Analysis and Boundary Control of Parabolic PDEs using Sum-of-Squares Polynomials

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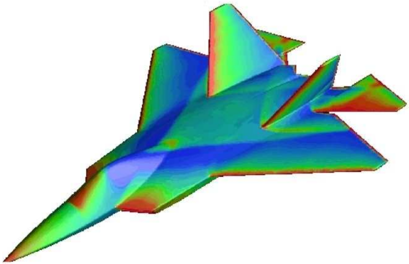
PhD Dissertation Defense

Parabolic PDEs

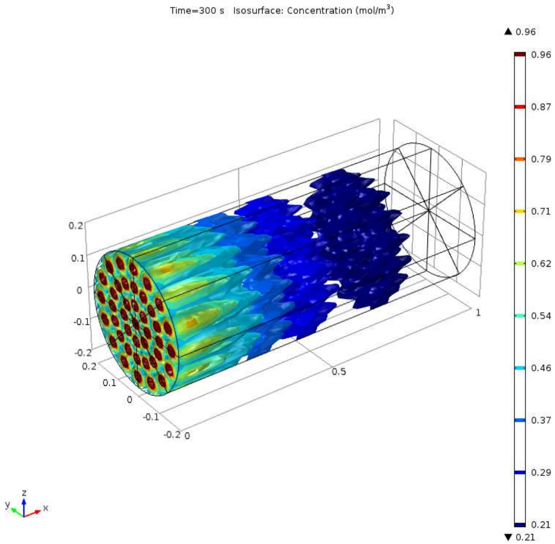
Parabolic PDEs are used to model processes involving heat distribution evolution[1].



[1] TAFSAM: Rice University and Waseda University

Parabolic PDEs

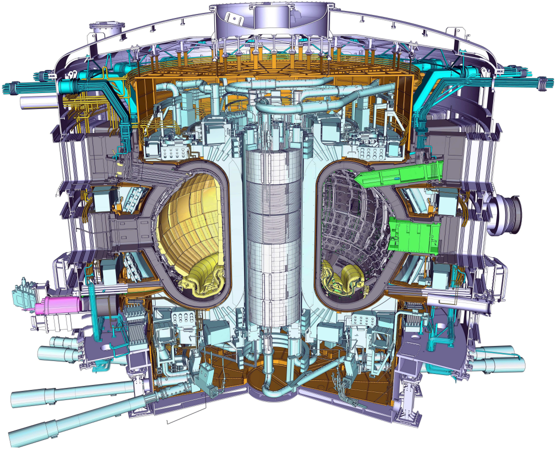
Chemical concentration evolution in chemical reactors[2].



[2] COMSOL Multiphysics

Parabolic PDEs

Magnetic flux evolution in Tokamaks (thermonuclear fusion reactors)[3].



[3] iter.org

Parabolic PDEs

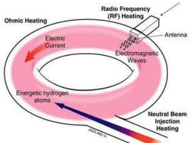
Oil spills and radiation leaks.[4]

[4] National Center for Atmospheric Research

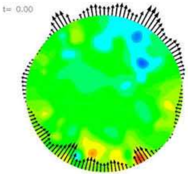
Control of Parabolic PDEs

Control of parabolic PDEs comes in two forms:

- Distributed Control: Control effort penetrates the domain of the PDE and is evenly distributed throughout the domain (similar with sensing).



- Boundary Control: Control actuation/sensing applied only through the boundary conditions.



Boundary control is generally considered to be more realistic.

- e.g., fluid flow with actuation from the walls of the flow domain.
- Intrusion of in-domain actuators/sensors can alter the dynamics of the system.

Control Methodologies

Boundary control of parabolic PDEs is fundamentally difficult.

- Input/output operators unbounded: mathematically challenging.
- Most literature dedicated to distributed control/sensing (bounded input/output operators) [Curtain & Zwart].

A large class of controller synthesis methods rely on numerical approximation.

- Model reduction (PDE $\Rightarrow$ ODE).
- Controller synthesis for the ODE.
- ODE controller applied to the PDE while proving robustness to neglected dynamics.
- Different model reduction methods lead to different ODEs.
- Controller performance is reduced due to some dynamics being neglected at the design stage.

Other methods which do not rely on numerical approximation include:

- Lyapunov methods [Coron, Bastin, d'Andrea-Novel 2007].
  - Searching for anything more complicated than rudimentary Lyapunov functions on infinite-dimensional spaces is difficult, if not entirely impossible.
  - Conservative.
- Backstepping method [Krstic 2008].
  - Fixed/pre-chosen target systems.
  - Not extendable to optimal controller design.

Goals and Related Work

Research Goal: Develop analysis and boundary controller synthesis methodologies which

- Take into account the infinite dimensional nature of the system dynamics (no model reduction).
- Allows us to construct controllers numerically and efficiently.
- Is applicable to a large class of parabolic PDEs with Dirichlet, Neumann, Robin and mixed boundary actuation/sensing.

Our Approach: We combine tools from control theory with techniques from mathematics and computer science.

- Lyapunov theory.
- Functional analysis.
- Convex optimization.
- Semi-Definite Programming (SDP).

Related Work

- Boundary controller synthesis for uncertain semi-linear parabolic and hyperbolic PDEs. [Fridman, Orlov 2009]
- Exponential stability of linear distributed parameter systems with time-varying delays. [Fridman, Orlov 2009]
- Stability analysis of PDEs using SOS polynomials. [Christodoulou, Peet 2006]
- Analysis of Polynomial Systems With Time Delays via the Sum-of-Squares Decomposition. [Christodoulou, Peet 2009]

Research Contribution

- Distributed controller synthesis for the safety factor regulation in Tokamaks. [IFAC 2011]
- Boundary controller synthesis using boundary measurement for the simple heat equation. [CDC 2011]
- Distributed controller synthesis for the bootstrap current density maximization in Tokamaks. [CDC 2012]
- Boundary controller synthesis for: [TAC 2014, under review]
  - A large class of possibly unsteady parabolic PDEs.
  - All possible configurations of boundary control/measurements.
- Boundary controller synthesis using boundary measurement of unsteady Parabolic PDEs subjected to external noise. [CDC 2015]
- Optimal state-feedback boundary controller synthesis of unsteady Parabolic PDEs subjected to external noise. [ACC 2016, under review]
- Optimal boundary controller synthesis using boundary measurement of unsteady Parabolic PDEs subjected to external noise. [Ongoing]

System Under Consideration

We consider the following class of parabolic PDEs

$$w_t(t,x) = a(x)w_{xx}(t,x) + b(x)w_x(t,x) + c(x)w(t,x),$$

where a, b and c are polynomials in $x \in [0,1]$ and $t \geq 0$, coupled to boundary conditions of the form

$$\rho_1 w(t,0) + \rho_2 w_x(t,0) = 0, \quad \mu_1 w(t,1) + \mu_2 w_x(t,1) = u(t).$$

- Not a specific boundary condition, it is a parametrization of all possible boundary conditions.
- Scalars ρ_i, μ_i can be chosen to represent Dirichlet, Neumann, Robin or mixed boundary conditions. These scalars satisfy

$$|\rho_1| + |\rho_2| > 0,$$
$$|\mu_1| + |\mu_2| > 0.$$

- $u(t) \in \mathbb{R}$ is the *control input*.
- Such PDEs are used to model processes of diffusion, convection and reaction over inhomogeneous media.

