# ANALYSIS OF ZENO STABILITY IN HYBRID SYSTEMS USING SUM-OF-SQUARES PROGRAMMING

BY

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LIST OF SYMBOLS

| Symbol | Definition |
|---|---|
| SOS | Sum-of-squares |
| $\Sigma_x$ | Cone of sum of squares polynomials in variable $x$ |
| $\mathbf{R}[x]$ | Commutative ring of polynomials in variable $x$ |

ABSTRACT

Hybrid dynamical systems are systems that combine continuous dynamics with discrete transitions. Such systems can exhibit many unique phenomena, such as Zeno behavior. Zeno behavior is the occurrence of infinite discrete transitions in finite time. This phenomenon has been likened to a form of finite-time asymptotic stability, wherein trajectories converge asymptotically to compact sets in finite time whilst undergoing infinite transitions. Corresponding Lyapunov theorems have been developed. The main objective of our research was to develop computational techniques to determine whether or not a given hybrid system exhibits this Zeno phenomenon. In this thesis, we propose a method to algorithmically construct Lyapunov functions to prove Zeno stability of compact sets in hybrid systems. We use sum-of-squares programming to construct Lyapunov functions that allow us to prove Zeno stability of compact sets for hybrid systems with polynomial vector fields. Examples illustrating the use of the proposed technique are also provided. Finally, we provide a method using sum-of-squares programming to show Zeno stability of compact sets for systems with parametric uncertainties in the vector field, guard sets and domains, and transition maps. We then discuss potential applications of the proposed methods, along with examples.

# CHAPTER 1
# INTRODUCTION

In recent years, there has been an increasing prevalence of systems that combine continuous dynamics with discrete behavior or logical switching. The reasons for this are twofold. First, many systems that are of interest or importance require discontinuous controllers. Such controllers are called variable structure controllers, such as bang-bang and sliding mode controllers. Since the control inputs are discontinuous, the result is that the associated closed loop systems also reflect the discontinuous structure of the controller. One can think of such systems as having interacting continuous and logical components.

Secondly, with the advent of small, cheap, and efficient microprocessors, many systems are designed with computers interacting with physical phenomena - indeed, the entire field of embedded systems is built upon this idea. Embedded systems are physical systems managed or controlled by a dedicated microprocessor. With the increasing prevalence of such systems, a need arose for formal methods for modeling and analyzing them. It was from these twin causes that hybrid systems theory was developed.

Hybrid systems are systems crafted from the interaction between continuous dynamical systems and logical switching rules. The formal paradigms for modeling hybrid systems can be applied to myriad physical and man made phenomena. Models of hybrid systems are referred to as hybrid automata, or, more simply, hybrid models. Simple physical systems, such as Newton's pendulum, can be modeled as hybrid systems. Newton's pendulum can be thought of as a hybrid system with two discrete modes, each corresponding to one of the bobs being stationary. Extremely complex systems, such as networks, and air traffic control can also be modeled as hybrid systems. Moreover, the discontinuous nature of hybrid systems means that they exhibit many unique phenomena. For instance, as shown in Figure 1.1, with correct switching rules, it is possible to generate a stable system from a collection of both stable and unstable subsystems. In the case of Figure 1.1, we partition $\mathbb{R}^2$ into 4 separate domains (corresponding to each quadrant of $\mathbb{R}^2$), where each vector field is active. We can define the system with the following differential equations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(x,y) = \begin{pmatrix} -1 \\ 1.1 \end{pmatrix} \qquad \text{when } x \geq 0, y > 0$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix} \qquad \text{when } x < 0, y \geq 0$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(x,y) = \begin{pmatrix} y^3 \\ x^3 \end{pmatrix} \qquad \text{when } x \leq 0, y < 0$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(x,y) = \begin{pmatrix} y^2/10 \\ x^2 \end{pmatrix} \qquad \text{when } x > 0, y \leq 0.$$

One of the most important phenomena unique to hybrid systems is Zeno behavior. The Zeno phenomenon is that of infinite discrete transitions occurring in a finite time period. It is similar to the chattering phenomenon observed in variable structure control; however, it must be noted here that the two are distinct. Zeno behavior arises in models of a variety of hybrid systems. A simple example of a hybrid system which exhibits Zeno behavior is that of the bouncing ball. We model a bouncing ball as a hybrid system with one domain and vector field, and resets occur

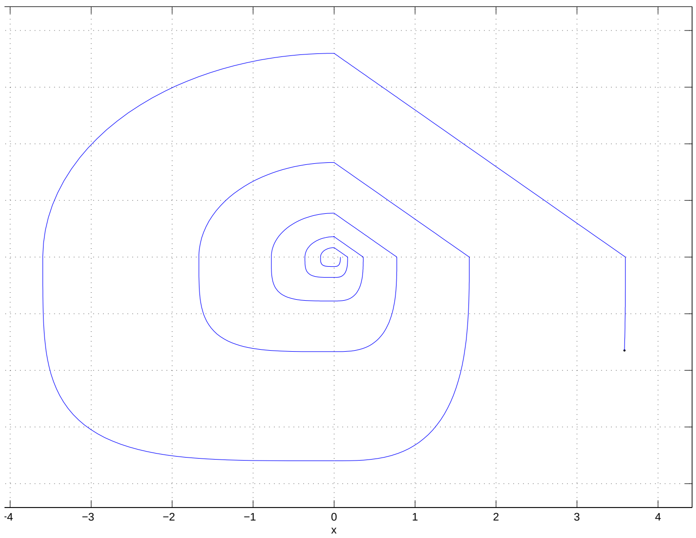


Figure 1.1. A hybrid system constructed from 4 dynamical systems

when the ball impacts with the surface. The dynamics of the ball are given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -g \end{pmatrix} \tag{1.1}$$

We assume that a coefficient of restitution $c \in (0, 1)$ reduces the value of $x_2$ at each impact (when $x_1(t) = 0$ when $t > 0$). First, since the coefficient of restitution never allows $x_2$ (the velocity of the ball) to become 0 at the impact, the model allows for infinite such impacts.

Second, we need to show that these transitions occur in finite time. Consider an initial condition $x_1(0), x_2(0)$. Suppose the velocity of the ball at the first impact is $-x_2^*$. Then, after $n$ impacts, the velocity of the ball immediately following the impact will be $c^n x_2^*$. We can then calculate the time until the $n+1$th impact as

$$t_{n+1} - t_n = \frac{2c^n x_2^*}{g}$$

If we sum the above expression over all $n \in \mathbb{N}$, we get

$$\begin{aligned}\sum_{n=1}^{\infty} &= \frac{2x_2^*}{g}\sum_{n=1}^{\infty} c^n \\ &= \frac{2x_2^*}{g}\frac{1}{1-c}\end{aligned}$$

which is finite. Furthermore, the initial interval $t_1 - t_0$ (that is, the time before the first impact) will also be finite as long as the initial conditions are finite. Hence, we see that every execution of the bouncing ball hybrid system will be Zeno.

While the Zeno phenomenon cannot occur in nature, it can occur even in highly sophisticated models of physical or man-made systems. It can be undesirable for several reasons - first, the fact that it involves infinite transitions means that it can often cause simulations to fail. Second, if a system whose model exhibits Zeno behavior is physically implemented, the fast transitions may lead to physical failures as well.

The main goal of this thesis is to address the problem of detecting Zeno behavior. We do so by framing the Zeno phenomenon as a form of stability. From there, we modify existing Lyapunov based conditions for this stability, and create algorithmic methods for testing those conditions. We then demonstrate our technique on various example hybrid systems.

## 1.1 Background and Related Work

**1.1.1 Hybrid Systems.** There is a large body of research into the modeling and analysis of hybrid systems. The use of finite state machines to capture the interaction between continuous dynamics and discrete transitions is described in detail in [1]. This modeling framework is used throughout this document, and is described in detail in Chapter 3.

Alternatively, much work has gone into using set valued maps to describe hybrid systems. A mapping $A : X \rightrightarrows Y$ is called a set valued mapping if for each $x \in X$, $A(x) \subset Y$. Typically, hybrid systems have piecewise continuous vector fields (that is, continuous almost everywhere). This in turn translates into different vector fields being active in different regions of the state space. As such, set valued maps between collections of domains and collections of vector fields are apt models of hybrid systems. A more comprehensive treatment of these modeling paradigms can be found in [2].

As mentioned previously, numerous physical and man-made systems can be modeled using hybrid models. For instance, in [3], hybrid systems were used to model systems with embedded microprocessors. Similarly, circuits with switches, notably power electronic converters, can be modeled with hybrid systems, such as in [4] and [5]. Hybrid systems are also used successfully to model communications networks, as is the case in [6] and [7]. Hybrid systems are used in the development of air traffic control management, as described in [8].

Apart from modeling paradigms, there has been a great deal of work into the field of non-smooth dynamical systems. The analysis of discontinuous vector fields is critical to the study of hybrid systems. A seminal work in this field is [9], which lays the groundwork for the analysis of discontinuous dynamical systems and differential inclusions. In that work, generalizations to concepts of continuity, existence and uniqueness, and solutions are described for discontinuous differential equations. Other important work is the generalizations of classical derivatives (notably gradients) to discontinuous vector fields, such as the development of generalized gradients by Clarke [10].

