that $\|f-r\|_\infty \leq \delta$, $\|\dot f - \dot r\|_\infty \leq \delta$, $L_i r = L_i f$ for $i = 1, \ldots, k$, and $K_i \dot r = K_i \dot f$ for $i = 1, \ldots, l$.*

Proof. Define the bounded linear operators $G_i : \mathcal{C}([a,b], \mathbb{R}^{n\times m}) \to \mathbb{R}^{p\times q}$ by

$$G_i z := L_i(\phi z), \quad (\phi z)(s) := \int_a^s z(t)dt.$$

By Theorem 2, there exists a polynomial v such that $\|\dot f - v\|_\infty \leq \delta/(b-a)$, $G_i \dot f = G_i v$ for $i = 1, \ldots, k$, and $K_i \dot f = K_i v$ for $i = 1, \ldots, m$. Let

$$r(s) = f(a) + \int_a^s v(t)dt.$$

Then r is polynomial, $\dot r = v$,

$$\|\dot f - \dot r\|_\infty = \|\dot f - v\|_\infty \leq \delta/(b-a),$$

and

$$\begin{aligned} \|f-r\|_\infty &= \sup_{s\in[a,b]} \| \int_a^s \dot f(t) - v(t)dt \| \\ &\leq \sup_{t\in[a,b]} \|\dot f(t) - v(t)\|(b-a) = (b-a)\|\dot f - v\|_\infty \leq \delta. \end{aligned}$$

Furthermore, since $f - r = \phi(\dot{f} - v)$,

$$L_i(f-r) = L_i(\phi(\dot{f}-v)) = G_i(\dot{f}-v) = 0 \quad \text{for } i = 1, \dots, k.$$

□

5 Optimization

Consider the optimization problem (2) where $L_i, K_i : \mathcal{C}([a,b], \mathbb{R}^{n\times m}) \to \mathbb{R}^{a\times b}$ for $i = 0, \dots, k_1$ and $G_i, H_i : \mathcal{C}([a,b], \mathbb{R}^{n\times m}) \to \mathcal{C}([a,b], \S^c)$ for $i = 1, \dots, k_2$ are bounded linear operators and $E \in \mathcal{C}([a,b], \S^c)$.

$$\begin{aligned} &\max\ L_0 f + K_0 \dot{f} : \\ &C_i + L_i f + K_i \dot{f} = 0, \qquad\qquad \text{for } i = 1, \dots, k_1 \\ &G_i f(s) + H_i \dot{f}(s) \succ E(s), \quad \text{for } s \in [a,b] \quad \text{and } i = 1, \dots, k_2 \end{aligned} \tag{2}$$

Theorem 4. *Consider optimization problem 2. Suppose $f \in \mathcal{C}^1([a,b], \mathbb{R}^{n\times m})$ is feasible with objective value h. Then there exists a feasible polynomial with objective value h.*

Proof. Suppose f is feasible with objective value h. Let β be a uniform bound for the K_i and L_i. By feasibility of f, there exists a $\gamma > 0$ such that

$$L_i f(s) + K_i \dot{f}(s) \succ \gamma + E(s), \quad \text{for all } s \in [a,b] \quad \text{and for } i = 1, \dots, k_2$$

Let $\delta = \frac{\gamma}{4\beta}$. By theorem 3, there exists a polynomial q with $\|f - q\|_\infty \le \delta$ and $\|\dot{f} - \dot{q}\|_\infty \le \delta$ such that

$$\begin{aligned} L_0 q + K_0 \dot{q} &= L_0 f + K_0 \dot{f} = h \\ L_i q + K_i \dot{q} &= L_i f + K_i \dot{f} = C_i, \quad \text{for } i = 1, \dots, m \end{aligned}$$

Then we have,

$$\begin{aligned} &L_i q(s) + K_i \dot{q}(s) = L_i f(s) + K_i \dot{f}(s) + L_i(q - f)(s) + K_i(\dot{q} - \dot{f})(s) \\ &\succeq \gamma E(s) - \beta\|f - q\|_\infty - \beta\|\dot{f} - \dot{q}\|_\infty \succeq \gamma - \gamma/2 - E(s) \succ E(s) \end{aligned}$$

Thus q is feasible with objective value h □

If the derivative condition is not included, then Theorem 4 is easily extended to arbitrary compact $X \subset \mathbb{R}^n$. For $X \subset \mathbb{R}^n$, the derivative can also be included in a manner similar to the work in [10].