Analysis and Boundary Control:Goals

We wish to accomplish the following:

- Test exponential stability of the autonomous PDE $(u(t) = 0)$.
- Develop exponentially stabilizing full state feedback boundary controllers.
- Develop exponentially stabilizing output feedback boundary controllers.
  - Only the boundary measurement of the state available. Most restrictive case.

Stability Analysis of ODEs

To motivate the approach we take, let us consider the stability analysis of ODEs. Consider the following autonomous ODE

$$\dot{x}(t) = Ax(t), \quad A \in \mathbb{R}^{n\times n}, \quad x(t) \in \mathbb{R}^n.$$

Let us define a quadratic Lyapunov function as

$$V(t) = \frac{1}{2} x(t)^T P x(t).$$

If for any $\epsilon > 0$, we have

$$P \geq \epsilon I,$$

then,

$$V(t) = \frac{1}{2} = x(t)^T P x(t) \geq \frac{\epsilon}{2} \|x(t)\|^2.$$

Additionally, if for any $\delta > 0$ we have

$$A^T P + PA + \delta P \leq 0,$$

then

$$\dot{V}(t) + 2\delta V(t) = x(t)^T \left(A^T P + PA + \delta P \right) x(t) \leq 0.$$

Stability Analysis LMIs

In such a case, the autonomous ODE is exponentially stable since $V(t)$ is a Lyapunov function satisfying

$$\begin{aligned} V(t) \geq& \frac{\epsilon}{2}\|x(t)\|^2, \\ \dot{V}(t) \leq& -2\delta V(t). \end{aligned}$$

Therefore, the exponential stability problem for the ODE may be expressed as:

$$\begin{aligned} \text{Find}: \quad & P \in \mathbb{S}^n, \\ \text{such that}: & P - \epsilon I \geq 0, \quad A^T P + PA + \delta P \leq 0. \end{aligned}$$

- These inequalities are linear in the variable $P \in \mathbb{S}^n$, and thus are known as Linear Matrix Inequalities.
- The search for P is performed over the cone of Positive Semi-Definite (PSD) matrices.
- Therefore, this search can be setup as a Semi-Definite Programming (SDP) problem and can be solved efficiently using interior point algorithms.

Stability Analysis for PDEs

Now let us look at the exponential stability conditions for the PDE.

Let us rewrite the PDE as

$$w_t(t,x) = \mathcal{A}w(t,x),$$

where we define the differential operator $\mathcal{A} : \mathcal{D} \subset L_2(0,1) \to L_2(0,1)$ as

$$(\mathcal{A}y)(x) = a(x)y_{xx}(x) + b(x)y_x(x) + c(x)y(x), \quad y \in \mathcal{D},$$

- The differential operator $\mathcal{A}$ defines the PDE.

and

$$\begin{aligned}\mathcal{D} =&\{y \in L_2(0,1) : \quad y, y_x \text{ are absolutely continuous and } y \in L_2(0,1) \\ &\text{and } \rho_1 y(0) + \rho_2 y_x(0) = 0, \quad \mu_1 y(1) + \mu_2 y_x(1) = 0\}.\end{aligned}$$

- $\mathcal{D}$ is the set over which the PDE is defined and contains the boundary conditions.

- With this representation we see that the PDE can be represented as a differential equation on the infinite dimensional space $L_2(0,1)$.
- Compare to an ODE which is a differential equation on the finite dimensional space $\mathbb{R}^n$.

Stability Analysis LOIs

For an operator $\mathcal{P} : L_2(0,1) \to L_2(0,1)$, let us define the quadratic Lyapunov function

$$V(t) = \frac{1}{2} \langle w(t,x), \mathcal{P}w(t,x) \rangle .$$

If for any $\epsilon > 0$ we have

$$\mathcal{P} \geq \epsilon \mathcal{I},$$

then

$$V(t) = \frac{1}{2} \langle w(t,x), \mathcal{P}w(t,x) \rangle \geq \frac{\epsilon}{2} \|w(t,x)\|^2.$$

- The condition $\mathcal{P} \geq \epsilon \mathcal{I}$ implies strict positivity of $\mathcal{P}$, i.e.

$$\langle y, \mathcal{P}y \rangle \geq \epsilon \langle y, y \rangle \Rightarrow \int_0^1 y(x) \left(\mathcal{P}y \right)(x) dx \geq \epsilon \int_0^1 y(x)^2 dx.$$

Additionally, if for any $\delta > 0$, we have

$$\mathcal{A}^\star \mathcal{P} + \mathcal{P} \mathcal{A} + \delta \mathcal{P} \leq 0,$$

then

$$\dot{V}(t) + 2\delta V(t) = \langle w(t,x), \left(\mathcal{A}^\star \mathcal{P} + \mathcal{P} \mathcal{A} + \delta \mathcal{P} \right) w(t,x) \rangle \leq 0.$$

Stability Analysis LOIs

In such a case, the autonomous PDE is exponentially stable since $V(t)$ is a Lyapunov function satisfying

$$\begin{aligned} V(t) \geq & \frac{\epsilon}{2}\|w(t,x)\|^2, \\ \dot{V}(t) \leq & -2\delta V(t). \end{aligned}$$

Thus, the test for the exponential stability for the PDE can be stated as: Given any $\epsilon, \delta > 0$

$$\begin{aligned} \text{Find :} \quad & \mathcal{P} : L_2(0,1) \to L_2(0,1), \\ \text{Such that :} \quad & \mathcal{P} - \epsilon\mathcal{I} \geq 0, \quad \mathcal{A}^\star\mathcal{P} + \mathcal{P}\mathcal{A} + \delta\mathcal{P} \leq 0. \end{aligned}$$

- These inequalities are linear in the operator $\mathcal{P}$ and thus are known as Linear Operator Inequalities (LOIs).