The analysis of hybrid systems has also progressed a great deal. There are extensions to the notions of controllability and observability, as described in [11]. Similarly, there is a rich body of work on the stability of hybrid systems, particularly in Lyapunov stability theorems for hybrid systems, such as [12], [13], and [14]. Moreover, research has been done on means to construct Lyapunov functions for hybrid systems, such as in [15], where a technique to construct piecewise quadratic Lyapunov functions is discussed. More recently, techniques using sum-of-squares programming have been used to construct Lyapunov functions for hybrid systems, as described in [16] and [17].

Zeno behavior has also garnered significant interest. Zeno hybrid systems are described in detail in, for example, [18]. Zeno behavior can cause simulations to halt or fail, since infinitely many transitions would need to be simulated, as noted in, e.g., [18]. This problem was addressed in [19] and [20], which describe methods to "regularize" hybrid systems to ensure that trajectories continue after the Zeno equilibrium. Sufficient conditions for Zeno behavior for a limited class of hybrid systems - first quadrant hybrid systems - were given in [21], and further sufficient conditions for systems with nonlinear vector fields based on constant approximations were given in [22]. More recently, necessary and sufficient Lyapunov conditions for the existence of isolated Zeno equilibrium were first given in [23]. These results were extended in [24], where the concept of Zeno stability was described as an extension of finite-time asymptotic stability. Moreover, [24] provided Lyapunov conditions for Zeno stability of compact sets. Ames and Lamperski also extended the results in [23] to non-isolated Zeno equilibrium were given in [25]. In that paper, the authors map hybrid systems to a 2-dimensional first quadrant hybrid system, which serve as two dimensional analogs to Lyapunov functions. Note that this technique cannot be implemented using sum-of-squares programming, as the conditions require the solution of a problem with a rational expression in the decision variables. Also, a Lyapunov characterization of Filippov solutions (see [9] for more details) was given in [26].

**1.1.2 Positive Polynomials and Sum-of-Squares Programming.** One of the key areas of study in algebraic geometry is the study of positivity of polynomials over semialgebraic sets. Positivstellensatz theorems are results that allow us to certify positivity of polynomials over semialgebraic sets. Notable positivstellensatz theorems are given in [27], [28], and [29]. Other notable results on the positivity of polynomials can be found in the survey [30].

Positive polynomials are integral to numerous disciplines, including systems analysis, optimization, and control. However, as noted by Blum in [31], determining whether a polynomial is positive is undecidable in polynomial time. To overcome the intractability of checking polynomial positivity, we can check whether a polynomial is a sum-of-squares (SOS) of other polynomials. As noted in, say, [32], checking whether a polynomial of degree $2d$ is a sum of squares is equivalent to finding a positive semidefinite matrix $Q$ such that

$$p(x) = Z(x)^T Q Z(x)$$

where $Z$ is a vector of monomials of degree $d$ or less. With the advent of efficient solvers for Linear Matrix Inequalities (LMIs), verifying whether a polynomial is a sum of squares is possible in polynomial time. This in turn led to the creation of the popular MATLAB toolbox, SOSTOOLS [33][34].

Using sum-of-squares techniques in systems analysis and control has led to numerous advances in those fields. The key reason for this is that finding Lyapunov functions for nonlinear systems can be accomplished using SOS programming. Sum of squares techniques have been applied to stability analysis of nonlinear systems [35], time delay systems [36], and, as noted earlier, hybrid systems [16][17]. Furthermore, SOS techniques can even be used in controller synthesis, as noted in [37][38]. This technique has been used in the analysis of various real world systems, such as [39] and [40]. SOS techniques have also been successfully applied to stochastic hybrid systems [41][42][43].

### 1.2 Summary of Contributions and Outline

The key contributions of this thesis are as follows. First, we provide a method to construct Lyapunov functions to prove Zeno stability using sum-of-squares programming. This result is useful because it enables us to verify whether or not a hybrid system exhibits Zeno behavior algorithmically. Furthermore, we also provide a means to verify Zeno stability for systems with parametric uncertainties. We also provide illustrative examples and important uses of the given techniques.

The outline of the thesis is as follows:

- Chapter 2 provides a discussion of some of the key results used in this research. Specifically, we discuss sum-of-squares polynomials, and their connection to semidefinite programming. We also discuss the Stengle's positivstellensatz, a theorem in algebraic geometry that is key to our analysis. Chapter 2 also provides a brief discussion on continuous dynamical systems. Most importantly, we discuss Lyapunov stability, and the use of Lyapunov functions to prove stability.

- In Chapter 3, we lay the groundwork for our analysis of hybrid systems. We provide formal definitions for hybrid systems, and introduce executions of hybrid systems, which are analogous to solutions or flows of continuous dynamical systems. We state results for the existence and uniqueness of executions. Lastly, Chapter 3 provides basic definitions pertinent to Zeno behavior - namely, Zeno equilibria, and Zeno stability.

- In Chapter 4, we give the main results. Lyapunov characterizations of Zeno stability are presented. We then provide our technique for constructing Lyapunov functions to prove Zeno stability. Simple numerical examples illustrating the use of the technique are also provided. We then provide a technique to verify Zeno stability for hybrid systems with parametric uncertainties. Again, numerical examples are provided.

- In Chapter 5, we discuss applications of the given technique. We apply our technique to a hybrid system with a nonlinear vector field, as well as a system controlled with a sliding mode controller with imperfect switching. We also demonstrate the use of our robustness result by analyzing Zeno stability of a hybrid system with uncertain switching.

# CHAPTER 2

# MATHEMATICAL BACKGROUND

The goal of this chapter is to define the mathematical tools used in this research. First, an overview of Sum-of-Squares programming is provided. Next, basic definitions and conditions of stability of continuous-time dynamical systems are discussed. We discuss these concepts because much of the analysis of hybrid systems is requires an understanding of the analysis of continuous-time dynamical systems.

## 2.1 Sum of Squares Polynomials and the Positivstellensatz

One of the key problems in the study of systems and controls is showing the global nonnegativity of polynomials, possibly under some constraints. This problem is, as described in [32], NP-hard. This means that no algorithm exists to verify global positivity of polynomials in polynomial time. As such, finding tractable relaxations for such problems becomes valuable. One technique of accomplishing this is showing that polynomials are sums of squares, which, as will be demonstrated shortly, can be accomplished in polynomial time.

### 2.1.1 Sum of Squares Polynomials.

**Definition 1. (Sum of Squares Polynomial)** A polynomial $p(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be Sum of Squares (SOS) if there exist polynomials $f_i(x) : \mathbb{R}^n \to \mathbb{R}$ such that

$$p(x) = \sum_i (f_i(x))^2$$

We use $p \in \Sigma_x \subset \mathbf{R}[x]$ to denote that $p$ is SOS.

We now present a theorem that gives a polynomial-time complexity test to determine whether a polynomial is SOS.

**Theorem 1.** *Suppose $p \in \mathbf{R}[x]$ has degree $2d$. The following two statements are equivalent:*

1. *$p(x)$ is SOS*
2. *There exists a positive semidefinite matrix $Q$, such that*

   $$p(x) = Z(x)^T QZ(x)$$

   *where $Z(x)$ is a vector of all monomials of degree $d$ or less*

We now present a proof of the above theorem. Similar proofs can be found in, say, [32].

**Proof:**

1. First, we show that (2) implies (1).

   We have

   $$p(x) = Z(x)^T QZ(x)$$

   Since $Q$ is positive semidefinite, there exists a matrix $A$ such that $Q = A^*A$, where $A^*$ denotes the conjugate transpose of $A$. Thus, we get

   $$\begin{aligned} p(x) &= Z(x)^T A^* AZ(x) \\ &= (AZ(x))^*(AZ(x)) \end{aligned}$$

   We can write

   $$AZ(x) = F(x)$$

   where $F(x)$ is a vector of polynomials. Hence,

   $$p(x) = F(x)^* F(x) \in \Sigma_x$$

Thus, if there exists a positive semidefinite matrix $Q$, and a vector $Z(x)$ of monomials up to degree $d$, and $p(x) = Z^T(x)QZ(x)$, then $p$ is a sum of squares.

2. To show (1) implies (2):

   If $p \in \Sigma_x$, and is of degree $2d$ then $p(x) = \sum_{i=1}^{n} f_i(x)^2$, where $\{f_i\}_{i=1,..,n}$ are polynomials of degree less than or equal to $d$. Define

$$F(x) = [f_1(x), ..., f_n(x)]^T.$$

   Thus, we can write

$$p(x) = F(x)^T F(x).$$

   Furthermore, let $\{m_j(x)\}_{j=1,...,M}$ be monomials in $x$ of degree less than or equal to $d$, and let $\{a_{ij}\}_{i=1,..,N,j=1,...,M}$ be constants. Then,

$$Z(x) = (m_1, ..., m_M)^T$$

   and

$$f_i(x) = \sum_{j=1}^{M} a_{ij} m_j(x).$$

   It follows that

$$F(x) = \begin{pmatrix} \sum_{j=1}^{M} a_{1j} m_j(x) \\ \vdots \\ \sum_{j=1}^{M} a_{Nj} m_j(x) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & \dots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{N1} & \dots & a_{NM} \end{pmatrix} \begin{pmatrix} m_1(x) \\ \vdots \\ m_M(x) \end{pmatrix} = AZ(x)$$

where

$$A = \begin{pmatrix} a_{11} & \dots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{N1} & \dots & a_{NM} \end{pmatrix}.$$

Thus, $p(x) = F(x)^T F(x) = Z(x)^T A^T A Z(x)$. Since $A^T A$ is positive semidefinite, we have (2).

