A Convex Approach for Stability Analysis of Partial Differential Equations

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MS Thesis Defense

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Research Goals

FOCUS: Application of convex optimization for stability analysis of Partial Differential Equations (PDEs).

MOTIVATION:

- Consider **infinite dimensional feature** of the dynamics, i.e. do not rely on model reduction techniques.
- Develop algorithms for performing analysis in **polynomial time**.
- Capture classes of PDEs with **spatially distributed coefficients**.
- Stability analysis is a step towards developing computational methods of **controlling** PDEs.

Our Approach

We use **synergy** of results from control theory, functional analysis and computer science.

- Lyapunov theory
- Semigroup approach for PDEs
- Convex optimization
- Semi-Definite Programming (SDP)
- Sum of Squares (SOS) polynomials

This thesis is a continuation of work done by Dr. Gahlawat and presented in his PhD Dissertation: “Analysis and control of parabolic partial differential equations with application to tokamaks using sum-of-squares polynomials”, 2015.

Who Else is Using SPD for Stability Analysis of PDEs?

Group of Antonis Papachristodoulou – SySOS lab in Oxford.

- M. Ahmadi, G. Valmorbida and A. Papachristodoulou, “Dissipation Inequalities for the Analysis of a Class of PDEs”, IFAC Automatica, 2016.
- G. Valmorbida, M. Ahmadi and A. Papachristodoulou, “Stability Analysis for a Class of Partial Differential Equations via Semi-Definite Programming”, IEEE Transactions on Automatic Control, To Appear, 2016.
- M. Ahmadi, G. Valmorbida and A. Papachristodoulou, “A Convex Approach to Hydrodynamic Analysis”, IEEE CDC, 2015.

Group of Emilia Fridman – Tel-Aviv University.

- E. Fridman and Y. Orlov, “Exponential Stability of Linear Distributed Parameter Systems with Time-Varying Delays”, Automatica, 2009.
- S. Nicaise, J. Valein and E. Fridman, “Stability of the heat and of the wave equations with boundary time-varying delays”, Discrete and Continuous Dynamical Systems, 2009.

Contribution

Group of Matthew Peet – CSCL in ASU.

- A. Papachristodoulou and M. M. Peet, “On the Analysis of Systems Described by Classes of Partial Differential Equations”, IEEE CDC, 2006.
- A. Gahlawat and M. M. Peet, “A Convex Sum-of-Squares Approach to Analysis, State Feedback and Output Feedback Control of Parabolic PDEs”, IEEE Transactions on Automatic Control, To Appear, 2016.
- E. Meyer and M. Peet, “Stability Analysis and Control of Parabolic Linear PDEs with two Spatial Dimensions Using Lyapunov Methods and SOS”, IEEE CDC, 2015.
- E. Meyer, “Stability Analysis of PDEs with Two Spatial Dimensions Using Lyapunov Methods and SOS”, SIAM Conference on Application of Dynamical Systems, 2015.
- E. Meyer and M. Peet, “A Convex Approach for Stability Analysis of Coupled PDEs with Spatially Dependent Coefficients”, Submitted to IEEE CDC, 2016.

We Consider Two Classes of PDEs

Class 1: **Linear coupled PDEs.** $x \in (a, b) \subset \mathbb{R}$,

$$u_t(t, x) = A(x)u_{xx}(t, x) + B(x)u_x(t, x) + C(x)u(t, x)$$

$A, B, C : (a, b) \to \mathbb{R}^{n \times n}$ – polynomials, $u(t, x) \in \mathbb{R}^n$.

Class 2: **Linear PDEs with 2 spatial variables.** $x \in \Omega \subset \mathbb{R}^2$,

$$u_t(t, x) = a(x)u_{x_1x_1}(t, x) + b(x)u_{x_1x_2}(t, x) + c(x)u_{x_2x_2}(t, x) + d(x)u_{x_1}(t, x) + e(x)u_{x_2}(t, x) + f(x)u(t, x)$$

$a, b, c, d, e, f : \Omega \to \mathbb{R}$ – polynomials, $u(t, x) \in \mathbb{R}$.

Why?!