Stability Analysis: LMIs and LOIs

To conclude

- Lyapunov equations for ODEs appear in the form of LMIs. The stability problem is reduced to

$$\begin{aligned} \text{Find}: &\quad P \in \mathbb{S}^n \\ \text{such that}: &\quad P - \epsilon I \geq 0, \quad A^T P + PA + \delta P \leq 0. \end{aligned}$$

- Can be solved using semidefinite programming (SDP).
- Lyapunov equations for PDEs appear in the form of LOIs. The stability problem is reduced to

$$\begin{aligned} \text{Find}: &\quad \mathcal{P} : L_2(0,1) \to L_2(0,1) \\ \text{such that}: &\quad \mathcal{P} - \epsilon \mathcal{I} \geq 0, \quad \mathcal{A}^\star \mathcal{P} + \mathcal{P}\mathcal{A} + \delta \mathcal{P} \leq 0. \end{aligned}$$

- Question: How to find the solution $\mathcal{P}$ of the LOI?
- Answer: Use SOS polynomials to parametrize positive operators on $L_2(0,1)$ so that we can use SDP to search for $\mathcal{P}$.

SOS Polynomials

Naturally, three questions arise:

- What are SOS polynomials?
- What properties of SOS polynomials we wish to exploit?
- How to parametrize positive operators using SOS polynomials?

A polynomial $p(x)$ is SOS if there exist polynomials $g_i(x)$ such that

$$p(x) = \sum_i g(x)^2.$$

- SOS polynomials are (semi) positive.
- Question: Why is the positivity of SOS polynomials important, as opposed to the positivity of any polynomial/function?
- Answer: A polynomial can be tested to be an SOS polynomial using SDP.

SOS Polynomial Decomposition

Following is the important property of SOS polynomials:

A polynomial $M(x)$ is SOS if and only if there exists a positive semidefinite matrix U such that

$$M(x) = Z(x)^T U Z(x),$$

where $Z(x)$ is a vector of monomials.

- Therefore, given a polynomial $M(x)$, to determine if it SOS, and hence positive, the following problem must be solved

$$\begin{aligned} \text{Find}: &\quad U \in \mathbb{S}^n \\ \text{Such that}: &\quad U \geq 0, \quad M(x) = Z(x)^T U Z(x). \end{aligned}$$

- This is an LMI subjected to equality constraints, can be solved using SDP!
- Conversely, every positive semidefinite matrix defines an SOS polynomial.

Positive Operators Parametrized by SOS Polynomials

Therefore, we have established

$$\underset{\text{Positive Semidefinite matrix}}{U \geq 0} \underset{\text{SDP}}{\longleftrightarrow} \underset{\text{SOS Polynomial (Positive)}}{M(x)}$$

- Given a polynomial, we can use SDP to test if it is SOS and hence positive.
- Given a positive definite matrix, we can construct an SOS, and hence a positive polynomial.

- Now we will make the final connection:

$$\underset{\text{Positive Semidefinite matrix}}{U \geq 0} \longleftrightarrow \underset{\text{SOS Polynomial (Positive)}}{M(x)} \longleftrightarrow \underset{\text{Positive Operator on } L_2(0,1)}{\mathcal{P}}$$

A Simple Parametrization

We need to establish how to use SOS polynomials to define positive operators on $L_2(0,1)$.

Let us explain using an illustrative example.

Given a polynomial $M(x)$, let us define the operator

$$(\mathcal{P}y)(x) = M(x)y(x), \quad y \in L_2(0,1).$$

For any scalar $\epsilon > 0$ if

$$\langle y, \mathcal{P}y \rangle = \int_0^1 M(x)y(x)^2 dx \geq \epsilon \int_0^1 y(x)^2 = \epsilon \|y\|^2,$$

then the operator $\mathcal{P}$ is positive since $\mathcal{P} - \epsilon\mathcal{I} \geq 0$.

Therefore, $\mathcal{P}$ is positive if $M(x) - \epsilon \geq 0$.

We already know that a sufficient condition is that $M(x) - \epsilon$ be SOS.

- Given the polynomial $M(x)$, we can search, using SDP, for a matrix $U \geq 0$ such that

$$M(x) - \epsilon = Z(x)^T U Z(x).$$

- Conversely, we can choose any $U \geq 0$ and set

$$M(x) - \epsilon = Z(x)^T U Z(x).$$

Positive Operators Parametrized by SOS Polynomials

- We have established that SOS polynomials can be used to parametrize positive operators on $L_2(0,1)$.
- The question now is what type of SOS parametrization should we use?
- Claim: Operators of the form

$$(\mathcal{P}y)(x) = M(x)y(x), \quad M(x) \text{ is SOS},$$

  are too restrictive.
- Such operators generalize diagonal matrices on $\mathbb{R}^n$.
- Positive diagonal matrices form a small subset of positive matrices on $\mathbb{R}^n$.
  - A general positive matrix does not have the constraint of off-diagonal elements being zero.
- Searching for the solution of Lyapunov LMIs for ODEs over the set of only positive diagonal matrices may not work.
  - There may be a non-diagonal positive matrix which solves the LMIs.

Positive Operators Parametrized by SOS Polynomials

- Therefore, this hints to the fact that operators of the form

$$(\mathcal{P}y)(x) = M(x)y(x), \quad M(x) \text{ is SOS},$$

  form a small subset of all positive operators on $L_2(0,1)$.
- What types of operators should we use then?
- Let us again look at matrices for clues.
- The most general form of positive matrix P on $\mathbb{R}^n$ can be decomposed as

$$P = \underbrace{P_1}_{\text{Positive diagonal matrix}} + \underbrace{P_2}_{\text{Lower triangular matrix}} + \underbrace{P_3}_{\text{Upper triangular matrix}}.$$

- This alludes to the fact that we should use operators that generalize the sum of diagonal, upper and lower triangular matrices.

Positive Operators Parametrized by SOS Polynomials

We use operators of the form

$$(\mathcal{P}y)(x) = \underbrace{M(x)y(x)}_{\mathcal{P}_1} + \underbrace{\int_0^x N_1(x,\xi)y(\xi)d\xi}_{\mathcal{P}_2} + \underbrace{\int_x^1 N_2(x,\xi)y(\xi)d\xi}_{\mathcal{P}_3}.$$

- As we already know $\mathcal{P}_1$ generalizes diagonal matrices.
- Operator $\mathcal{P}_2$ generalizes a lower triangular matrix.
- Operator $\mathcal{P}_3$ generalizes an upper triangular matrix.

Question: How to enforce positivity of such operators.

Answer: As before, define M, N_1 and N_2 to be SOS polynomials defined using a positive semidefinite matrix.

Parametrization of Choice

How to enforce positivity using SOS?