□

Therefore, checking whether a polynomial is SOS is equivalent to checking the existence of a positive-semidefinite matrix $Q$ under some affine constraints which has been shown to be decidable in polynomial time. Thus, we see that while checking polynomial positivity is NP-hard, checking whether a polynomial is SOS is decidable in polynomial time. We also provide a simple example that illustrates the use of Theorem 1:

**Example 1.**

Consider the polynomial $p(x) = x^2 + 2x + 1 = (x + 1)^2$. Obviously, this polynomial is SOS. We choose $Z = [1, x]^T$, and we search for a $2 \times 2$ matrix $Q$. We can then find

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

which is positive-semidefinite. We see that

$$\begin{pmatrix} 1 \\ x \end{pmatrix}^T \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = x^2 + 2x + 1$$

**Positive Polynomials are not always Sum-of-Squares**

While sum-of-squares polynomials provide a tractable relaxation for problems of positivity of polynomials, such techniques are conservative. This is because not every nonnegative polynomial can be decomposed to a sum-of-squares form. Further discussion on this matter can be found in [32].

A classic example of a polynomial that is positive but not SOS is the Motzkin polynomial

$$M(x_1, x_2, x_3) = x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2 \tag{2.1}$$

which has been shown to be globally nonnegative. However, as demonstrated in [32], 2.1 admits no SOS decomposition.

### 2.1.2 The Positivstellensatz.

A positivstellensatz is a theorem from real algebraic geometry which provides a means to verify the positivity of a polynomial over a semialgebraic set. For this section, we use $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$

**Definition 2. (Semialgebraic Set)** A semialgebraic set is a set of the form

$$S := \{x \in \mathbb{R}^n : f_i(x) \geq 0, i = 1, ..., n_1,\ g_i(x) = 0, i = 1, ..., n_2\}$$

where each $f_i \in \mathbf{R}[x]$, and $g_i \in \mathbf{R}[x]$.

In simple terms, a semialgebraic set is a set defined by polynomial equalities and inequalities. Examples of semialgebraic sets include:

1. The unit disc in $\mathbb{R}^2$:

$$\mathbb{B} := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 - 1 \leq 0\},$$

2. The half circle $\mathcal{H}$ formed by intersection of the unit disc and a half-plane:

$$\mathcal{H} := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 - 1 \leq 0, x_1 - 2x_2 \geq 0\}.$$

**Definition 3. (Multiplicative Monoid)** A multiplicative monoid $\mathcal{M}$ generated by elements $\{f_1, ..., f_n\} \in \mathbf{R}[x]$ is the set

$$\mathcal{M} := \left\{ p \in \mathbf{R}[x] : p = \prod_{i=1}^{n} f_i^{k_i},\ k_i \in \mathbb{N}^0 \right\}$$

Thus, $\mathcal{M}$ is the set of of finite products of $\{f_1, ..., f_n\}$.

An illustrative example is given below:

**Example 2.**

Consider the polynomials $f_1$ and $f_2$. The multiplicative monoid generated by $f_1$ and $f_2$ is the set

$$\mathcal{M} := \{p \in \mathbb{R}[x] : p = \prod f_1^i f_2^j, i, j \in \mathbb{N}^0\}$$

As such, elements of $\mathcal{M}$ will include 1, $f_1$, $f_2$, $f_1 f_2$, $f_1^2$, $f_2^2$, $f_1^3 f_2^4$ and so on. This set contains all possible products of $f_1$ and $f_2$

**Definition 4. (Cone)** For given elements $\{f_1, ..., f_n\} \in \mathbf{R}[x]$, let $\bar{\mathcal{M}} \subset \mathcal{M}$ be the set of products defined by

$$\bar{\mathcal{M}} := \left\{ p \in \mathbf{R}[x] : p = \prod_{i=1}^{n} f_i^{k_i}, k_i \in \{0, 1\} \right\}$$

and let $M$ denote the cardinality of $\bar{\mathcal{M}}$. We say the **cone** $\mathcal{P}$ generated by $\{f_1, ..., f_n\} \in \mathbf{R}[x]$ is the subset of $\mathbf{R}[x]$ defined as

$$\mathcal{P} := \left\{ p \in \mathbf{R}[x] : p = s_0 + \sum_{i=1}^{M} s_i m_i,\ m_i \in \bar{\mathcal{M}},\ s_i \in \Sigma_x \right\}.$$

$\mathcal{P}$ satisfies the following properties:

1. $a, b \in \mathcal{P}$ implies $a + b \in \mathcal{P}$

2. $a, b \in \mathcal{P}$ implies $a \cdot b \in \mathcal{P}$

3. $a \in \mathbf{R}[x_1, ..., x_n]$ implies $a^2 \in \mathcal{P}$

Below, we provide some illustrative exampes:

**Example 3.**

Consider the polynomials $f_1$ and $f_2$. The cone $\mathcal{P}$ generated by these polynomials is given by

$$\mathcal{P} := \{p \in \mathbb{R}[x] : p = s_0 + s_1 f_1 + s_2 f_2 + s_3 f_1 f_2; s_i \in \Sigma_x\}$$

**Example 4.**

The cone generated by the empty set is $\Sigma_x$. It is the smallest such cone possible.

**Definition 5.** **(Ideal)** The Ideal $\mathcal{I}$ generated by $\{f_1, ..., f_n\} \in \mathbf{R}[x]$ is defined as

$$\mathcal{I} := \left\{ p \in \mathbf{R}[x] : p = \sum_{i=1}^{n} q_i f_i,\ q_i \in \mathbf{R}[x] \right\}$$

Note that $\mathcal{I}$ must satisfy

1. $a, b \in \mathcal{I}$ implies $a + b \in \mathcal{I}$

2. $a \in \mathcal{I}; b \in \mathbf{R}[x]$ implies $ab \in \mathcal{I}$

Intuitively, the ideal generated by a collection of polynomials is the set of polynomials that vanishes when *all* of the generating polynomials vanishes. Provided below is an illustrative example:

**Example 5.**

Consider the polynomials $f_1 = x_1 + x_2$ and $f_2 = x_1^2 + x_1$. The ideal generated by $f_1$ and $f_2$ is the set

$$\mathcal{I} := \{p \in \mathbb{R}[x] : p = s_1 f_1 + s_2 f_2; s_1, s_2 \in \mathbb{R}[x]\}$$

Clearly, every element of $\mathcal{I}$ vanishes when both $f_1$ and $f_2$ vanish.

**Theorem 2. (Stengle's Positivstellensatz)** *Consider the polynomials* $\{f_1, f_2, ..., f_{n_1}\} \in \mathbf{R}[x]$, $\{g_1, g_2, ..., g_{n_2}\} \in \mathbf{R}[x]$, *and* $\{h_1, h_2, ..., h_{n_3}\} \in \mathbf{R}[x]$. *Let* $\mathcal{P}$ *be the cone generated by* $\{f_i\}_{i=1,2,...,n_1}$, $\mathcal{M}$ *be the multiplicative monoid generated by* $\{g_j\}_{j=1,2,...,n_2}$, *and* $\mathcal{I}$ *be the Ideal generated by* $\{h_k\}_{k=1,2,...,n_3}$. *Then, the following statements are equivalent:*

1. $\{x \in \mathbb{R}^n : f_i(x) \geq 0, g_j(x) \neq 0, h_k(x) = 0, i = 1, ...n_1, j = 1, ..., n_2, k = 1, ..., n_3\} = \emptyset$

2. $\exists f \in \mathcal{P}$, $\exists g \in \mathcal{M}$, $\exists h \in \mathcal{I}$ *s.t.*

$$f + g^2 + h \equiv 0$$

The positivstellensatz has been described as a generalization of the S-procedure (described in [44]). However, while the S-procedure provides information regarding the positivity of quadratic forms such that other quadratic forms are also positive, the positivstellensatz can be used to obtain certificates of positivity for polynomials of arbitrary degree over semialgebraic sets. We use the positivstellensatz extensively in this thesis to construct Lyapunov functions which are positive on bounded sets (see chapter 4).

For further details and proofs, we refer to [27] and [32]. Also, provided below are examples illustrating the use of the positivstellensatz.

**Example 6.**

Consider the sets given by

$$D_1 := \{x \in \mathbb{R}^2 : 1 - x_1^2 - x_2^2 \geq 0\}$$

and

$$D_2 := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 - 2 = 0\}$$

We can use the positivstellensatz to show that these two sets do not intersect. We need to show that the set

$$\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 - 2 \geq 0, 1 - x_1^2 - x_2^2 \geq 0\} = \emptyset$$

We then apply the positivstellensatz- if the above is true, then we can find SOS polynomials $s_0, s_1$, and a polynomial $t$ such that

$$s_0(x) + s_1(x)(1 - x_1^2 - x_2^2) + t(x)(x_1^2 + x_2^2 - 2) = 0$$

We can choose $s_0(x) = s_1(x) = t(x) = 1$ to give us the desired result.