Some Types of PDEs Can be Reformulated as Coupled PDEs

A second order linear PDE in two variables can be written in the form

$$au_{xt} + 2bu_{xt} + cu_{tt} + du_x + eu_t + fu = 0. \tag{1}$$

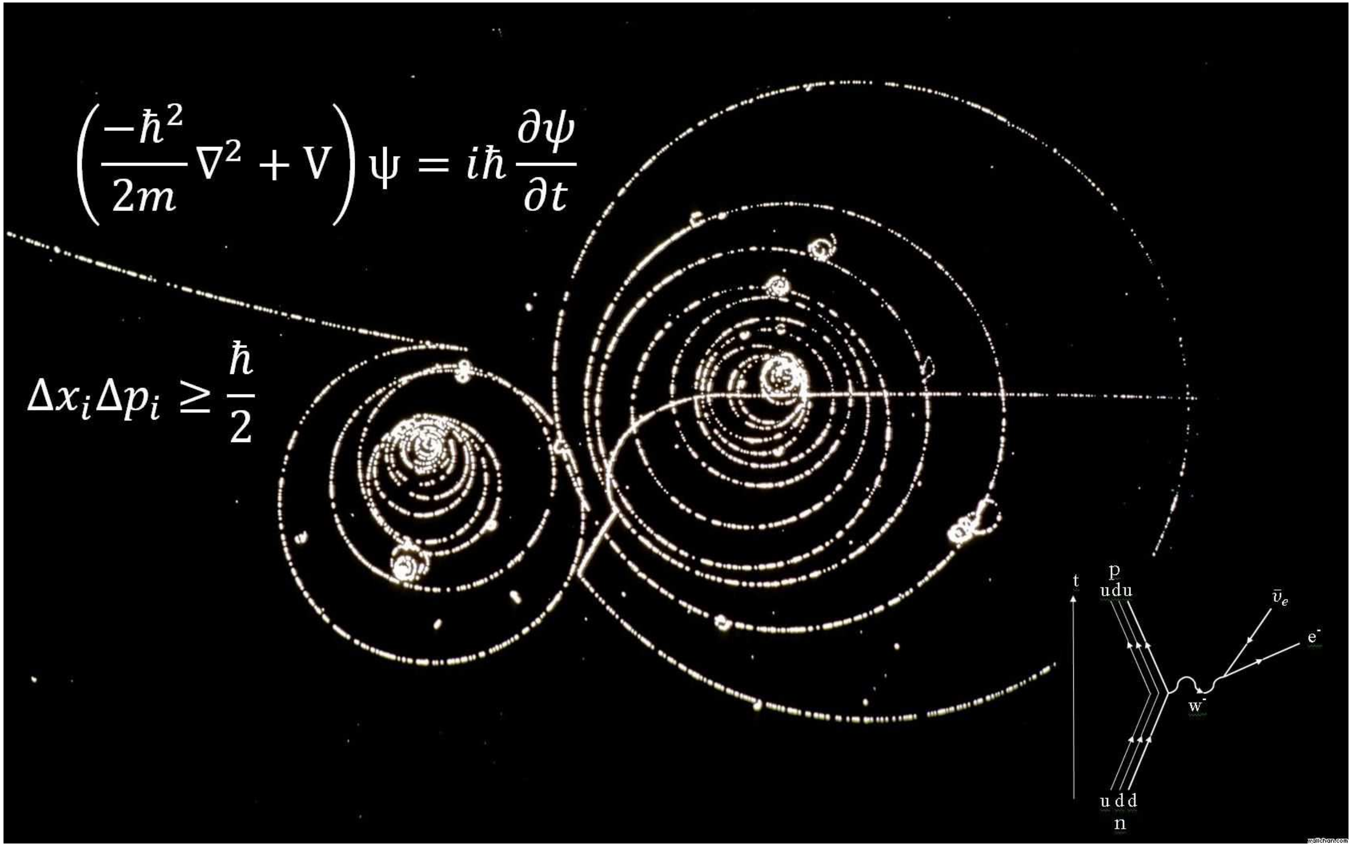
Then

- if $b^2 - ac < 0$, then (1) is of elliptic type.
- if $b^2 - ac = 0$, then (1) is of parabolic type.
- if $b^2 - ac > 0$, then (1) is of hyperbolic type.

Now we present examples of **parabolic and hyperbolic** PDEs which can be stated as

$$u_t(t, x) = A(x)u_{xx}(t, x) + B(x)u_x(t, x) + C(x)u(t, x).$$

Schrödinger Equation can be Formulated as Coupled PDEs*

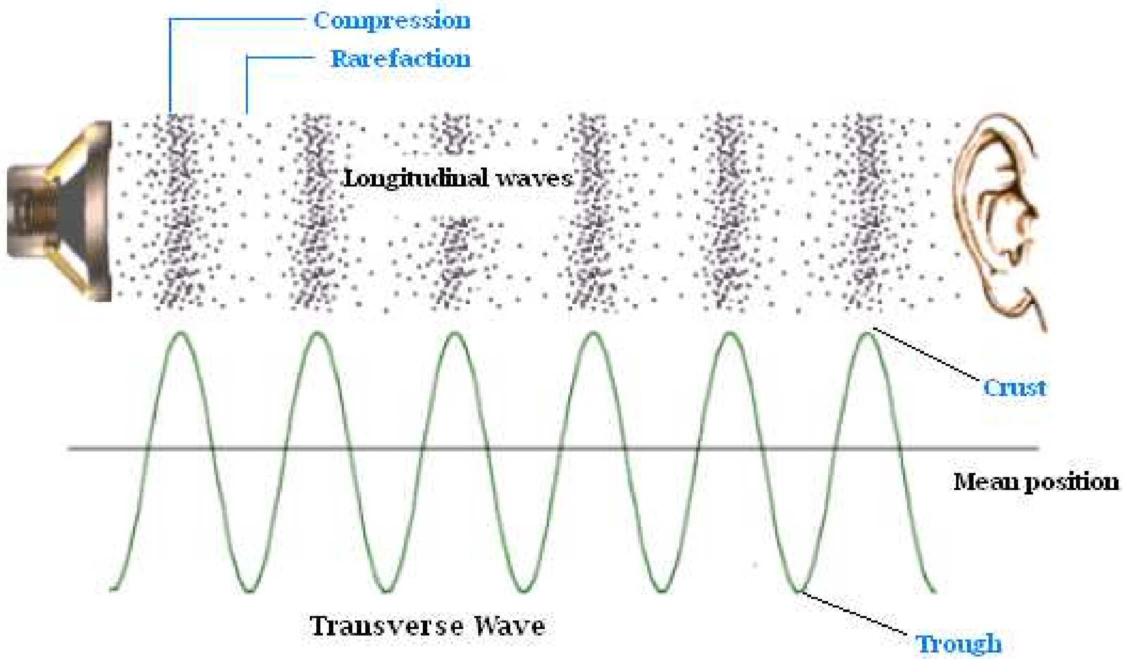


* Decompose real and imaginary parts of Schrödinger PDE and then couple the result. Details are **here** or in the thesis.