6 Background on Time-Delay Systems

The motivating problem for this paper arises from analysis of linear systems with discrete delays. Specifically, we are interested in systems of the form

$$\dot{x}(t) = \sum_{i=0}^{k} A_i x(t - h_i), \tag{3}$$

where the trajectory $x : [-h, \infty) \to \mathbb{R}^n$. In the simplest case we are given information about the the delays $0 = h_0 < h_1 < \cdots < h_k = h$ and the matrices $A_0, \ldots, A_k \in \mathbb{R}^{n \times n}$ and we would like to determine whether the system is stable.

For these types of systems, the boundary conditions are specified by a given function $\phi : [-h, 0] \to \mathbb{R}^n$ and the constraint

$$x(t) = \phi(t) \quad \text{for all } t \in [-h, 0]. \tag{4}$$

Let $\phi \in C[-h, 0]$. Then there exists a unique function x satisfying (3) and (4). The system is called ***exponentially stable*** if there exists $\sigma > 0$ and $a \in \mathbb{R}$ such that for every initial condition $\phi \in C[-h, 0]$ the corresponding solution x satisfies

$$\|x(t)\| \leq a e^{-\sigma t} \|\phi\| \quad \text{for all } t \geq 0.$$

We write the solution as an explicit function of the initial conditions using the map $G : C[-h, 0] \to \Omega[-h, \infty)$, defined by

$$(G\phi)(t) = x(t) \quad \text{for all } t \geq -h,$$

where x is the unique solution of (3) and (4) corresponding to initial condition ϕ. Also for $s \geq 0$ define the *flow map* $\Gamma_s : C[-h, 0] \to C[-h, 0]$ by

$$\Gamma_s \phi = H_s G \phi,$$

which maps the state of the system x_t to the state at a later time $x_{t+s} = \Gamma_s x_t$.

6.1 Lyapunov Functionals

Suppose $V : C[-h, 0] \to \mathbb{R}$. We use the notion of derivative as follows. Define the ***Lie derivative*** of V with respect to Γ by

$$\dot{V}(\phi) = \limsup_{r \to 0^+} \frac{1}{r} \big(V(\Gamma_r \phi) - V(\phi) \big).$$

We will use the notation $\dot{V}$ for both the Lie derivative and the usual derivative, and state explicitly which we mean if it is not clear from context. We will consider the set X of quadratic functions, where $V \in X$ if there exist piecewise continuous functions $M : [-h, 0) \to \S^{2n}$ and $N : [-h, 0) \times [-h, 0) \to \mathbb{R}^{n \times n}$ such that

$$V(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds + \int_{-h}^{0} \int_{-h}^{0} \phi(s)^T N(s, t) \phi(t) \, ds \, dt. \tag{5}$$

The following important result shows that for linear systems with delay, the system is exponentially stable if and only if there exists a quadratic Lyapunov function.

Theorem 5. *The linear system defined by equations (3) and (4) is exponentially stable if and only if there exists a Lie-differentiable function $V \in X$ and $\varepsilon > 0$ such that for all $\phi \in C[-h, 0]$*

$$\begin{aligned} V(\phi) &\geq \varepsilon \|\phi(0)\|^2 \\ \dot{V}(\phi) &\leq -\varepsilon \|\phi(0)\|^2. \end{aligned} \tag{6}$$

Further $V \in X$ may be chosen such that the corresponding functions M and N of equation (5) have the following smoothness property: $M(s)$ and $N(s,t)$ are continuous at all s, t such that $s \in H^c$ and $t \in H^c$ and are bounded.

Proof. See [2] for a recent proof. $\square$

7 Application to Time-Delay Systems

We wish to prove stability by constructing functionals of the form of Equation (5) using polynomial optimization. The derivative of a quadratic functional $V \in X$ has a similar structure to V and is also defined by matrix functions which are linear transformations of M and N. In [11], a necessary and sufficient condition was given for positivity of the first part of the functional. A version of this is as follows.