## 2.2 Continuous Time Dynamical Systems

In this subsection, we briefly describe properties of finite dimensional continuous dynamical systems. Understanding the properties of such systems is crucial to the understanding of more complex systems, including hybrid systems. Here, we describe basic notions of stability.

**2.2.1 Stability and Continuous Time Dynamical Systems.** In this subsection, we define stability for continuous time dynamical systems, and provide theorems for the stability of continuous time dynamical systems.

**Definition 6. (Solution map)** Consider a dynamical system

$$\dot{x} = f(x(t)) \quad x(0) = x_0 \tag{2.2}$$

where $x_0 \in D$, and $f \in C^1(D, \mathbb{R}^n)$. A function $\phi : \mathbb{R}^n \times \mathbb{R}$ is a solution map of 2.2 if

$$\frac{d}{dt}\phi(x_0, t) = f(\phi(x_0, t)) \tag{2.3}$$

$$\phi(\phi(x_0, t), s) = \phi(x_0, t + s) \tag{2.4}$$

$$\phi(x_0, 0) = x_0 \quad \text{for all } x_0 \in D. \tag{2.5}$$

**Definition 7. (Stability)** Consider the dynamical system

$$\dot{x}(t) = f(x(t)); \quad x(0) = x_0 \tag{2.6}$$

where $x_0 \in D$, and $f \in C^1(D, \mathbb{R}^n)$, and let $f(x^*) = 0$. Suppose 2.6 generates the solution map $\phi(x_0, t)$.

- $x^*$ is said to be a Lyapunov stable equilibrium of $f$ if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x_0$ satisfying $|x_0 - x^*| < \delta$, $|\phi(x_0, t) - x^*| < \varepsilon$.
- $x^*$ is said to be a locally asymptotically stable equilibrium of $f$ if it is Lyapunov stable, and there exists some $\delta > 0$ such that for any $x_0$ satisfying $|x_0 - x^*| < \delta$, $\lim_{t\to\infty} \phi(x_0, t) = x^*$.
- $x^*$ is said to be a globally asymptotically stable equilibrium of $f$ if it is Lyapunov stable, and for all $x_0 \in D$, $\lim_{t\to\infty} \phi(x_0, t) = x^*$.
- $x^*$ is said to be a locally exponentially stable equilibrium of $f$ if there exist $\alpha, \beta,\ \gamma > 0$ such that for any $x_0$ satisfying $|x_0 - x^*| < \delta$, $|\phi(x_0, t) - x^*| \leq \alpha|x_0 - x^*|e^{-\beta t}$ .
- $x^*$ is said to be a globally exponentially stable equilibrium of $f$ if there exist $\alpha, \beta,\ \gamma > 0$ such that for all $x_0 \in D$, $|\phi(x_0, t) - x^*| \leq \alpha|x_0 - x^*|e^{-\beta t}$ for all $x_0 \in D$.

To prove stability of dynamical systems, we use Lyapunov's method, which is described below in Theorem 3 :

**Theorem 3.** *Consider the dynamical system*

$$\dot{x}(t) = f(x(t)); \quad x(0) = x_0 \tag{2.7}$$

*where* $x_0 \in D \subset \mathbb{R}^n$*, and* $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$*, and let* $x^*$ *be an equilibrium of the system. Assume a unique solution* $x(t) = \phi(x_0, t)$ *to 2.7 exists for each* $x_0 \in D$*. If there exists a function* $V : D \to \mathbb{R}$ *that satisfies*

$$\begin{aligned} V(x(t)) > 0 \quad \forall x(t) \in D \backslash x^* \\ V(x^*) = 0 \\ \langle \boldsymbol{\nabla} V(x(t)), f(x(t)) \rangle \leq 0 \quad \forall x \in \mathbb{R}^n \end{aligned} \tag{2.8}$$

*then* $x^*$ *is a Lyapunov stable equilibrium of the system. Furthermore, if*

$$\langle \boldsymbol{\nabla} V(x(t)), f(x(t)) \rangle < 0 \quad \forall x \in \mathbb{R}^n$$

*then* $x^*$ *is an asymptotically stable equilibrium. If there exist constants* $\alpha, \beta$*, and* $\gamma$*, and* $p \geq 0$ *such that*

$$\alpha |x(t)|^p \leq V(x(t)) \leq \beta |x(t)|^p \quad \forall x \in \mathbb{R}^n$$
$$\langle \boldsymbol{\nabla} V(x(t)), f(x(t)) \rangle \leq \gamma V(x(t)) \quad \forall x \in \mathbb{R}^n$$

*then* $x^*$ *is an exponentially stable equilibrium.*

Further modifications to the above are also possible. For instance, it is possible to find Lyapunov functions that prove that system trajectories converge to the equilibrium in finite time. For a more detailed discussion of Lyapunov stability, refer to [45] or [46].

We now provide a simple example that demonstrates the application of Theorem 3 to a nonlinear dynamical system.

**Example 7.**

We want to prove stability of the vector field

$$\dot{x} = f(x) = \begin{pmatrix} -x_1^3 - x_1 x_2^2 \\ \\ -x_2 - 3x_1^2 x_2^3 \end{pmatrix}$$

We see that $x^* = (0, 0)$ is an equilibrium of the system. If we choose $V(x) = x^2 + y^2$, we notice that

- $V(0) = 0$
- $V(x) > 0$ for all $x \in \mathbb{R}^2 \backslash \{0\}$.
- $\langle \boldsymbol{\nabla} V(x), f(x) \rangle = (-x_1^4 - x_1^2 x_2^2) + (-x_2^2 - 3x_1^2 x_2^4)$

Since $-(-x_1^4 - x_1^2 x_2^2) - (-x_2^2 - 3x_1^2 x_2^4) \in \Sigma_x$, we note that $\langle \boldsymbol{\nabla} V(x), f(x) \rangle \leq 0$. Thus, we satisfy the conditions of Theorem 3 and prove Lyapunov stability of the equilibrium of the given vector field.

There are several interesting points to note regarding Lyapunov functions. A Lyapunov function can be considered to be a measure of energy for a given system. Thus, Theorem 3 can be interpreted as showing that the energy of the system decreases to 0 until the equilibrium, thus showing that the system is stable. Also of note is the fact that Lyapunov functions are not unique - there may exist multiple functions that satisfy the conditions of Theorem 3 for a given vector field.

However, in recent years, the use of Lyapunov functions and techniques associated with them have become especially important. The availability of efficient optimization algorithms means the ability to find Lyapunov functions no longer depends on the analytical skill of the investigator. For example, for the linear system

$$\dot{x} = Ax$$

the Lyapunov function can be found by finding a positive semidefinite matrix $P$ that satisfies

$$A^T P + PA \preceq 0. \tag{2.9}$$

In this case, the Lyapunov function $V(x)$ is given by $V(x) = x^T P x$. With the advent of efficient solvers for Linear Matrix Inequalities (LMIs), it is relatively simple to find Lyapunov functions for linear systems. For a more detailed treatment of such techniques, refer to, say, [47]. Moreover, using SOS techniques, it is possible to find Lyapunov functions for polynomial vector fields [35].

# CHAPTER 3
# HYBRID SYSTEMS

Hybrid systems are, as mentioned previously, systems that exhibit both "flow" and "jump" properties. As such, it becomes necessary to define a modeling paradigm that allows us to capture both aspects of the system's behavior. Furthermore, this extension of the model of a dynamical system also necessitates an extension of the idea of a solution. Extending the idea of the solution to one that captures both continuous flows and discrete transitions raises further questions. For instance, under what conditions can the existence and uniqueness of solutions be guaranteed? What does it mean for a hybrid system to be stable? How can we know whether or not a hybrid system is stable?

The aim of this chapter is to answer four questions. First, how can we model systems that exhibit both continuous flows and discrete transitions? Second, how do we define solutions to hybrid systems? Third, under what conditions do solutions exist for such systems, and when are these solutions unique? Lastly, how can we verify stability for such systems?

## 3.1 Modeling Hybrid Systems

In this section, we define hybrid systems and their executions. We use similar notation to that given in [1] and, more recently, [23].

**Definition 8. (Hybrid System)** A hybrid system **H** is a tuple:

$$H = (Q, E, D, F, G, R)$$

where

- $Q$ is a finite collection of discrete states or indices.

- $E \subset Q \times Q$ is a collection of edges. For any edge $e = (q, q')$ we use the functions $s$ and $t$ to denote the start and end, so that for $e = (q, q')$, $s(e) = q$ and $t(e) = q'$.

- $D = \{D_q\}_{q \in Q}$ is a collection of Domains, where for each $q \in Q$, $D_q \subseteq \mathbb{R}^n$.

- $F = \{f_q\}_{q \in Q}$ is a collection of vector fields, where for each $q \in Q$, $f_q : D_q \to \mathbb{R}^n$.