PDE for Acoustic Waves can be Stated as Coupled PDEs*

$x \in (0, L)$, c – speed of sound, p – pressure

$$p_{tt}(t,x) = c^2 p_{xx}(t,x) + \frac{2c^2}{x} p_x(t,x)$$



* Use an auxiliary function $q = p_t$, similarly to derivation of a state space form for ODEs. Details are **here** or in the thesis.

Hyperbolic (Relativistic) Heat Conduction Equation (HHCE)

Classical form (Euclidean space, $ds^2 = dx^2 + dy^2 + dz^2$)

$$u_t = \alpha \bigtriangledown^2 u = \alpha \left(\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] u \right). \quad (2)$$

Problem: (2) assumes that **the speed of information propagation is higher than the speed of light in vacuum** c, which is physically unacceptable. [Y. Ali, L. Zhang, "Relativistic heat conduction", International Journal of Heat and Mass Transfer, 2005]

HHCE (Minkowski space, $ds^2 = dx^2 + dy^2 + dz^2 + d\tau^2$, $d\tau = ict$)

$$u_t = \alpha \square^2 u = -\frac{\alpha}{c^2} u_{tt} + \alpha \bigtriangledown^2 u. \quad (3)$$

Using an auxiliary function $w = u_t$, (3) can be reformulated as linear coupled PDEs. Details are **here**.

Capturing Dirichlet, Neumann, Robin and Mixed Boundary Conditions (BC)

We parameterize different BC for

$$u_t(t,x) = A(x)u_{xx}(t,x) + B(x)u_x(t,x) + C(x)u(t,x)$$

through elements of matrix D such that for all $t > 0$,

$$D \begin{bmatrix} u(t,a) \\ u(t,b) \\ u_x(t,a) \\ u_x(t,b) \end{bmatrix} = 0.$$

Examples:

$$\text{Mixed BC} \begin{cases} u(t,a) = 0 \\ u_x(t,b) = 0 \end{cases} \Longrightarrow D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix},$$

$$\text{Robin BC} \begin{cases} H_1 u(t,a) - H_2 u_x(t,a) = 0 \\ H_3 u(t,b) + H_4 u_x(t,b) = 0 \end{cases} \Longrightarrow D = \begin{bmatrix} H_1 & 0 & -H_2 & 0 \\ 0 & H_3 & 0 & H_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

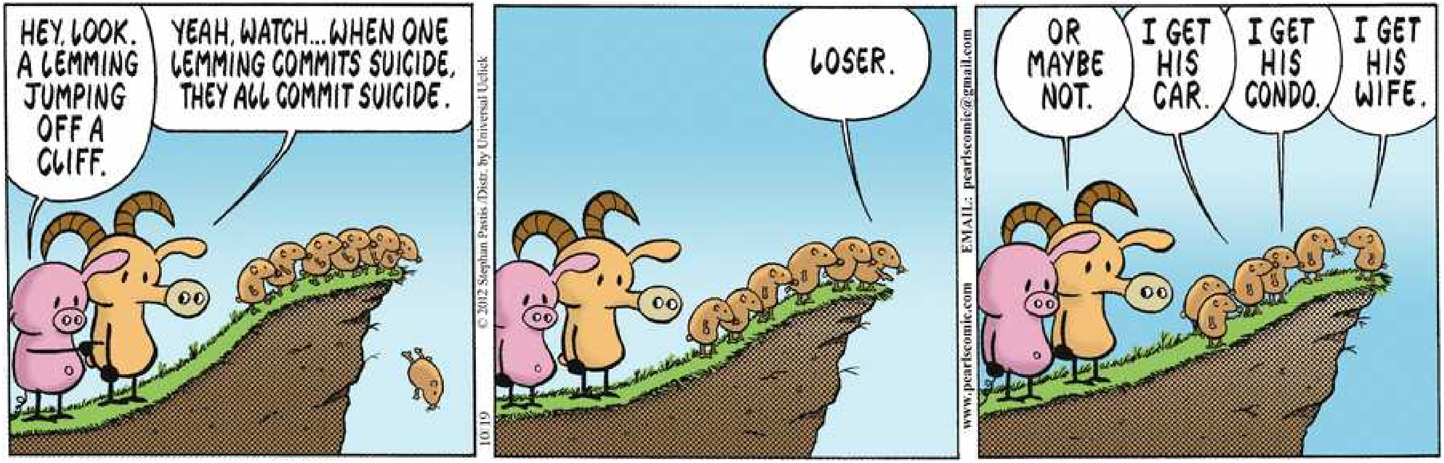
Extra Spatial Variable Allows Study Dynamics of Population

$x_1, x_2 \in (0,1)$, $u(t, x_1, x_2) \in \mathbb{R}$ – population density

$$u_t = h(u_{x_1x_1} + u_{x_2x_2}) + ru.$$

Model independently introduced by Kierstead and Slobodkin (1953) and Skellam (1951) to look for a smallest patch that can sustain a population.

Thus Stability Analysis Can Save Lemmings!