- Given any scalar $\epsilon > 0$ and a positive definite matrix U such that

$$U = \begin{bmatrix} U_{11} + \epsilon I & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} \geq 0.$$

- Let

$$\begin{aligned} M(x) =& Z_1(x)^T U_{11} Z_1(x), \\ N_1(x,\xi) =& Z_1(x)^T U_{12} Z_2(x,\xi) + Z_2(\xi,x)^T U_{31} Z_1(\xi) \\ & + \int_0^\xi Z_2(\eta,x)^T U_{33} Z_2(\eta,\xi) d\eta + \int_\xi^x Z_2(\eta,x)^T U_{32} Z_2(\eta,\xi) d\eta \\ & + \int_x^1 Z_2(\eta,x)^T U_{22} Z_2(\eta,\xi) d\eta, \\ N_2(x,\xi) =& N_1(\xi,x). \end{aligned}$$

- If we define the operator

$$(\mathcal{P}y)(x) = M(x)y(x) + \int_0^x N_1(x,\xi)y(\xi)d\xi + \int_x^1 N_2(x,\xi)y(\xi)d\xi.$$

- Then

$$\langle \mathcal{P}y, y\rangle \geq \epsilon \|y\|^2 \Rightarrow \mathcal{P} - \epsilon\mathcal{I} \geq 0.$$

- Positivity is implied by the existence of a square root operator $\mathcal{P}^{\frac{1}{2}}$.

Parametrization of Choice

Given a scalar $\epsilon > 0$ and polynomials $M(x)$, $N_1(x,\xi)$ and $N_2(x,\xi)$, we say that

$$\{M, N_1, N_2\} \in \Xi_\epsilon,$$

if

$$(\mathcal{P}y)(x) = M(x)y(x) + \int_0^x N_1(x,\xi)y(\xi)d\xi + \int_x^1 N_2(x,\xi)y(\xi)d\xi,$$

satisfies $\mathcal{P} \geq \epsilon\mathcal{I}$.

- Given any $U \geq 0$, we can construct $M(x)$, $N_1(x,\xi)$ and $N_2(x,\xi)$ such that

  $$\{M, N_1, N_2\} \in \Xi_\epsilon \text{ and thus } \mathcal{P} > 0.$$

- Conversely, given any operator $\mathcal{P}$ as defined above, we can always search for a matrix $U \geq 0$ such that

  $$\{M, N_1, N_2\} \in \Xi_\epsilon \text{ and determine if } \mathcal{P} > 0.$$

- Therefore, the construction and the determination of positivity of such operators depend on the existence of positive semidefinite matrices subject to linear constraints.
  - Can use SDP to accomplish these tasks.

Stability Analysis

Let us return to the problem of exponential analysis.

Recall that the PDE is exponentially stable if, for any scalars $\epsilon, \delta > 0$, there exists an operator $\mathcal{P}$ such that

$$\mathcal{P} - \epsilon\mathcal{I} \geq 0, \quad \mathcal{A}^\star\mathcal{P} + \mathcal{P}\mathcal{A} + \delta\mathcal{P} \leq 0.$$

As discussed, for polynomials $M(x)$, $N_1(x,\xi)$ and $N_2(x,\xi)$, let us define

$$\left(\mathcal{P}y\right)(x) = M(x)y(x) + \int_0^x N_1(x,\xi)y(\xi)d\xi + \int_x^1 N_2(x,\xi)y(\xi)d\xi.$$

- For such operators we know how to enforce

$$\{M, N_1, N_2\} \in \Xi_\epsilon.$$

- Therefore,

$$\mathcal{P} \geq \epsilon\mathcal{I}.$$

- Consecutively

$$V(t) = \frac{1}{2}\left\langle w(t,x), \mathcal{P}w(t,x)\right\rangle \geq \frac{\epsilon}{2}\|w(t,x)\|^2.$$

Stability Analysis

Now,

$$\dot{V}(t) + 2\delta V(t) = \langle w(t,x), (\mathcal{A}^\star \mathcal{P} + \mathcal{P}\mathcal{A} + \delta\mathcal{P})\, w(t,x)\rangle\,.$$

After tedious manipulations using integration by parts, integral inequalities and Fubini's theorem, we get

$$\langle w(t,x), (\mathcal{A}^\star \mathcal{P} + \mathcal{P}\mathcal{A} + \delta\mathcal{P})\, w(t,x)\rangle \le \left\langle w(t,x), \hat{\mathcal{P}} w(t,x)\right\rangle + \text{ boundary terms,}$$

where

$$\left(\hat{\mathcal{P}} y\right)(x) = \hat{M}(x)y(x) + \int_0^x \hat{N}_1(x,\xi)y(\xi)d\xi + \int_x^1 \hat{N}_2(x,\xi)y(\xi)d\xi.$$

- $\hat{M}$, $\hat{N}_1$ and $\hat{N}_2$ are linear functions of M, N_1 and N_2.

Observation: Both $\mathcal{P}$ and $\hat{\mathcal{P}}$ have the same structure.

- We can enforce negativity.

Stability Analysis

What we have so far.

If $\{M, N_1, N_2\} \in \Xi_\epsilon$ then

- $\mathcal{P} \geq \epsilon\mathcal{I}$, where

$$(\mathcal{P}y)(x) = M(x)y(x) + \int_0^x N_1(x,\xi)y(\xi)d\xi + \int_x^1 N_2(x,\xi)y(\xi)d\xi.$$

- Thus,

$$V(t) = \frac{1}{2}\langle w(t,x), \mathcal{P}w(t,x)\rangle \geq \frac{\epsilon}{2}\|w(t,x)\|^2.$$

Additionally, if $\{-\hat{M}, -\hat{N}_1, -\hat{N}_2\} \in \Xi_0$ then

- $\hat{\mathcal{P}} \leq 0$ where

$$\left(\hat{\mathcal{P}}y\right)(x) = \hat{M}(x)y(x) + \int_0^x \hat{N}_1(x,\xi)y(\xi)d\xi + \int_x^1 \hat{N}_2(x,\xi)y(\xi)d\xi.$$

- Since $\mathcal{A}^\star\mathcal{P} + \mathcal{P}\mathcal{A} + \delta\mathcal{P} \leq \hat{\mathcal{P}}$

$$\dot{V}(t) + 2\delta V(t) = \left\langle w(t,x), \left(\mathcal{A}^\star\mathcal{P} + \mathcal{P}\mathcal{A} + \delta\mathcal{P}\right)w(t,x)\right\rangle \leq 0.$$

$V(t)$ is a Lyapunov function proving exponential stability of the autonomous system.