- $G = \{G_e\}_{e \in E}$ is a collection of guard sets, where for each $e = (q, q') \in E$, $G_e \subset D_q$

- $R = \{\phi_e\}_{e \in E}$ is a collection of Reset Maps, where for each $e = (q, q') \in E$, $\phi_e : G_e \to D_{q'}$.

This framework modeling hybrid systems combines differential equations and finite state machines, which simultaneously describe the continuous evolution and discrete evolution.

We now provide two illustrative examples for the modeling of hybrid systems.

**Example 8.**

The first example is the bouncing ball on a flat surface. The bouncing ball (in an ideal, frictionless universe) can be easily modeled with a hybrid system. In this case, the bouncing ball system **B** is modeled by a tuple:

$$\mathbf{B} = (Q, E, D, F, G, R)$$

where

- $Q = \{q_0\}$

- $E = \{(q_0, q_0)\}$

- $D := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$

- $F = \{f\}$, where

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ \\ -g \end{pmatrix}$$

where $g$ is the acceleration caused by gravity.

- $G := \{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \leq 0\}$

- $R = \phi(x) = [0, -cx_2]^T$. Here, $c$ is a coefficient of restitution.

In this model of the bouncing ball, there is only one discrete state, corresponding to the ball bouncing, and one transition that occurs every time the ball reaches the surface. The only domain corresponds to the space above the surface, and the only system dynamics are those caused by force of gravity on the ball. The guard set corresponds to the surface. The reset maps reflects the behavior of the bounce - the velocity of the ball when it reaches the surface is reversed, and scaled by the coefficient of restitution.

**Example 9.**

Another example of a simple physical system that can be modeled with a hybrid system is Newton's pendulum. The system can be simplified to 2 pendula, with one motionless while the other swings. The bobs collide at the trough, whereupon momentum is transferred elastically from one pendulum to the other. In this model, we take $x_1$ and $x_2$ to be the angular position and velocity of the first bob, and $x_3$ and $x_4$ to be the angular position and velocity of the second bob. With this framework, we model Newton's pendulum with the hybrid system $\mathbf{P} = (Q, E, D, F, G, R)$ where

- $Q = \{q_1, q_2\}$. Here, $q_1$ describes the scenario when the first bob is moving and the second is stationary. The opposite occurs in $q_2$,

- $E = \{(q_1, q_2), (q_2, q_1)\}$
- $D := \{D_1, D_2\}$ where

$$D_1 = \{x \in \mathbb{R}^4 : x_1 < 0, x_2 > 0, x_3 = x_4 = 0\}$$
$$D_2 = \{x \in \mathbb{R}^4 : x_3 < 0, x_4 > 0, x_1 = x_2 = 0\}$$

- $F = \{f_1, f_2\}$, where

$$f_1(x) = (-k_1 x_2, k_2 x_1, 0, 0)^T$$
$$f_1(x) = (0, 0, k_3 x_4, -k_4 x_3)^T$$

  where $k_1$, $k_2$, $k_3$ and $k_4$ are constants. where $g$ is the acceleration caused by gravity.

- $G := \{G_{12}, G_{21}\}$ where

$$G_{12} := \left\{x \in \mathbb{R}^4 : x_1 = 0, x_2 \geq 0, x_3 = 0, x_4 = 0\right\}$$
$$G_{21} := \left\{x \in \mathbb{R}^4 : x_1 = 0, x_2 = 0, x_3 = 0, x_4 \leq 0\right\}$$

- $R = \{\phi_{12}(x), \phi_{21}(x)\}$ where

$$\phi_{12} = [0, 0, 0, cx_2]$$
$$\phi_{21} = [0, cx_4, 0, 0]$$

  and $c$ is a coefficient of restitution.

In the Newton's cradle hybrid model, there are two discrete modes, 1 and 2, each corresponding to the motion of a single bob. There are transitions form mode 1 to 2 and 2 to 1. Each domain corresponds to one bob moving - $D_1$ corresponds to the first bob moving, and $D_2$ corresponds to the second bob moving. The guard sets are reached when the two bobs collide. Each reset maps the velocity of the moving bob to the stationary bob, scaled by the coefficient of restitution.

There are other means of defining hybrid systems as well. In the next definition, we use set valued maps to model hybrid systems, as described in, say, [2].

**Definition 9. (Set Valued Map)** Consider two sets $X$ and $Y$. If for each $x \in X$, there is a corresponding set given by $F(x) \subset Y$, then we say $F$ is a set valued map. We use $F : X \rightrightarrows Y$ to denote that $F$ is a set valued map.

**Definition 10. (Alternative Definition for Hybrid Systems)** A hybrid system $H$ is the tuple $(Q, F, G, C, D)$ where

- $C \subset \mathbb{R}^n$
- $D \subset \mathbb{R}^n$
- $F : C \times Q \rightrightarrows \mathbb{R}^n$
- $G : D \times Q \rightrightarrows \mathbb{R}^n \times Q$

and which satisfy:

$$\begin{aligned} \dot{x} &\in F(x) & & x \in C \\ \begin{pmatrix} x^+ \\ q^+ \end{pmatrix} &\in G(x) & & x \in D, q \in Q \end{aligned}$$

A third form of hybrid systems are switched systems. Switched systems are hybrid systems without discrete jumps when mode transitions occur. Consequently, such systems admit continuous solutions (described in detail in [9]). A formal definition for switched systems is provided below:

**Definition 11. (Switched system)** A switched system is a hybrid system $H_S = (Q, E, D, F, G, R)$, which satisfies:

- For each $q, q' \in Q$, $\text{int}D_q \cap \text{int}D_{q'} = \emptyset$
- For each $e = (q, q') \in Q,\ G_e \in \partial D_q \cap \partial D_{q'}$
- For each $e = (q, q') \in E,\ \phi_e(x) = x$

We now introduce cyclic hybrid systems. Cyclic hybrid systems are those hybrid systems for which the pair $(Q, E)$ forms a directed graph - $Q$ is the collection of vertices, and $E$ the collection of edges.

**Definition 12.** A cyclic hybrid system $\mathbf{H}_c$ is a hybrid system where for each domain $q \in Q$, we can associate a unique edge $e(q) = (q, q_i) \in E$ such that $s(e(q)) = q$ and such that for any $q \in Q$, $q = t(e(t(e(\cdots t(e(t(e(q))))))))$. That is, the set of edges forms a directed graph.

### 3.2 Executions, Existence and Uniqueness, and Stability

We now consider the notion of a solution to a hybrid system. The classical definitions for solutions of dynamical systems are insufficient. A generalization of the idea of a solution is required. To address this issue, we use the idea of the execution of hybrid systems, described below.

**Definition 13. (Hybrid System Execution)** The tuple

$$\chi = (I, T, p, C)$$

where

- $I \subseteq \mathbb{N}$ is index of intervals.

- $T = \{T_i\}_{i \in I}$ are a set of open time intervals associated with points in time $\tau_i$ as $T_i = (\tau_i, \tau_{i+1}) \subset \mathbb{R}^{n+}$ where $T_{i+1} = (\tau_{i+1}, \tau_{i+2})$.
- $p : I \to Q$ maps each interval to a domain.
- $C = \{c_i(t)\}_{i \in I}$ is a set of continuously differentiable

is an execution of a hybrid system $H = (Q, E, D, F, G, R)$ starting from $(q_0, x_0) \in Q \times \cup_{q \in Q} D_q$ if

1. $c_{p(1)}(0) = x_0$ and $p(1) = q_0$
2. $\dot{c}_i(t) = f_{p(i)}(c_i(t))$ for $t \in T_i$ and for all $i \in I$.
3. $c_i(t) \in D_{p(i)}$ for $t \in T_i$ and for all $i \in I$.
4. $c_i(\tau_{i+1}) \in G_{(p(i),p(i+1))}$ for all $i \in I$.
5. $c_{i+1}(\tau_{t+1}) = \phi_{(p(i),p(i+1))}(c_i(\tau_i))$ for all $i \in I$.

Executions of hybrid systems are analogous to solutions of classical dynamical systems. However, since a classical solution map may not correctly model jumps caused by discrete maps.

We now introduce some concepts necessary for understanding existence and uniqueness of hybrid systems.

**Definition 14. (Infinite Execution)** An execution is called *infinite* if $I \equiv \mathbb{N}$, and *finite* if $\max_{i \in I} i < \infty$

An infinite execution is one where infinite transitions occur (as opposed to finite executions, where only a finite number of transitions occur).

**Definition 15. (Maximal Executions)** Consider a hybrid system $H = (Q, E, D, F, G, R)$ with execution $\chi = (I, T, p, C)$. $\chi$ starting from $(q_0, x_0) \in Q \times \cup_{q \in Q} D_q$ is called a maximal execution of $H$ if there exists no execution $\chi' = (I', T', p', C')$ starting from $(q_0, x_0)$ such that $\Sigma_{i \in I} \tau_{i+1} - \tau_i < \Sigma_{i' \in I'} \tau_{i'+1} - \tau_i'$.