When population density reaches critical value, lemmings may migrate in large groups and possibly die. This led to a misconception that lemmings commit mass suicide when they migrate, by jumping off cliffs.

What Does Stability Analysis of PDEs Mean?

To answer that question, first recall stability analysis of linear ODEs and some definitions.

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0.$$

Here, for each instant, $t \geq 0$, $x(t)$ is the **state of a system** and belongs to $\mathbb{R}^n$ – the **state space**.

$A \in \mathbb{R}^{n \times n}$ describes the dynamics and, with **initial condition** x_0, defines the **trajectory** of the system $x : [0, \infty) \to \mathbb{R}^n$.

Stability of Linear ODEs

Definition 1 *: If for*

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0. \tag{4}$$

there exist $k > 0, (\delta > 0)\delta \geq 0$ *such that*

$$\|x(t)\|_{\mathbb{R}^n} \leq k \exp(-\delta t) \qquad \text{for all } t > 0$$

then (4) is called (exponentially) stable.

In other words, (4) is stable (δ can be 0) if the norm $\|\cdot\|_{\mathbb{R}^n}$ of the state $x(t)$ is bounded for all $t > 0$.

If $\delta > 0$ then $x(t) \to \bar{0}$ (zero vector of $\mathbb{R}^n$) exponentially as $t \to \infty$.

State and State Space of Dynamics Modeled by PDEs

Consider general linear PDE with t representing time and x – spatial variable.

$$F(t, x, u_t, u_{x_1}, \dots, u_{x_n}, u_{x_1 x_1}, u_{x_1 t}, \dots) = 0, \tag{5}$$

$x \in \Omega \subseteq \mathbb{R}^n$, F is linear and assume that solution, $u : [0, \infty) \times \Omega \to \mathbb{R}^m$, exists and unique.

Now for each instant solution belongs to a space of functions

$$L_2^m(\Omega) := \left\{ f : \Omega \to \mathbb{R}^m \,\middle|\, \sqrt{\int_\Omega \|f(x)\|_{\mathbb{R}^m} dx} < \infty \right\}.$$

Thus, $L_2^m(\Omega)$ is the **state space of PDE** (5).

$u(t, \cdot)$ denotes the **state of PDE** (5) at moment t.

Stability of PDEs

Definition 2 *If for linear PDE*

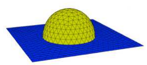
$$F(t, x, u_t, u_{x_1}, \dots, u_{x_n}, u_{x_1x_1}, u_{x_1t}, \dots) = 0, \tag{6}$$

there exist $k > 0, (\delta > 0)\delta \geq 0$ *such that*

$$\|u(t,\cdot)\|_{L_2^m} \leq ke^{-\delta t} \quad \text{for all } t > 0$$

then (6) is called (exponentially) stable in the sense of L_2*.*

In this sense, stability of linear PDEs is similar to stability of linear ODEs, except the state space is different.



What can be used to answer if PDE is stable? Well, let's see what we have for ODEs.

Stability Condition from Lyapunov Theorem for Linear ODEs

A linear ODE

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \tag{7}$$

is stable if there exists $P > 0$ such that

$$A^T P + PA \leq 0.$$

If such P exists then $V(x) := x^T Px$ is a **quadratic Lyapunov function** for (7) with negative time derivative

$$\dot{V}(x(t)) = x^T (A^T P + PA)x \leq 0.$$

Question: how to find P?

Answer: use **Semi-Definite Programming!**

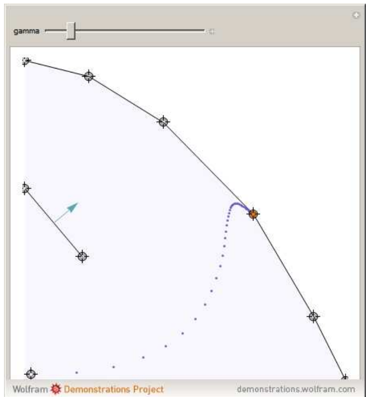
Recall General Form of Semi-Definite Programming (SDP)

For some given $c \in \mathbb{R}^n$ and $F_i \in \mathbb{S}^m$

$$\min_{x \in \mathbb{R}^n} c^T x$$

$$\text{such that } F_0 + \sum_{i=1}^{n} x_i F_i \leq 0.$$

SDPs are **convex optimization problems** and, thus, can be solved computationally using **interior point method**.



LMIs are Useful for Stability Analysis of $\dot{x} = Ax$

The feasibility problem of an SDP

$$F_0 + \sum_{i=1}^{n} x_i F_i \leq 0.$$

is known as **Linear Matrix Inequalities (LMIs)**.

Existence of $P \in \mathbb{S}^n$ such that

$$P > 0 \quad \text{and} \quad A^T P + PA < 0$$

can be cast* as an LMI and then be solved computationally.

Question: can we use LMIs to find Lyapunov functionals for PDEs?
Answer: yes, we can use **Sum of Squares (SOS) polynomials!**

* Details are **here**.

What are Sum of Squares (SOS) Polynomials?

A polynomial p is SOS if there exist polynomials g_i such that

$$p = \sum_i g_i^2.$$

Useful properties

- SOS polynomials are **semi-positive**.
- A condition for polynomial p being SOS is an SDP constrain.

A polynomial p of degree d is SOS if and only if there exists $U \geq 0$ such that

$$p(x) = z(x)^T U z(x),$$

where z is a vector of monomials up to degree d.