Stability Analysis

The stability analysis problem is reduced to:

Given scalars $\epsilon, \delta > 0$, find polynomials $M(x)$, $N_1(x,\xi)$ and $N_2(x,\xi)$ such that

$$\{M, N_1, N_2\} \in \Xi_\epsilon \quad \text{Can be solved by SDP!},$$
$$\{-\hat{M}, -\hat{N}_1, -\hat{N}_2\} \in \Xi_0 \quad \text{Can be solved by SDP!},$$

where $\hat{M}$, $\hat{N}_1$ and $\hat{N}_2$ are linear functions in M, N_1 and N_2.

- By parametrizing positive operators using SOS polynomials, the question of stability analysis is reduced to the feasibility problem of an SDP.

State Feedback Controller Synthesis

The PDE with boundary input is

$$w_t(t,x) = a(x)w_{xx}(t,x) + b(x)w_x(t,x) + c(x)w(t,x) = \mathcal{A}w(t,x),$$
$$\rho_1 w(t,0) + \rho_2 w_x(t,0) = 0, \quad \mu_1 w(t,1) + \mu_2 w_x(t,1) = u(t).$$

- TASK: Find a boundary controller $\mathcal{F}$ such that if

$$u(t) = \mathcal{F}w(t,x),$$

then the PDE is exponential stable: a Lyapunov function exists.

State Feedback Controller Synthesis for ODEs

To motivate, let us look at the LMI conditions for ODEs for the state feedback controller synthesis.

Consider the following ODE with an input

$$\dot{x}(t) = Ax(t) + Bu(t).$$

TASK: Find a controller F such that if

$$u(t) = Fx(t),$$

then the system is exponentially stable: a Lyapunov function exists.

Let us define a Lyapunov function as

$$V(t) = \frac{1}{2}x(t)^T P^{-1} x(t).$$

If for any $\epsilon > 0$, we have

$$P > \epsilon I.$$

Then

$$V(t) = \frac{1}{2}x(t)^T P^{-1} x(t) > 0.$$

State Feedback Controller Synthesis LMIs

Let us define, for a matrix Y, the control input as

$$u(t) = Fx(t) = YP^{-1}x(t).$$

If for any $\delta > 0$,

$$AP + PA^T + BY + Y^T B^T + \delta P \leq 0.$$

Then

$$\dot{V}(t) + 2\delta V(t) = \left(P^{-1}x(t)\right)^T \left(AP + PA^T + BY + Y^T B^T + \delta P\right) \left(P^{-1}x(t)\right) \leq 0.$$

Therefore, if

$$u(t) = Fx(t) = YP^{-1}x(t),$$

then $V(t)$ is a Lyapunov function proving the exponential stability of the controlled system.

The controller synthesis problem can be stated as

$$\begin{aligned} \text{Find:} \quad & P \text{ and } Y \\ \text{such that:} \quad & P \geq \epsilon I, \quad AP + PA^T + BY + Y^T B^T + \delta P \leq 0. \end{aligned}$$

- For stability analysis we were only searching for P.

State Feedback Controller Synthesis

Back to controller synthesis for the PDE.

As before, we define $\mathcal{P}$ as

$$\left(\mathcal{P}y\right)(x) = M(x)y(x) + \int_0^x N_1(x,\xi)y(\xi)d\xi + \int_x^1 N_2(x,\xi)y(\xi)d\xi.$$

We know how to enforce

$$\{M, N_1, N_2\} \in \Xi_\epsilon.$$

Therefore

$$\mathcal{P} \geq \epsilon\mathcal{I}.$$

Consecutively

$$V(t) = \frac{1}{2}\left\langle w(t,x), \mathcal{P}^{-1}w(t,x)\right\rangle > 0.$$

State Feedback Controller Synthesis

Now let us define

$$u(t) = \mathcal{F}w(t,x) = \mathcal{Y}\mathcal{P}^{-1}w(t,x),$$

where, we define

$$\mathcal{Y}y(x) = \mu_1 Y_1 y_x(1) + \mu_2 Y_2 y(1) + \int_0^1 Y_3(x)y(x)dx,$$

for scalars Y_1 and Y_2 and polynomial $Y_3(x)$.

- The choice of the structure of $\mathcal{Y}$ reveals itself since it is chosen to cancel out all terms at the controlled boundary $x = 1$.

We obtain

$$\dot{V}(t)+2\delta V(t) = \left\langle \left(\mathcal{P}^{-1}w(t,x)\right), \left(\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^\star + \mathcal{B}\mathcal{Y} + \mathcal{Y}^\star\mathcal{B}^\star + \delta\mathcal{P}\right)\left(\mathcal{P}^{-1}w(t,x)\right)\right\rangle.$$

Applying similar manipulation as done for stability analysis

$$\begin{aligned}&\left\langle \left(\mathcal{P}^{-1}w(t,x)\right), \left(\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^\star + \mathcal{B}\mathcal{Y} + \mathcal{Y}^\star\mathcal{B}^\star + \delta\mathcal{P}\right)\left(\mathcal{P}^{-1}w(t,x)\right)\right\rangle \\ &\le \left\langle \left(\mathcal{P}^{-1}w(t,x)\right), \hat{\mathcal{P}}\left(\mathcal{P}^{-1}w(t,x)\right)\right\rangle + \text{boundary terms},\end{aligned}$$

where

$$\left(\hat{\mathcal{P}}y\right)(x) = \hat{M}(x)y(x) + \int_0^x \hat{N}_1(x,\xi)y(\xi)d\xi + \int_x^1 \hat{N}_2(x,\xi)y(\xi)d\xi,$$

where $\hat{M}$, $\hat{N}_1$ and $\hat{N}_2$ are linear functions in M, N_1, N_2, Y_1, Y_2 and Y_3.

- For stability analysis, $\hat{M}$, $\hat{N}_1$ and $\hat{N}_2$ were linear functions in M, N_1 and N_2 only.

- Observation: Both $\mathcal{P}$ and $\hat{\mathcal{P}}$ have the similar structure.
  - We can enforce negativity.

State Feedback Controller Synthesis

What we have so far.