**Definition 16. (Reachable State)** Consider a hybrid system $H = (Q, E, D, F, G, R)$. The pair $(q^*, x^*) \in Q \times \mathbb{R}^n$ is a reachable state of $H$ if, for some execution $\chi = (I, T, p, C)$ of $H$, starting from some initial condition $(q_0, x_0) \in Q \times \cup_{q \in Q} D_Q$, there exists an interval $(t_{f-1}, t_f) \in T$ such that $c_{p(f)}(t_f) = x^*$ and $p(f) = q^*$.

We use $\text{Reach}_H$ to denote the set of all states reachable by a hybrid system $H$.

Next, we define the set of all points from which no further continuous evolution can occur for any given $q \in Q$.

**Definition 17. (Out(H))** Consider a hybrid system $H = (Q, E, D, F, G, R)$, with executions $\chi \in \Psi_H(q_0, x_0)$, where $\chi = (I, T, p, C)$. The set of all reachable $c_q(t) \in D_q$ from which no continuous evolution can occur, denoted by $\{\text{Out}_q\}_{q \in Q}$, is given by

$$\text{Out}_q := \{c_q(t) \in D_q : \forall\, \varepsilon > 0,\ \exists \tau \in [0, \varepsilon) \text{ such that } \dot{c}_q(t + \tau) \notin D_q\}$$

Note that for switched systems (defined in the previous section), we consider $\text{Out}(q) \subset \partial D_q$, since for switched systems, $G_{q,q'} \subseteq \partial D_q \cap \partial D_{q'}$.

**Definition 18. (Non-blocking Hybrid System)** A hybrid system $H = (Q, E, D, F, G, R)$ is said to be non-blocking if an infinite execution exists for each initial condition $(q_0, x_0) \in Q \times \cup_{q \in Q} D_q$.

**Lemma 1.** *We say a hybrid system $H = (Q, E, D, F, G, R)$ is non-blocking if for any reachable state $(q, x) \in \text{Out}(q)$, there exists a $(q, q') \in E$ such that*

*1.* $x \in G_{q,q'}$

*2.* $R_{q,q'}(x) \neq \emptyset$

Refer to [48] for a proof.

**Definition 19. (Deterministic Hybrid System)** A hybrid system is said to be deterministic if for each initial condition $(q_0, x_0) \in Q \times \cup_{q \in Q} D_q$, only one maximal execution exists.

We now present a theorem for the existence and uniqueness of hybrid system executions:

**Theorem 4.** *Consider a hybrid system* $H = (Q, E, D, F, G, R)$*. For any initial condition* $(q_0, x_0) \in Q \times \mathbb{R}^n$*, there exists a unique infinite execution* $\chi = (I, T, p, C)$ *if* $H$ *is non-blocking and deterministic.*

For a proof, refer to [48].

### 3.3 Stability of Hybrid Systems

As with continuous dynamical systems, determining stability of hybrid systems is a leading area. For the most part, we are able to extend basic concepts of continuous dynamical systems, such as stability to hybrid systems. In the last 30 years, much work has gone into determining means to prove stability of hybrid systems. Of particular importance is the extension of Lyapunov theorems. First, Branicky showed in [14] that the existence of a Lyapunov function that is globally nonnegative and monotonically decreasing along system trajectories, and non-increasing at discrete events, was sufficient to show stability of a hybrid system.This result was then extended to the use of multiple Lyapunov functions for hybrid systems, such as in [49]. The use of multiple Lyapunov functions for stability analysis of hybrid systems can be summarized in Theorem 5 given below:

**Theorem 5.** *Consider a non-blocking and deterministic hybrid system $H = (Q, E, D, F, G, R)$, with executions $\chi = (I, T, p, C)$. Let $x^* \in \cup_{q \in Q} D_q$ be an equilibrium of the system $H$. If for each $q \in Q$, there exists a function $V_q : D_q \to \mathbb{R}$ that satisfies*

$$V_q(c_q(t)) > 0 \quad \textit{for all } c_q(t) \in D_q \backslash x^*, q \in Q \tag{3.1}$$

$$V_q(x^*) = 0, \quad \textit{for some } q \in Q \tag{3.2}$$

$$\langle \boldsymbol{\nabla} V_q, (c_q(t)) f_q(c_q(t)) \rangle \leq 0 \quad \textit{for all } c_q(t) \in D_q, \ , q \in Q \tag{3.3}$$

$$V_q(c_q(t)) \geq V_{q'}(\phi_e(c_q(t))) \textit{for all } e = (q, q') \in E \textit{ and } c_q(t) \in G_e \tag{3.4}$$

*then $x^*$ is an asymptotically stable equilibrium of $H$.*

Note that the Lyapunov function is not necessarily continuous. However, it still satisfies the conditions given in [13] and [14]. That is, the Lyapunov function is decreasing everywhere, including over transitions.

As noted previously, Lyapunov functions provide a useful deductive means to verify stability of dynamical systems. Computational techniques for the computation of multiple Lyapunov functions have been developed, such as in [15]. More recently, techniques to construct multiple Lyapunov functions for hybrid systems using sum-of-squares techniques were provided in [17] and [16]. Much of the work presented in the aforementioned papers has been the inspiration for the current work.

## 3.4 Zeno Executions and Equilibria

We now present definitions of Zeno executions, equilibria, and stability.

**3.4.1 Zeno Executions.** In this section, we describe Zeno executions.

**Definition 20. (Zeno Execution)** We say an execution $\chi = (I, T, p, C)$ of a hybrid System $\mathbf{H} = (Q, E, D, F, G, R)$ is Zeno if

1. $I = \mathbb{N}$
2. $\lim_{i \to \infty} \tau_i < \infty$

We see from the above that Zeno executions undergo infinite transitions in finite time.

**3.4.2 Zeno Stability.** In this subsection, we describe Zeno equilibria and stability. Unlike the equilibria of continuous dynamical systems, or, for that matter, hybrid systems without Zeno executions, Zeno equilibria are similar to limit cycles. We also frame Zeno behavior as a form of finite-time asymptotic stability.

**Definition 21. (Zeno Equilibrium)** A set $z = \{z_q\}_{q \in Q}$ is a Zeno equilibrium of a Hybrid System $\mathbf{H} = (Q, E, D, F, G, R)$ if it satisfies

1. For each edge $e = (q, q') \in E$, $z_q \in G_e$ and $\phi_e(z_q) = z_{q'}$.
2. $f_q(z_q) \neq 0$ for all $q \in Q$.

Note that for any $z \in \{z_q\}_{q \in Q}$, where $\{z_q\}_{q \in Q}$ is a Zeno equilibrium of a cyclic hybrid system $\mathbf{H}_c$,

$$(\phi_{i-1} \circ \cdots \circ \phi_0 \cdots \phi_i)(z) = z$$

We now define isolated Zeno equilibria.

**Definition 22. (Isolated Zeno Equilibrium)** A set $\{z_q\}_{q \in Q}$ is called an isolated Zeno equilibrium if there exists a collection of neighborhoods $\{W_q \subset D_q\}_{q \in Q}$, with $z_q \in W_q$, and such that for any $q, q' \in Q$, $z_{q'} \notin W_q$ unless $z_q = z_{q'}$.

Next, we define Zeno stability:

**Definition 23. (Zeno Stability)** Let $\mathbf{H} = (Q, E, D, F, G, R)$ be a hybrid system, and let $z = \{z_q\}_{q \in Q}$ be a compact set. The set $z$ is Zeno stable if, for each $q \in Q$, there exist neighborhoods $Z_q$, where $z_q \in Z_q$, such that for any initial condition $x_0 \in \bigcup_{q \in Q} Z_q$, the execution $\chi = (I, T, p, C)$, with $c_o(t_0) = x_0$ is Zeno, and converges to $z$.

Note that this definition of Zeno stability is equivalent to both the definition of "bounded time asymptotic Zeno stability" given in [25] and the definition of "Zeno asymptotic stability" given in [24]. This definition is applicable to the analysis of isolated Zeno equilibria (as noted in [23]) or compact sets [24]. A study of non-isolated Zeno equilibria is given in [25].

# CHAPTER 4

# ZENO STABILITY IN HYBRID SYSTEMS

In this chapter, we describe the Zeno phenomenon, which, as mentioned previously, occurs only in hybrid systems. In the previous chapter, we described Zeno executions of hybrid systems, and since Zeno executions asymptotically converge to Zeno equilibria, we are able to frame the phenomenon as a form of asymptotic stability. In this chapter, we present Lyapunov characterizations of Zeno stability (based on the work in [23] and [24]). Last, we present a method to algorithmically construct Lyapunov functions to prove Zeno stability utilizing sum-of-squares programming. Moreover, we extend the result to the verification of Zeno stability for systems with parametric uncertainties, where the uncertainties lie in semialgebraic sets.

## 4.1 A Lyapunov Characterization of Zeno Stability

The motivation for a Lyapunov methods for the analysis Zeno stability is similar to that of Lyapunov methods for "classical" stability analysis; that is, it is often difficult to verify stability of dynamical systems by directly solving the associated differential equations. Similarly, to ascertain Zeno stability using this method not only requires the solution of a hybrid system, but also a verification that the Zeno equilibrium is asymptotically approached in finite time. And while it is not difficult to verify Zeno stability for elementary systems such as the bouncing ball, even slightly more complex systems provide significant challenges in this regard. As such, developing Lyapunov methods to verify Zeno stability becomes a valuable undertaking.