SOS Polynomials Define Positive Functionals over $L_2(a,b)$

If polynomial p is SOS then for some $\epsilon > 0$ quadratic functional $V : L_2(a,b) \to \mathbb{R}$, defined as

$$V(w) := \int_a^b w(x)(p(x)+\epsilon)w(x)\,dx \geq \epsilon \int_a^b w^2(x)\,dx > 0$$

for all $w \in L_2(a,b)$, except $w = 0$.

$V > 0$	$\Longleftarrow$	p is SOS	$\Longleftrightarrow$	SDP constraint

This was an example of how positive matrices parameterize a subset of positive quadratic functionals using SOS polynomials.

In our work we use more "complicated" parameterization based on [M. Peet, "LMI parametrization of Lyapunov functions for infinite-dimensional systems: A framework", IEEE ACC, 2014].

Parameterization of Positive Functionals over $L_2^n(a,b)$

Given any positive semi-definite matrix $P \in \mathbb{S}^{\frac{n}{2}(d+1)(d+4)}$, let

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},\ Z_1(x) := z_d(x) \otimes I_n,\ Z_2(x,y) := z_d(x,y) \otimes I_n$$

where $P_{11} \in \mathbb{S}^{n(d+1)}$, $x, y \in (a,b)$, z_d is a vector of monomials up to degree d and $\otimes$ denotes Kronecker product. If for some $\epsilon > 0$

$$M(x) = Z_1(x)^T P_{11} Z_1(x) + \epsilon I_n,$$

$$N(x,y) = Z_1(x)^T P_{12} Z_2(x,y) + Z_2(y,x)^T P_{21} Z_1(y) + \int_a^b Z_2(z,x)^T P_{22} Z_2(z,y)\, dz,$$

then functional $V : L_2^n(a,b) \to \mathbb{R}$, defined as

$$V(w) := \int_a^b w(x)^T M(x) w(x)\, dx + \int_a^b w(x)^T \int_a^b N(x,y) w(y)\, dy dx,$$

is positive over $L_2^n(a,b)$. Proof can be found in the thesis.

Positive and Negative Functionals over $L_2^n(a,b)$

The idea remains the same: positive semi-definite matrix P and some $\epsilon > 0$ defines a pair of polynomials (M, N) such that

$$V(w) := \int_a^b w(x)^T M(x) w(x)\, dx + \int_a^b w(x)^T \int_a^b N(x,y) w(y)\, dy dx$$

is positive over $L_2^n(a,b)$. For brevity, define a parameterized set

$$\Xi^{n,d,\epsilon}$$

of all such polynomial pairs (M, N).

NOTE: we also can parameterize **negative functionals** over $L_2^n(a,b)$

$V > 0$	$\Longleftarrow$	$(M,N) \in \Xi^{n,d,\epsilon}$	$\Longleftrightarrow$	SDP constraint
$V < 0$	$\Longleftarrow$	$(-M,-N) \in \Xi^{n,d,\epsilon}$	$\Longleftrightarrow$	SDP constraint

Suitable Form of Time Derivative of Lyapunov Functional

Linearity of coupled PDEs,

$$u_t(t,x) = A(x)u_{xx}(t,x) + B(x)u_x(t,x) + C(x)u(t,x), \tag{8}$$

allows to derive* **quadratic form** of the time derivative of the Lyapunov functional V along trajectories of (8), i.e.

$$\frac{d}{dt}[V(u(t,x))] = \int_a^b q(t,x)^T K(x) q(t,x)\, dx + \int_a^b q(t,x)^T \int_a^b L(x,y) q(t,y)\, dy dx,$$

where K, L are defined by M, N, A, B, C and

$$q(t,x) := \begin{bmatrix} u(t,x) \\ u_x(t,x) \\ u_{xx}(t,x) \end{bmatrix} \quad \Longrightarrow \quad q(t,\cdot) \in L_2^{3n}(a,b).$$

NOTE: if $(-K,-L) \in \Xi^{3n,d,0}$, then $\frac{d}{dt}[V(u(t,x))] \leq 0$.

BUT: Condition $(-K,-L) \in \Xi^{3n,d,0}$ alone is **conservative**. Why?!

* Details are **here** or in thesis.

To Reduce Conservatism We Use Spacing Functions

REASON: **elements of q are not independent**, since they are constructed using u and its partial derivatives, u_x and u_{xx}.

SOLUTION: Using **fundamental theorem of calculus and matrix D**, which represents boundary conditions of coupled PDEs, we parameterize (details are in thesis) a set of **spacing functions** – polynomial pairs (T, R) such that

$$\int_a^b w(x)^T T(x) w(x)\, dx + \int_a^b w(x)^T \int_a^b R(x,y) w(y)\, dy dx = 0$$

for all $w \in \Lambda$ where

$$\Lambda := \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in L_2^{3n}(a,b) \;\middle|\; \underbrace{D \begin{bmatrix} w_1(a) \\ w_1(b) \\ w_2(a) \\ w_2(b) \end{bmatrix} = 0,}_{\text{PDE boundary conditions}} \; \begin{matrix} w_2 = w_1', \\ w_3 = w_1'' \end{matrix} \right\}.$$

Condition $(-K + T, -L + R) \in \Xi^{3n,d,\epsilon}$ is less restrictive for $\frac{d}{dt}[V(u(t,x))] \leq 0$.