If $\{M, N_1, N_2\} \in \Xi_\epsilon$ then

- $\mathcal{P} \geq \epsilon\mathcal{I}$, where

$$\left(\mathcal{P}y\right)(x) = M(x)y(x) + \int_0^x N_1(x,\xi)y(\xi)d\xi + \int_x^1 N_2(x,\xi)y(\xi)d\xi.$$

- Thus,

$$V(t) = \frac{1}{2}\left\langle w(t,x), \mathcal{P}^{-1}w(t,x)\right\rangle > 0.$$

Additionally, if $\{-\hat{M}, -\hat{N}_1, -\hat{N}_2\} \in \Xi_0$ then

- $\hat{\mathcal{P}} \leq 0$ where

$$\left(\hat{\mathcal{P}}y\right)(x) = \hat{M}(x)y(x) + \int_0^x \hat{N}_1(x,\xi)y(\xi)d\xi + \int_x^1 \hat{N}_2(x,\xi)y(\xi)d\xi.$$

- Since $\mathcal{AP} + \mathcal{PA}^\star + \mathcal{BY} + \mathcal{Y}^\star\mathcal{B}^\star + \delta\mathcal{P} \leq \hat{\mathcal{P}}$,

$$\dot{V}(t) + 2\delta V(t) = \left\langle \left(\mathcal{P}^{-1}w(t,x)\right), \left(\mathcal{AP} + \mathcal{PA}^\star + \mathcal{BY} + \mathcal{Y}^\star\mathcal{B}^\star + \delta\mathcal{P}\right)\left(\mathcal{P}^{-1}w(t,x)\right)\right\rangle \leq 0$$

$V(t)$ is a Lyapunov function proving exponential stability of the controlled system with input

$$u(t) = \mathcal{F}w(t,x) = \mathcal{Y}\mathcal{P}^{-1}w(t,x).$$

State Feedback Controller Synthesis

The controller synthesis problem is reduced to:

Given scalars $\epsilon, \delta > 0$, find scalars Y_1 and Y_2 and polynomials $M(x)$, $N_1(x,\xi)$, $N_2(x,\xi)$ and $Y_3(x)$ such that

$$\{M, N_1, N_2\} \in \Xi_\epsilon \quad \text{Can be solved by SDP!},$$
$$\{-\hat{M}, -\hat{N}_1, -\hat{N}_2\} \in \Xi_0 \quad \text{Can be solved by SDP!},$$

where $\hat{M}$, $\hat{N}_1$ and $\hat{N}_2$ are linear functions in M, N_1, N_2, Y_1, Y_2 and Y_3.

- By parametrizing positive operators and controllers using SOS polynomials, the problem of controller synthesis is reduced to the feasibility problem of an SDP.

Observer Design

- In most cases, only a partial measurement of state $w(t,x)$ is available.
  - State feedback control requires the measurement of the complete state.
- We now consider the most restrictive case when we can only measure the boundary value of the state

$$y(t) = \gamma_1 w(t,1) + \gamma_2 w_x(t,1),$$

  where γ_i are scalars satisfying

$$|\gamma_1| + |\gamma_2| > 0.$$

- TASK: Use the boundary measurement to design an exponentially stabilizing control input $u(t)$.
- We design an observer whose state $\hat{w}$ estimates the state of the system w exponentially fast.
  - $\|\hat{w}(\cdot,t) - w(\cdot,t)\| \to 0$ exponentially in t.
- Use the controller designed for state feedback $\mathcal{F}$ and set

$$u(t) = \mathcal{F}\hat{w}(t,x).$$

- Controller and observer can be designed separately: Separation principle.

Observer Design

We wish to design a Luenberger observer: copy of the plant plus output injection.

The observer is given as

$$\hat{w}_t(t,x) = \underbrace{a(x)\hat{w}_{xx}(t,x) + b(x)\hat{w}_x(t,x) + c(x)\hat{w}(t,x)}_{\text{Copy of plant}} + \underbrace{O_1(x)\left(\hat{y}(t) - y(t)\right)}_{\text{Output injection}},$$

$$\rho_1\hat{w}(t,0) + \rho_2\hat{w}_x(t,0) = 0, \quad \mu_1\hat{w}(t,1) + \mu_2\hat{w}_x(t,1) = u(t) + \underbrace{O_2\left(\hat{y}(t) - y(t)\right)}_{\text{Output injection}}.$$

- $O_1(x)$ and O_2 are observer gains to be determined.

Similar to controller design, the observer synthesis problem can be stated as: Given scalars $\epsilon, \delta > 0$, find scalar O_2 and polynomials $M(x)$, $N_1(x,\xi)$, $N_2(x,\xi)$ and $R(x)$ such that

$$\{M, N_1, N_2\} \in \Xi_\epsilon \quad \text{Can be solved by SDP!},$$
$$\{-\hat{M}, -\hat{N}_1, -\hat{N}_2\} \in \Xi_0 \quad \text{Can be solved by SDP!},$$

where $\hat{M}$, $\hat{N}_1$ and $\hat{N}_2$ are linear functions in M, N_1, N_2, R and O_2. The observer gain $O_1(x)$ can then be recovered as

$$O_1(x) = \left(\mathcal{P}^{-1}R\right)(x).$$

Numerical Results: Stability Analysis

We consider the following two systems

$$
\begin{aligned}
w_t(x,t) =& w_{xx}(x,t) + \lambda w(x,t), \text{ and} && (1)\\
w_t(x,t) =& \left(x^3 - x^2 + 2\right) w_{xx}(x,t) + \left(3x^2 - 2x\right) w_x(x,t) \\
& + \left(-0.5x^3 + 1.3x^2 - 1.5x + 0.7 + \lambda\right) w(x,t), && (2)
\end{aligned}
$$

where λ is a scalar which may be chosen freely.

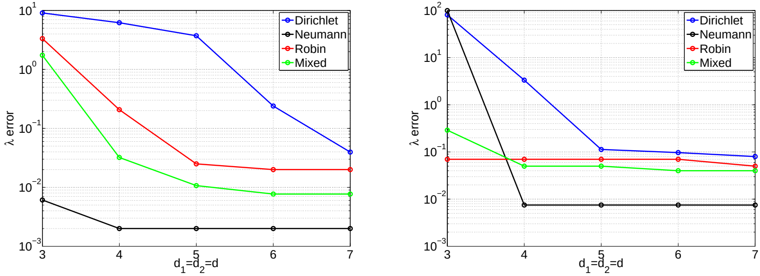
We consider the following boundary conditions

$$
\begin{aligned}
\text{Dirichlet:} &= w(0) = 0, \quad & w(1) = 0,\\
\text{Neumann:} &= w_x(0) = 0, \quad & w_x(1) = 0,\\
\text{Mixed:} &= w(0) = 0, \quad & w_x(1) = 0,\\
\text{Robin:} &= w(0) = 0, \quad & w(1) + w_x(1) = 0.
\end{aligned}
$$

Increasing λ shifts the eigenvalues of the differential operator to the right in the complex plane. Therefore the systems become unstable for larger values of λ.

TASK: Determine maximum λ for which we prove the stability of these equations and compare it to the maximum calculated/predicted value for stability.