First, consider Assumption 1 below:

*Assumption 1:*

In this work, we consider hybrid systems with polynomial vector fields and resets, and semialgebraic domains and guard sets. We implicitly assume every hybrid system is of this form and that associated with every hybrid system is a set of polynomials $g_{qi}$, $h_{e,k}$ for $q \in Q$, $e \in E$, $i = k = 1, \cdots, K_q$ and $k = 1, \cdots, N_q$ for some $K_q, N_q > 0$. In this framework, the domains of the hybrid system $\mathbf{H}$ are defined as

$$D_q = \{x \in \mathbb{R}^n \ : \ g_{qk}(x) \geq 0, \ k = 1, 2, \cdots, K_q\} \tag{4.1}$$

where $g_{qk} \in \mathbf{R}[x]$, and $K_q \in \mathbb{N}$. Similarly, the guard sets are defined as

$$G_e = \{x \in \mathbb{R}^n \ : \ h_{e,0}(x) = 0, \ h_{e,k}(x) \geq 0, \ k = 1, 2, \cdots, N_q\} \tag{4.2}$$

where each $h_{ek} \in \mathbf{R}[x]$, and $N_q \in \mathbb{N}$. Lastly, for each $e = (q, q') \in E$, the reset map $\phi_e$ is given by the vector-valued polynomial function

$$\phi_e = [\phi_{e,1}, \cdots, \phi_{e,n}]^T \tag{4.3}$$

where $\phi_{e,j} \in \mathbf{R}[x]$ for $j = 1, \cdots, n$.

We now reiterate the Lyapunov conditions for the stability of Zeno equilibria in cyclic hybrid systems presented in [23], which are as follows:

**Theorem 6. (Lamperski and Ames)** *Consider a hybrid system* $\mathbf{H} = (Q, E, D, F, G, R)$, *with an isolated Zeno equilibrium* $\{z_q\}_{q \in Q}$. *Let* $\{W_q\}_{q \in Q}$ *be a collection of open neighborhoods of* $\{z_q\}_{q \in Q}$. *Suppose there exist continuously differentiable functions* $V_q : \mathbb{R}^n \to \mathbb{R}$ *and* $B_q : \mathbb{R}^n \to \mathbb{R}$, *and non-negative constants* $\{r_q\}_{q \in Q}$, $\gamma_a$, *and* $\gamma_b$, *where* $r_q \in [0,1]$*, and* $r_q < 1$ *for some* $q$ *and such that*

$$V_q(x) > 0 \quad \textit{for all } x \in W_q \backslash z_q, q \in Q \tag{4.4}$$

$$V_q(z_q) = 0, \quad \textit{for all } q \in Q \tag{4.5}$$

$$\boldsymbol{\nabla} V_q^T(x) f_q(x) \leq 0 \quad \textit{for all } x \in W_q,\ q \in Q \tag{4.6}$$

$$B_q(x) \geq 0 \quad \textit{for all } x \in W_q,\ q \in Q \tag{4.7}$$

$$\boldsymbol{\nabla} B_q^T(x) f_q(x) < 0 \quad \textit{for all } x \in W_q,\ q \in Q \tag{4.8}$$

$$V_{q'}(R_{(q,q')}(x)) \leq r_q V_q(x), \tag{4.9}$$

$$\textit{for all } e = (q, q') \in E \textit{ and } x \in G_e \cap W_q$$

$$B_q(R_{(q',q)}(x)) \leq \gamma_b \left(V_q(R_{(q,q')}(x))\right)^{\gamma_a} \tag{4.10}$$

$$\textit{for all } e = (q, q') \in E \textit{ and } x \in G_e \cap W_q.$$

*Then* $\{z_q\}_{q \in Q}$ *is Zeno stable.*

Theorem 6 is used to prove stability of isolated Zeno equilibria. In [24, Propositions 5.1,5.2], sufficient conditions for Zeno stability of compact sets were presented. Moreover, the conditions of [24, Proposition 5.2] which prove Zeno stability of a compact set, were shown to be equivalent to those in [23]. As noted in [24], satisfying 4.4-4.10 guarantees asymptotic Zeno stability of a collection of compact sets. To simplify notation, we will use the sufficient conditions of the previous theorem as follows in Theorem 7. Note that our subsequent theorem can be easily applied directly to the conditions of Theorem 6; and in our numerical examples we have tested both sets of conditions and they yield similar results.

**Theorem 7.** *Let* $\mathbf{H} = (Q, E, D, F, G, R)$ *be a cyclic hybrid system, and let* $z = \{z_q\}_{q \in Q}$ *be a collection of compact sets. Let* $\{W_q \subset D_q\}_{q \in Q}$*, be a collection of neighborhoods of the* $\{z_q\}_{q \in Q}$*. Suppose that there exist continuously differentiable functions* $V_q : W_q \to \mathbb{R}$*, and positive constants* $\{r_q\}_{q \in Q}$ *and* $\gamma$*, where* $r_q \in (0,1]$*, and* $r_q < 1$ *for* *some $q$ and such that*

$$V_q(x) > 0 \quad \textit{for all } x \in W_q \backslash z_q, q \in Q \tag{4.11}$$

$$V_q(z_q) = 0, \quad \textit{for all } q \in Q \tag{4.12}$$

$$\nabla V_q^T(x) f_q(x) \leq -\gamma \quad \textit{for all } x \in W_q,\ q \in Q \tag{4.13}$$

$$r_q V_q(x) \geq V_{q'}(\phi_e(x)) \tag{4.14}$$

$$\textit{for all } e = (q, q') \in E \textit{ and } x \in G_e \cap W_q.$$

*then $z$ is a stable Zeno equilibrium.*

**Proof:**

We show that if for each $q \in Q$, we can find a $V_q$ such that (4.11)-(4.14) are satisfied, then the same $V_q$ also satisfies (4.4)-(4.10). From inspection, it is clear that if $V_q$ satisfies (4.11)-(4.14), then (4.4)-(4.6) and (4.9) are satisfied. Second, choose $B_q = V_q$ for each $q \in Q$. From inspection, it is clear that $V_q$ also satisfies (4.7) and (4.8). Last, if $\gamma_a = \gamma_b = 1$, we get $V_q \leq V_q$, where the equality holds. From this, we see that for each $q \in Q$, $V_q$ also satisfies (4.10). Thus, the theorem is proved. □

### 4.2 Using Sum-of-Squares Programming to Verify Zeno Stability

While using Lyapunov functions to verify Zeno stability is certainly simpler than directly solving the associated differential equations, it may still be difficult to find a Lyapunov function that satisfies the conditions listed in the theorems above. This necessitates an algorithmic method for the construction of Lyapunov functions that satisfy Theorem 7. To this end, we use SOS programming. SOS programming allows us to search for polynomial Lyapunov functions using semidefinite programming, whose associated problems can be solved in finite time.

Theorem 8 provides sufficient conditions for Zeno stability in cyclic hybrid systems. We now show that these conditions can be enforced using SOS.

Let $\mathbf{H} = (Q, E, D, F, G, R)$ be a hybrid system, and let $\{z_q\}_{q\in Q}$ be a collection of compact sets. Let $\{W_q\}_{q\in Q}$ be a collection of neighborhoods of $\{z_q\}_{q\in Q}$. Moreover, suppose that each $W_q$ is a semialgebraic set defined as

$$W_q := \{x \in \mathbb{R}^n : w_{qk}(x) > 0, k = 1, 2, ..., K_{qw}\}$$

where $w_{qk} \in \mathbf{R}[x]$.

We define feasibility problem 1:

**Feasibility Problem 1:**

For hybrid system $\mathbf{H} = (Q, E, D, F, G, R)$, find

- $a_{qk}$, $c_{qk}$, $i_{qk}$, $\in \Sigma_x$, for $k = 1, 2, ..., K_{qw}$ and $q \in Q$;
- $b_{qk}$, $d_{qk}$, $j_{qk} \in \Sigma_x$, for $k = 1, 2, ..., K_q$ and $q \in Q$.
- $m_{e,l} \in \Sigma_x$ for $e \in E$ and $l = 1, 2, ..., N_q$
- $V_q$, $m_{e,0} \in \mathbf{R}[x]$ for $e \in E$ and $q \in Q$.
- Constants $\alpha, \gamma > 0$, $\{r_q\}_{q\in Q} \in (0, 1]$ such that $r_q < 1$ for some $q \in Q$.

such that

$$V_q - \alpha x^T x - \sum_{k=1}^{K_{qw}} a_{qk} w_{qk} - \sum_{k=1}^{K_q} b_{qk} g_{qk} \in \Sigma_x \quad \text{for all } q \in Q \tag{4.15}$$

$$V_q(z_q) = 0 \qquad \text{for all } q \in Q \tag{4.16}$$

$$-\boldsymbol{\nabla} V_q^T f_q - \gamma - \sum_{k=1}^{K_{qw}} c_{qk} w_{qk} - \sum_{k=1}^{K_q} d_{qk} g_{qk} \in \Sigma_x \qquad \text{for all } q \in Q \tag{4.17}$$

$$r_q V_q - V_{q'}(\phi_e) - m_{e,0} h_{e,0} - \sum_{l=1}^{N_q} m_{e,l} h_{e,l}$$
$$-\sum_{k=1}^{K_{qw}} i_{qk} w_{qk} - \sum_{k=1}^{K_q} j_{qk} g_{qk} \in \Sigma_x \qquad \text{for all } e = (q, q') \in E \tag{4.18}$$

**Theorem 8.** *Consider a hybrid system* $\mathbf{H} = (Q, E, D, F, G, R)$ *and let* $z = \{z_q\}_{q \in Q}$ *a compact set. If Feasibility Problem 1 has a solution, then* $z$ *is Zeno stable.*

**Proof:**

To prove the theorem we show that if $V_q$, $q \in Q$ are elements of a solution of Feasibility Problem 1, then for each $q \in Q$, the same $V_q$ also satisfy (4.4)-(4.9) of Theorem 4. That is, we show that if the $V_q$ satisfy (4.15)-(4.18), then the same $V_q$ also satisfies (4.11)-(4.14).