Summarizing Results for Stability Test of Coupled PDEs

1. Input parameters n, ϵ, d to define a set of (M, N), which parameterize V

$$V(w) = \int_a^b w(x)^T M(x) w(x)\, dx + \int_a^b w(x)^T \int_a^b N(x,y) w(y)\, dy dx$$

2. (M, N) with coefficients A, B, C of

$$u_t(t,x) = A(x) u_{xx}(t,x) + B(x) u_x(t,x) + C(x) u(t,x), \tag{9}$$

   determine polynomials K and R in the quadratic form of the Lyapunov functional time derivative.

3. Boundary conditions define elements of D, which parameterize spacing functions (T, R).

4. Lyapunov functional can be found for (9) by solving feasibility problem of the resultant SPD.

$V > 0$	$\Longleftarrow$	$(M, N) \in \Xi^{n,d,\epsilon}$	$\Longleftrightarrow$	SDP constraint
$\dot{V} \leq 0$	$\Longleftarrow$	$(-K + T, -L + R) \in \Xi^{3n,d,0}$	$\Longleftrightarrow$	SDP constraint

5. If SDP is infeasible, then increase d and repeat the test.

Numerical Validation: Example with Decoupled and Coupled PDEs

$u : [0, \infty) \times [0, 1] \to \mathbb{R}^2$ satisfies homogeneous Dirichlet boundary conditions and

$$u_t(t, x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_{xx}(t, x) + C(\lambda) u(t, x) \qquad (10)$$

where C is defined by parameter $\lambda > 0$, which **value affects stability** of (10). Also C represents two cases.

$$\underbrace{C = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}_{\text{(10) is Decoupled}} \qquad \text{and} \qquad \underbrace{C = \begin{bmatrix} \lambda & \lambda \\ \lambda & \lambda \end{bmatrix}}_{\text{(10) is Coupled}}.$$

- Using the proposed algorithm and bisection search over λ we searched for an **upper bound on λ** for which (10) is stable.
- We also studied the **dependence of the feasibility problem on degree d**.
- We **compared our results with λ_{num}** – an upper bound on λ, which was calculated using MATLAB solver PDEPE and bisection search over λ.

Results of Numerical Validation for Decoupled and Coupled PDEs

Decoupled PDE $\Longrightarrow$

d	1	2	3	4	5	6	λ_{num}
λ	5	5.8	7.4	8.1	8.1	8.1	9.8

Coupled PDE $\Longrightarrow$

d	1	2	3	4	5	6	λ_{num}
λ	4	5.8	6.9	7.2	7.4	7.4	8.8

We repeated the numerical experiment for the coupled case using **mixed boundary conditions (BC)**, i.e.

$$u_x(t,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } u(t,1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Coupled PDE with mixed BC $\Longrightarrow$

d	1	2	3	4	5	6	λ_{num}
λ	8.6	12.7	13.9	14.4	14.6	14.7	15.9

Coupled PDEs with Spatially Distributed Coefficients

We proved stability of the following, randomly generated example.

$u : [0, \infty) \times [0, 1] \to \mathbb{R}^2$ satisfies homogeneous Dirichlet BC and

$$u_t(t,x) = \begin{bmatrix} 5x^2+4 & 0 \\ 2x^2+7x & 7x^2+6 \end{bmatrix} u_{xx}(t,x) + \begin{bmatrix} 1 & -4x \\ -3.5x^2 & 0 \end{bmatrix} u_x(t,x) - \begin{bmatrix} x^2 & 3 \\ 2x & 3x^2 \end{bmatrix} u(t,x)$$

With $d = 4, \epsilon = 0.001$ the resultant SPD based on the proposed method is feasible.

Scalar Linear PDEs with 2 Spatial Variables

Recall the 2^{nd} class of PDEs we consider. $x \in \Omega \subset \mathbb{R}^2$, $u(t,x) \in \mathbb{R}$

$$\boxed{\begin{aligned} u_t(t,x) = a(x)u_{x_1x_1}(t,x) + b(x)u_{x_1x_2}(t,x) + c(x)u_{x_2x_2}(t,x) \\ + d(x)u_{x_1}(t,x) + e(x)u_{x_2}(t,x) + f(x)u(t,x) \end{aligned}}$$

$a, b, c, d, e, f : \Omega \to \mathbb{R}$ – polynomials.

Now $u(t,\cdot) \in L_2(\Omega)$ with $\Omega \subset \mathbb{R}^2$, which makes the integration be performed over two variables.

Thus, we start with a simple form of Lyapunov functionals.

Given $\Omega \subset \mathbb{R}^2$, any polynomial $M : \Omega \to \mathbb{R}^{n \times n}$ defines a quadratic functional $V : L_2^n(\Omega) \to \mathbb{R}$ as

$$V(w) := \int_\Omega w(x)^T M(x) w(x)\, dx.$$

Choose an $\epsilon > 0$.

$$V > 0 \quad \Longleftarrow \quad \left\{ \begin{array}{c} \text{there exists} \\ \text{SOS polynomial } P \\ \text{such that} \\ M = P + \epsilon I_n \end{array} \right\} \quad \Longleftrightarrow \quad \text{SDP constraint}$$

$$V < 0 \quad \Longleftarrow \quad \left\{ \begin{array}{c} \text{there exists} \\ \text{SOS polynomial } P \\ \text{such that} \\ M = -(P + \epsilon I_n) \end{array} \right\} \quad \Longleftrightarrow \quad \text{SDP constraint}$$