Numerical Results: Stability Analysis



(a) PDE 1 (b) PDE 2

Error between calculated max. λ as a function of polynomial degree d using our method and calculated/estimated max. λ using Sturm-Liouville/finite-differences.

- Increasing the degree d means that we are searching over the larger set of polynomials of a higher degree.
- Searching over a set of positive semidefinite matrices of larger dimensions.

Numerical Results: Output Feedback Controller Synthesis

We now consider the same systems:

$$
\begin{aligned}
w_t(x,t) =& w_{xx}(x,t) + \lambda w(x,t), \text{ and} && (3)\\
w_t(x,t) =& \left(x^3 - x^2 + 2\right) w_{xx}(x,t) + \left(3x^2 - 2x\right) w_x(x,t)\\
&+ \left(-0.5x^3 + 1.3x^2 - 1.5x + 0.7 + \lambda\right) w(x,t), && (4)
\end{aligned}
$$

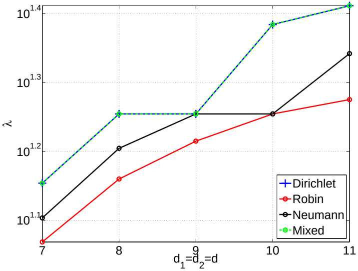
with the following configurations of boundary conditions, boundary inputs and boundary measurements:

$$
\begin{aligned}
\text{Dirichlet:} &= w(0) = 0, & w(1) &= u(t), & y(t) &= w_x(1),\\
\text{Neumann:} &= w_x(0) = 0, & w_x(1) &= u(t), & y(t) &= w(1),\\
\text{Mixed:} &= w(0) = 0, & w_x(1) &= u(t), & y(t) &= w(1),\\
\text{Robin:} &= w(0) + w_x(0) = 0, & w(1) + w_x(1) &= u(t), & y(t) &= w(1).
\end{aligned}
$$

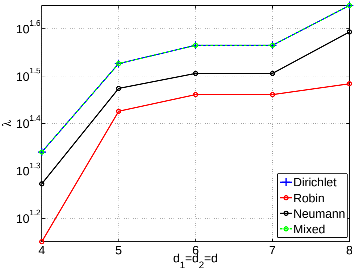
TASK: Determine the maximum $\lambda > 0$ such that we can construct exponentially stabilizing boundary controllers.

- Ideal scenario would be an ability to synthesize boundary controllers for any arbitrary $\lambda \in \mathbb{R}$.

Numerical Results: Output Feedback Controller Synthesis



(a) PDE 3



(b) PDE 4

Maximum λ as a function of polynomial degree d for which we can construct an exponentially stabilizing controller.

Numerical Results: Output Feedback Controller Synthesis

Conclusions from these numerical results:

- Increasing the degree d leads to synthesis of exponentially stabilizing boundary controllers for higher values of parameter λ.
- These numerical results indicate that the method is asymptotically accurate.
  - Given any controllable system, we can synthesize boundary controllers for a large enough d.
- Size of the underlying SDP problem scales as $O(d^2)$.
  - Increasing the degree d leads to increased memory requirements.

Numerical Simulation

We provide numerical simulation for the following system

$$\begin{aligned} w_t(x,t) = \left(x^3 - x^2 + 2\right) w_{xx}(x,t) + \left(3x^2 - 2x\right) w_x(x,t) \\ + \left(-0.5x^3 + 1.3x^2 - 1.5x + 0.7 + 35\right) w(x,t), \end{aligned}$$

coupled to the following boundary conditions and output

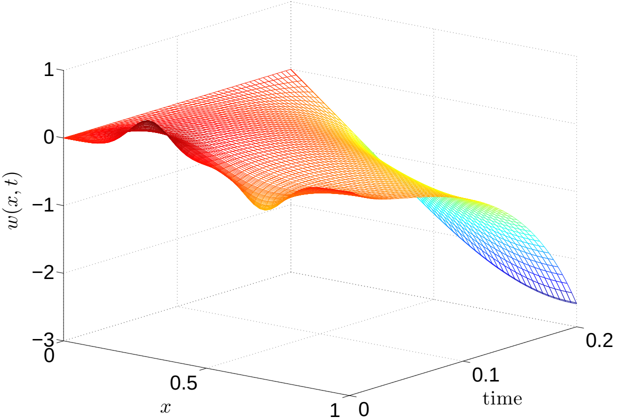
$$w(0,t) = 0, \quad w_x(1,t) = u(t), \quad y(t) = w(1,t).$$

The system is initiated by with an initial condition

$$w(0,x) = e^{-\frac{-(x-0.3)^2}{2(0.07)^2}} - e^{-\frac{-(x-0.7)^2}{2(0.07)^2}},$$

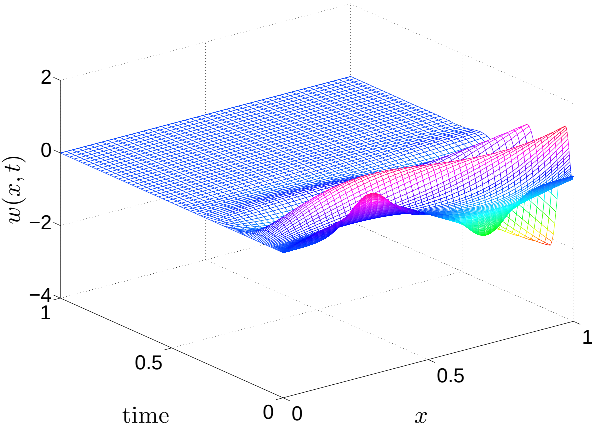
and the observer is initiated by a zero initial condition.

Numerical Simulation: Autonomous evolution



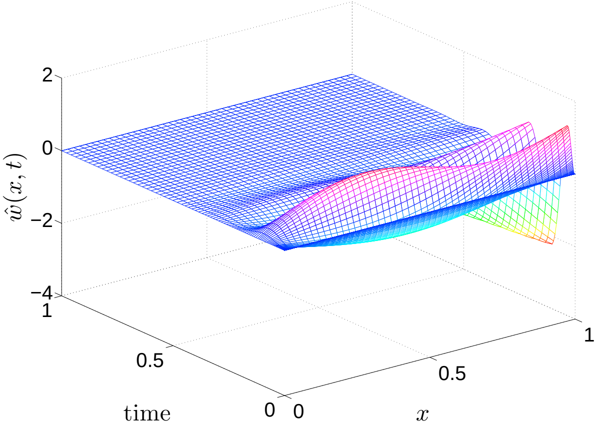
Autonomous state evolution.