Theorem 6. *Suppose $M : [-h, 0] \to \S^{n+m}$ is continuous except at points h_i and is bounded. Then the following are equivalent.*

(i) There exists an $\epsilon > 0$ so that for all $c \in \mathbb{R}^n$ and continuous $y : [-h, 0] \to \mathbb{R}^m$,

$$\int_{-h}^{0} \begin{bmatrix} c \\ y(t) \end{bmatrix}^T M(t) \begin{bmatrix} c \\ y(t) \end{bmatrix} dt \geq \epsilon \|y\|_{L_2} \tag{7}$$

(ii) There exist an $\eta > 0$ and a function $T : [-h, 0] \to \S^n$, continuous except at points h_i, which is bounded and satisfies

$$M(t) + \begin{bmatrix} T(t) & 0 \\ 0 & -\eta I \end{bmatrix} \geq 0 \quad \text{for all } t \in [-h, 0], \text{ and} \qquad \int_{-h}^{0} T(t)\, dt \quad = 0.$$

This theorem converts positivity of an integral to *pointwise* positivity of a function with a linear constraint. If we assume M and T are polynomial, pointwise positivity is equivalent to a sum-of-squares constraint. The constraint that T integrates to zero is a bounded linear constraint. The condition that the derivative of the Lyapunov function be negative has a similar structure. For details on formulating the semidefinite program, we refer to [11].

The important question in this case is whether one can assume that M and T are polynomials. The following theorem answers this question in the case of a single delay. Recall that we define $A : \mathcal{C}^1([-h,0],\S^{2n}) \to \mathcal{C}([-h,0],\S^{3n})$ by

$$(AM)(s) = \begin{bmatrix} A_0^T M_{11} + M_{11}A_0 & M_{11}A_1 & 0 \\ A_1^T M_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_0^T M_{12}(s) \\ 0 & 0 & A_1^T M_{12}(s) \\ M_{21}(s)A_0 & M_{21}(s)A_1 & 0 \end{bmatrix}$$
$$+ \frac{1}{h}\begin{bmatrix} M_{12}(0)+M_{21}(0)+M_{22}(0) & -M_{12}(-h) & 0 \\ -M_{21}(-h) & -M_{22}(-h) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\dot{M}_{12}(s) \\ 0 & 0 & 0 \\ -\dot{M}_{21}(s) & 0 & -\dot{M}_{22}(s) \end{bmatrix}.$$

Theorem 7. *Suppose there exist continuous functions $M \in \mathcal{C}^1([-h,0],\S^{2n})$, $Q \in \mathcal{C}([-h,0],\S^{2n})$ and $T \in \mathcal{C}([-h,0],\S^{n})$ such that*

$$M(s) + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h,0], \quad \int_{-h}^{0} T(s)ds = 0,$$
$$-(AM)(s) + \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h,0], \ \textit{and} \quad \int_{-h}^{0} Q(s)ds = 0.$$

Then there exist polynomials B, C, and D, of the same dimension, which satisfy

$$B(s) + \begin{bmatrix} C(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h,0], \quad \int_{-h}^{0} C(s)ds = 0,$$
$$-(AB)(s) + \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h,0], \ \textit{and} \quad \int_{-h}^{0} D(s)ds = 0.$$

Proof. The proof follows immediately from theorem 4. □

8 Multiple Delays

In the case of multiple delays, $h = h_k, \ldots, h_0 = 0$, we define the jump values of M at the discontinuities as follows.

$$\Delta M(h_i) = \lim_{t \to (-h_i)_+} M(t) - \lim_{t \to (-h_i)_-} M(t) \text{ for each } i = 1, \ldots, k-1$$

Definition 8. *Define the map L by $D = L(M)$ if for all $t \in [-h, 0]$ we have*

$$D_{11} = A_0^T M_{11} + M_{11} A_0 + \frac{1}{h}\big(M_{12}(0) + M_{21}(0) + M_{22}(0)\big)$$
$$D_{12} = \begin{bmatrix} M_{11}A_1 & \cdots & M_{11}A_{k-1} \end{bmatrix} - \begin{bmatrix} \Delta M_{12}(h_1) & \cdots & \Delta M_{12}(h_{k-1}) \end{bmatrix}$$
$$D_{13} = \frac{1}{h}\big(M_{11}A_k - M_{12}(-h)\big), \quad D_{22} = \frac{1}{h}\,\mathrm{diag}\big(-\Delta M_{22}(h_1), \ldots, -\Delta M_{22}(h_{k-1})\big)$$
$$D_{23} = 0, \quad D_{33} = -\frac{1}{h} M_{22}(-h), \quad D_{14}(t) = A_0^T M_{12}(t) - \dot{M}_{12}(t)$$
$$D_{24}(t) = \begin{bmatrix} A_1^T M_{12}(t) \\ \vdots \\ A_{k-1}^T M_{12}(t) \end{bmatrix}, \quad D_{34}(t) = A_k^T M_{12}(t), \quad D_{44}(t) = -\dot{M}_{22}(t).$$