Quadratic Form of the Lyapunov Functional Time Derivative

Using the form of Lyapunov functional

$$V(w) = \int_\Omega w(x)m(x)w(x)\,dx,$$

we convert the time derivative of V along the trajectories of

$$\begin{aligned} u_t(t,x) = a(x)u_{x_1x_1}(t,x) + b(x)u_{x_1x_2}(t,x) + c(x)u_{x_2x_2}(t,x) \\ + d(x)u_{x_1}(t,x) + e(x)u_{x_2}(t,x) + f(x)u(t,x) \end{aligned}$$

to a quadratic form

$$\frac{d}{dt}\left[V(u(t,\cdot))\right] = \int_\Omega q(t,x)^T Q(x) q(t,x)\,dx,$$

where Q is defined by PDE coefficients a, b, c, d, e, f and polynomial m

$$q(t,x) := \begin{bmatrix} u(t,x) \\ u_{x_1}(t,x) \\ u_{x_2}(t,x) \end{bmatrix} \quad \Longrightarrow \quad q(t,\cdot) \in L_2^3(\Omega).$$

Example 1: Testing Biological PDE

$$u_t(t,x) = h\Big(u_{x_1x_1}(t,x) + u_{x_2x_2}(t,x)\Big) + ru(t,x) \qquad (11)$$

where $h, r > 0$, $x \in (0,1)^2$. By analytic solution for $h > h_{cr}$, where

$$h_{cr} := r/2\pi^2,$$

the PDE (11) is stable.

Table 1: Predicted minimum stable h vs deg(s) for $r = 4$

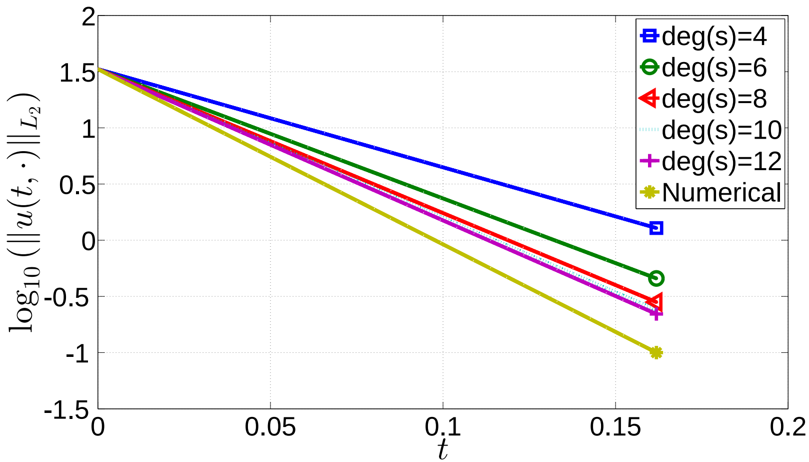
deg(s)	4	6	8	10	12	analytic
h	0.332	0.259	0.238	0.229	0.227	0.203

We used the MATLAB toolbox SOSTOOLS and SeDuMi.

Example 1: Predicted Rate of Exponential Decay

Table 2: Maxim rate of decay γ vs deg(s) for $h = 2$

deg(s)	4	6	8	10	12
γ	40.25	53	59	61	62



Numerical Solution for KISS PDE and Predicted Bounds with different deg(s) for $u(0,x) = 10^3 x_1 x_2 (1 - x_1)(1 - x_2)$.

Example 2: Randomly generated PDE

$$\begin{aligned} u_t(t,x) = (5x_1^2 - 15x_1x_2 + 13x_2^2)(u_{x_1x_1}(t,x) + u_{x_2x_2}(t,x)) \\ + (10x_1 - 15x_2)u_{x_1}(t,x) + (-15x_1 + 26x_2)u_{x_2}(t,x) \\ - (17x_1^4 - 30x_2 - 25x_1^2 - 8x_2^3 - 50x_2^4)u(t,x) \end{aligned}$$

with $u(0,x) = 10^3 x_1 x_2 (1 - x_1)(1 - x_2)$.

Table 3: Estimated Rate of Decay based on

Numerical Solution	13.07
Proposed SOS Method	12.5

THANK YOU FOR LISTENING

Appendix

Formulation of Schrödinger Equation with 1 spatial dimension as coupled PDEs

V – potential energy, i – imaginary unit, $\hbar$ – reduced Planck constant and ψ – wave function of the quantum system.

$$i\hbar\psi_t(t,x) = -\frac{\hbar^2}{m}\psi_{xx}(t,x) + V(x)\psi(t,x).$$

Decompose the solution into real and imaginary parts as

$$\psi(t,x) = \psi^{rl}(t,x) + i\psi^{im}(t,x),$$

then substitute in the initial PDE and separate the real and imaginary parts as

$$\begin{bmatrix} \psi_t^{rl}(t,x) \\ \psi_t^{im}(t,x) \end{bmatrix} = \underbrace{\frac{\hbar}{m}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \psi_{xx}^{rl}(t,x) \\ \psi_{xx}^{im}(t,x) \end{bmatrix} + \underbrace{\frac{V(x)}{\hbar}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{C(x)} \begin{bmatrix} \psi^{rl}(t,x) \\ \psi^{im}(t,x) \end{bmatrix}.$$

Back to **main**.