Numerical Simulation: Closed loop evolution



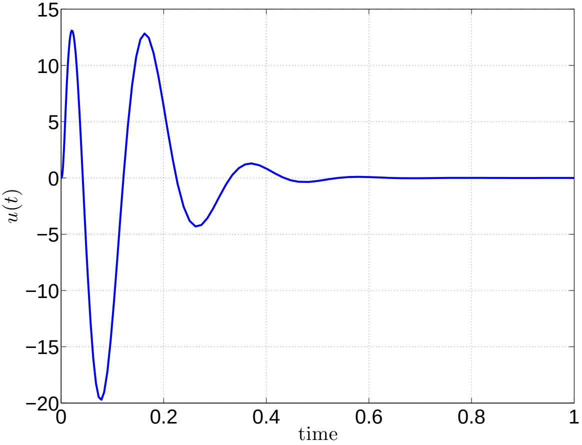
Closed loop state evolution.

Numerical Simulation: Observer evolution



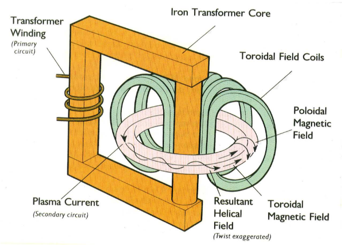
Observer state evolution.

Numerical Simulation: Control input evolution



Control effort evolution.

Tokamaks



- A tokamak uses a helical magnetic field to contain and compress the plasma (Hydrogen isotopes) to initiate and sustain fusion.
- The helical magnetic field is a result of the superposition of two magnetic fields:
  - Toroidal magnetic field: generated by external current carrying coils.
  - Poloidal magnetic field: generated by the plasma current.

Poloidal Magnetic Flux Gradient

Poloidal magnetic flux gradient is inversely proportional to:

- Safety factor profile (q-profile).
  - Magnetic field line pitch.
  - An important heuristic for the suppression of MHD instabilities.
- Bootstrap current density j_{bs}
  - Internally generated current density.
  - An increased proportion of j_{bs} would lead to lowered energy input for the Tokamak operation.

A simplified evolution model for the poloidal magnetic flux gradient

$$Z_t(x,t) = \frac{1}{\mu_0 a^2} \frac{\partial}{\partial x} \left(\frac{\eta_{\parallel}(x,t)}{x} \frac{\partial}{\partial x} \left(x Z(x,t) \right) \right) + R_0 \frac{\partial}{\partial x} \left(\eta_{\parallel}(x,t) \left(j_{lh}(x,t) + j_{bs}(x,t) \right) \right)$$

Here

$$\begin{aligned} Z &= \text{Poloidal magnetic flux gradient}, \\ j_{lh} &= \text{Lower Hybrid Current Drive (LHCD) deposit}, \\ j_{bs} &= \text{Booststrap current density}, \\ \eta_{\parallel} &= \text{Plasma resistivity}, \\ \mu_0 &= \text{Permeability of vacuum}, \\ R_0 &= \text{Plasma major radius}, \\ a &= \text{Plasma minor radius}. \end{aligned}$$

Poloidal Magnetic Flux Gradient Control: Tasks

TASK 1: Given a safety factor profile $q_{ref}(Z_{ref})$

- Design state feedback control input j_{lh} such that

$$q \to q_{ref} \quad (Z \to Z_{ref}) .$$

TASK 2: Design state feedback control input j_{lh} such that

- Design state feedback control input j_{lh} such that

$$\|j_{bs}\| \text{ is maximized} \quad (\|Z\| \text{ is minimized}) .$$

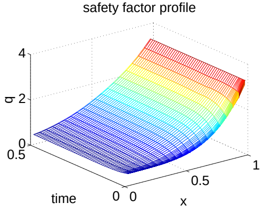
To achieve these tasks, we use control of the form

$$j_{lh}(x,t) = K_1(x)Z(x,t) + \frac{\partial}{\partial x}\left(K_2(x)Z(x,t)\right),$$

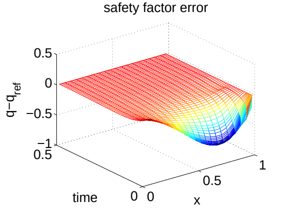
and Lyapunov functions of the form

$$\left\langle Z(\cdot,t), \mathcal{P}^{-1}Z(\cdot,t)\right\rangle = \int_0^1 M(x)^{-1}Z(x,t)^2 dx.$$

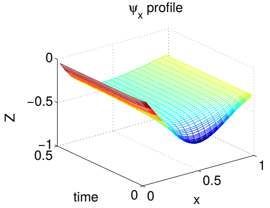
Safety Factor Regulation: Simulation



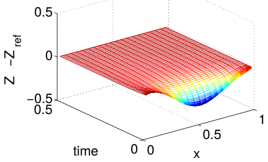
(a) Time evolution of the safety factor profile or the q-profile.



(b) Time evolution of the q-profile Error, $q(x,t) - q_{ref}(x)$.



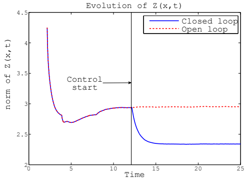
(c) Time evolution of Z-profile corresponding to the q-profile.



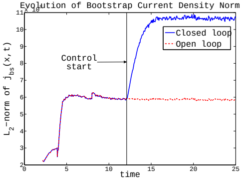
(d) Z-profile error, $Z - Z_{ref}$.

Time evolution of safety-factor and Z profiles and their corresponding error profiles

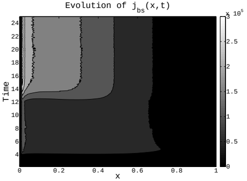
Bootstrap Current Density Maximization: Simulation



(a) Evolution of closed loop ($t \geq 12$) and open loop $\|Z(\cdot, t)\|$.



(b) Evolution of closed loop ($t \geq 12$) and open loop $\|j_{bs}(\cdot, t)\|$.



(c) Evolution of level sets of bootstrap current density $j_{bs}(x, t)$ in closed loop ($t \geq 12$).

Bootstrap current density evolution.

Conclusions and Future Work

Conclusions

- We presented a numerical method, using SOS and convex optimization, for boundary controller synthesis using boundary measurement.
- The method is asymptotically accurate.
- We can tweak the method to increase the rate of exponential decay of the state.
- Algorithm performance depends on the degree of polynomial representation.
- Memory requirements enforce a constraint on the method.
- Similar methodology used to demonstrate state boundedness in presence of an external noise.
- Design of optimal state feedback controllers.

Future Work

- Extension of the method to optimal boundary control using boundary measurement.
- Application to uncertain (non) linear systems.
- Boundary controller synthesis for hyperbolic PDEs (wave and beam equations).
- Observer synthesis for poloidal magnetix flux in Tokamaks.

Acknowledgments