Here M is partitioned according to

$$M(t) = \begin{bmatrix} M_{11} & M_{12}(t) \\ M_{21}(t) & M_{22}(t) \end{bmatrix}, \tag{8}$$

where $M_{11} \in \S^n$ and D is partitioned according to

$$D(t) = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14}(t) \\ D_{21} & D_{22} & D_{23} & D_{24}(t) \\ D_{31} & D_{32} & D_{33} & D_{34}(t) \\ D_{41}(t) & D_{42}(t) & D_{43}(t) & D_{44}(t) \end{bmatrix} \tag{9}$$

Theorem 9. *Suppose there exist functions $M : [-h, 0] \to \S^{2n}$, $Q : [-h, 0] \to \S^{2n}$ and $T : [-h, 0] \to \S^{nk+2}$ which are piecewise continuous with possible jumps at h_i. Suppose that that*

$$M(s) + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \text{for all } s \in [-h, 0], \quad \int_{-h}^{0} T(s)ds = 0,$$
$$-(LM)(s) + \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \text{for all } s \in [-h, 0], \text{ and} \quad \int_{-h}^{0} Q(s)ds = 0.$$

Then there exist B, C, and D of the same dimension as M, T, and Q, respectively, and defined by polynomials on the intervals $[-h_i, -h_{i-1}]$ with jumps at the h_i, and which satisfy

$$B(s) + \begin{bmatrix} C(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \text{for all } s \in [-h, 0], \quad \int_{-h}^{0} C(s)ds = 0,$$
$$-(LB)(s) + \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \text{for all } s \in [-h, 0], \text{ and} \quad \int_{-h}^{0} D(s)ds = 0.$$

Proof. Let M_i, T_i, and Q_i be defined by the functions M, T, and Q restricted to the interval $[-h_i, -h_{i-1}]$ with right and left-hand limits defined by continuity. Then M_i, T_i, and Q_i are continuous. Then by scaling and application of Theorem 4, there exist polynomials B_i, T_i, and Q_i for $i = 1, \dots, k$ such that

$$\begin{aligned} B_i(s) + \begin{bmatrix} C_i(s) & 0 \\ 0 & 0 \end{bmatrix} &\succ 0 \quad \text{for all } s \in [-h_i, -h_{i-1}], \\ -(LB)(s) + \begin{bmatrix} D_i(s) & 0 \\ 0 & 0 \end{bmatrix} &\succ 0 \quad \text{for all } s \in [-h_i, -h_{i-1}], \end{aligned}$$

for $i = 1, \dots, k$, and

$$\sum_{i=1}^{k} \int_{-h_i}^{-h_{i-1}} C_i(s) ds = 0, \qquad \sum_{i=1}^{k} \int_{-h_i}^{-h_{i-1}} D_i(s) ds = 0,$$

where B is the function which is defined by B_i on the interval $[-h_i, -h_{i-1}]$ for $i = 1, \dots, k$. If we similarly define C and D by C_i and D_i, respectively, on the interval $[-h_i, -h_{i-1}]$, then we recover the conditions of the theorem. □

9 Conclusion

Numerous extensions of the Weierstrass approximation theorem have been proposed in the literature. Typically, these results either alter the algebra used to approximate the continuous functions or consider continuous functions on spaces other than the reals. The contribution of this paper is to instead consider approximations on subsets of the continuous functions, and in particular those defined by affine constraints. The application we consider is the use of polynomials for constructing Lyapunov-Krasovskii functionals for time-delay systems. Alternatively, Theorem 4 may be used in an opposite sense. That is, this theorem can be used to prove that if a polynomial $\tilde{M}$ approximates a continuous M to sufficient accuracy, then if M proves exponential stability, then $\tilde{M}$ will prove exponential stability. This interpretation is relevant to solving Lyapunov equalities. See [2] for work in this area. In future work, we will consider the second half of the complete quadratic functional. This part of the quadratic form is fundamentally different than the first half and will require a different approach to approximation.

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