Formulation of Equation for Acoustic Waves as coupled PDEs

$t > 0, x \in (0, L), c > 0,$

$$p_{tt}(t,x) = c^2 p_{xx}(t,x) + \frac{2c^2}{x} p_x(t,x)$$

is equivalent to a system of two coupled first order PDEs as

$$\begin{bmatrix} q_t(t,x) \\ p_t(t,x) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & c^2 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} q_{xx}(t,x) \\ p_{xx}(t,x) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & \frac{2c^2}{x} \\ 0 & 0 \end{bmatrix}}_{B(x)} \begin{bmatrix} q_x(t,x) \\ p_x(t,x) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} q(t,x) \\ p(t,x) \end{bmatrix},$$

where q is an auxiliary function, $q = p_t$. Moreover, if the boundary conditions imply amplification of the waves, i.e.

$$p(t,0) = f_1 p(t,L) \quad \text{and} \quad p_x(t,0) = f_2 p_x(t,L)$$

for some $f_1, f_2 > 0$ and all $t > 0$, then the boundary conditions can be stated using matrix D with

$$D(1,2) = 1, D(1,4) = -f_1, D(2,6) = 1, D(2,8) = -f_2.$$

Back to **main**.

Example of HHCE with 1 spatial dimension

$$u_t = \alpha \Box^2 u = -\frac{\alpha}{c^2} u_{tt} + \alpha u_{xx}.$$

Define an auxiliary function w through $w = u_t$, then

$$w = -\frac{\alpha}{c^2} w_t + \alpha u_{xx}$$

resulting in representation of HHCE as coupled PDEs, i.e.

$$\begin{bmatrix} u_t \\ w_t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c^2 & 0 \end{bmatrix} \begin{bmatrix} u_{xx} \\ w_{xx} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & -\frac{c^2}{\alpha} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}.$$

Back to **main**.

Example of formulation $P > 0, A^T P + PA < 0$ as an LMI

For example, consider $A \in \mathbb{R}^{2\times 2}$ and denote

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}.$$

Let e_{ij} be basis for $\mathbb{R}^{2\times 2}$, then

$$P = p_1 e_{11} + p_2 e_{12} + p_2 e_{21} + p_3 e_{22} \tag{12}$$

and $P > 0$ can be stated as

$$-(p_1 e_{11} + p_2(e_{12} + e_{21}) + p_3 e_{22}) + \epsilon I_2 \leq 0$$

for some chosen $\epsilon > 0$ in order to satisfy strict definiteness.

Using (12) one can formulate $A^T P + PA < 0$ as

$$p_1(A^T e_{11} + e_{11}A) + p_2(A^T(e_{12} + e_{21}) + (e_{12} + e_{21})A) \\ + p_3(A^T e_{22} + e_{22}A) + \epsilon I_2 \leq 0$$

Example of formulation $P > 0, A^T P + PA < 0$ as an LMI

As a result $P > 0, A^T P + PA < 0$ can be formulated as an LMI

$$F_0 + \sum_{i=1}^{3} p_i F_i \leq 0$$

with

$$F_0 = \begin{bmatrix} \epsilon I_2 & 0 \\ 0 & \epsilon I_2 \end{bmatrix}, \; F_1 = \begin{bmatrix} -e_{11} & 0 \\ 0 & A^T e_{11} + e_{11} A \end{bmatrix},$$

$$F_2 = \begin{bmatrix} -(e_{12} + e_{21}) & 0 \\ 0 & A^T(e_{12} + e_{21}) + (e_{12} + e_{21})A \end{bmatrix},$$

$$F_3 = \begin{bmatrix} -e_{22} & 0 \\ 0 & A^T e_{22} + e_{22} A \end{bmatrix}.$$

Back to **main**.

Time Derivative of the Lyapunov Functional along Trajectories of Coupled PDEs

$$
\begin{aligned}
&\frac{d}{dt}\left[V(u(t,x))\right] \\
&\quad = \frac{d}{dt}\left[\int_a^b u(t,x)^T M(x) u(t,x)\, dx + \int_a^b u(t,x)^T \int_a^b N(x,y) u(t,y)\, dy dx\right] \\
&\quad = \int_a^b \underbrace{\begin{bmatrix} u(t,x)^T \\ u_x(t,x)^T \\ u_{xx}(t,x)^T \end{bmatrix}^T}_{q(t,x)^T} \underbrace{\begin{bmatrix} C^T M + MC & MB & MA \\ B^T M & 0 & 0 \\ A^T M & 0 & 0 \end{bmatrix}}_{K(x)} \underbrace{\begin{bmatrix} u(t,x) \\ u_x(t,x) \\ u_{xx}(t,x) \end{bmatrix}}_{q(t,x)} dx \\
&\qquad + \int_a^b \underbrace{\begin{bmatrix} u(t,x)^T \\ u_x(t,x)^T \\ u_{xx}(t,x)^T \end{bmatrix}^T}_{q(t,x)} \int_a^b \underbrace{\begin{bmatrix} C^T N + NC & NB & NA \\ B^T N & 0 & 0 \\ A^T N & 0 & 0 \end{bmatrix}}_{L(x,y)} \underbrace{\begin{bmatrix} u(t,y) \\ u_y(t,y) \\ u_{yy}(t,y) \end{bmatrix}}_{q(t,y)} dy dx \\
&\quad = \int_a^b q(t,x)^T K(x) q(t,x)\, dx + \int_a^b q(t,x)^T \int_a^b L(x,y) q(t,y)\, dy dx.
\end{aligned}
$$

Back to **main**.