First, we observe that (4.16) directly implies (4.12). Next, from (4.15), we know that

$$V_q(x) \geq \sum_{k=1}^{K_{qw}} a_{qk}(x) w_{qk}(x) + \sum_{k=1}^{K_q} b_{qk}(x) g_{qk}(x) + \alpha x^T x$$

Since $a_{qk}(x)$ and $b_{qk}(x)$ are SOS, and thus, always nonnegative, by the Positivstellensatz and the definitions of $W_q$ and $D_q$, we have that $V_q(x) \geq \alpha x^T x$ for all $x \in W_q \subset D_q$. Thus, (4.15) implies (4.11) is satisfied. Similarly, from (4.17),

$$-\boldsymbol{\nabla} V_q^T(x) f_q(x) - \gamma \geq \sum_{k=1}^{K_{qw}} c_{qk}(x) w_{qk}(x) + \sum_{k=1}^{K_q} d_{qk}(x) g_{qk}(x).$$

Since $c_{qk}(x)$ and $d_{qk}(x)$ are always nonnegative, by the definition of $D_q$ and $W_q$, $\nabla V_q(x)^T f_q(x) \leq -\gamma$ for $x \in \{x \in \mathbb{R}^n : g_{qk}(x) \geq 0,\ w_{qk}(x) \geq 0\} = D_q \cap W_q$ which implies (4.13) is satisfied. Next, from (4.18) we have that for all $e = (q, q') \in Q$,

$$r_q V_q(x) - V_{q'}(\phi_e(x)) \geq m_{e,0}(x) h_{e,0}(x) + \sum_{l=1}^{N_q} m_{e,l}(x) h_{e,l}(x)$$
$$+ \sum_{k=1}^{K_q} i_{qk}(x) w_{qk}(x) + \sum_{k=1}^{K_q} j_{qk}(x) g_{qk}(x).$$

First note that $h_{e,0}(x) = 0$ and hence $m_{e,0}(x) h_{e,0}(x) = 0$ on $G_e$. Since $m_{e,l} \in \Sigma_x$, we have $m_{e,l}(x) h_{e,l}(x) \geq 0$ on $G_e$. Similarly $j_{qk}(x) g_{qk}(x) \geq 0$ on $D_q$ and $i_{qk}(x) w_{qk}(x) \geq 0$ on $W_q$. It follows that $r_q V_q(x) - V_{q'}(\phi_e(x)) \geq 0$ when $x \in G_e \cap W_q \cap D_q$ for all $e = (q, q') \in E$. Thus, we have shown that (4.18) implies (4.14).

Thus we conclude that the solution elements $V_q$ of Feasibility Problem 1 satisfy the conditions (4.11)-(4.14) of Theorem 4. Thus by Theorem 4 we conclude Zeno stability of $z$. □

### 4.2.1 Numerical Examples.

**The Bouncing Ball:**

We first consider a very simple model of the bouncing ball:

**Example 10.**

A bouncing ball $\mathbf{B}$ is modeled by a tuple:

$$\mathbf{B} = (Q, E, D, F, G, R)$$

where

- $Q = \{q_0\}$
- $E = \{(q_0, q_0)\}$

- $D := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$
- $F = \{f\}$, where

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ -g \end{pmatrix}$$

- $G := \{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \leq 0\}$
- $R = \phi(x) = [0, -cx_2]^T$. Here, $c$ is a coefficient of restitution.

**Results:** The Zeno equilibrium is $z = (0,0)^T$. We consider stability on the unit ball

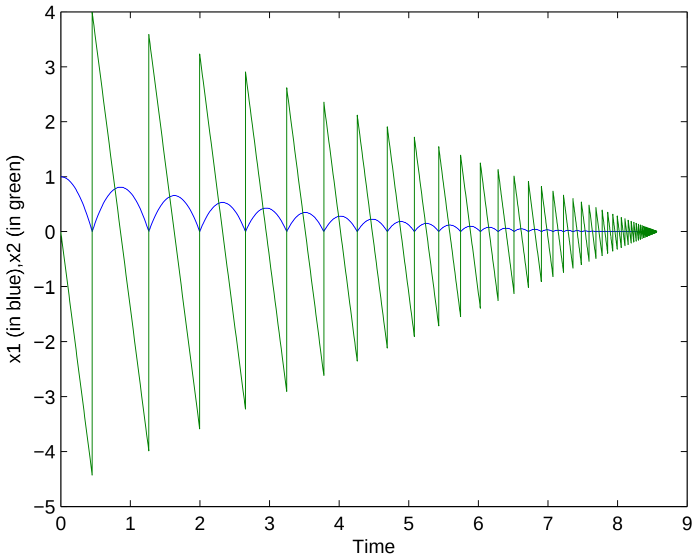


Figure 4.1. Bouncing Ball with $c = 0.9$

$W_q := \{x \in \mathbb{R}^n : x_1 \geq 0, 1 - x_1^2 - x_2^2 < 0\}$ for feasibility problem 1.

Using SOSTOOLS to implement the conditions given in Feasibility Problem 1, for values of $c \in [0, .999]$ we were able to find a 4th–order $V(x)$ which verifies stability of Zeno executions using SOS polynomial multipliers of second order.

**Water Tank Problem:**

A classic example of a hybrid system exhibiting Zeno behavior is the dual water tank, as described in [50]. The system consists of two water tanks sharing a single supply pipe, which pumps water at the constant rate $W$. Moreover, each tank leaks at a constant rate ($v_1$ and $v_2$). The supply pipe switches to tank $i$ when the water level falls below a prescribed level $r_i$.

**Example 11.**

The two-tank system can be modeled by a hybrid system $\mathbf{T}$, which is the tuple

$$\mathbf{T} = (Q, E, D, F, G, R)$$

where

- $Q = q_1, q_2$
- $E = \{(q_1, q_2), (q_2, q_1)\}$
- $D = \{D_{q_1}, D_{q_2}\}$ , where

  $$D_{q_1} := \{x \in \mathbb{R}^2 : x_2 - r_2 \geq 0, x_1 \geq 0\}$$

  and

  $$D_{q_2} := \{x \in \mathbb{R}^2 : x_1 - r_1 \geq 0, x_2 \geq 0\}$$

- $F = \{f_1, f_2\}$, where

  $$f_1 = \begin{pmatrix} W - v_1 \\ \\ -v_2 \end{pmatrix}$$

and

$$f_2 = \begin{pmatrix} -v_1 \\ \\ W - v_2 \end{pmatrix}$$

- $G = \{G_{q_1 q_2}, G_{q_2 q_1}\}$, where

$$G_{q_1 q_2} := \{x \in \mathbb{R}^2 : r_2 - x_2 \geq 0\}$$

and

$$G_{q_2 q_1} := \{x \in \mathbb{R}^2 : r_1 - x_1 \geq 0\}$$

- $R = \{R_{q_1 q_2}, R_{q_2 q_1}\}$, where

$$R_{q_1 q_2} = R_{q_2 q_1} = x$$

**Results:**

The Zeno equilibrium is $z = [r_1, r_2]^T$. For solving Feasibility problem 1, we again consider the unit ball in both domains.

$$W_q := \{x \in \mathbb{R}^2 : ||x - z||^2 \leq 1\}$$

We then obtain fourth order $V_1(x)$ and $V_2(x)$ by solving Feasibility Problem 1. Exploring values of the parameter space, we find our algorithm is able to prove stability when $v_1 + v_2 < W$.

**System with nonlinear resets and vector field:**

We now consider a more difficult model similar to that of the bouncing ball, but with a nonlinear vector field and a nonlinear reset.

**Example 12.**

The nonlinear hybrid system can be represented by **N**, which is the tuple:

$$\mathbf{N} = (Q, E, D, F, G, R)$$

where

- $Q = \{q_0\}$
- $E = \{(q_0, q_0)\}$
- $D := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$
- $G := \{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \leq 0\}$
- $F = \{f\}$, where

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ \\ -g + c_1 x_2^2 \end{pmatrix}$$

- $R = \phi(x) = [0, -c_2 x_2(1 - c_3 x_2^2)]^T$. Here, $c_1$, $c_2$, and $c_3$ are positive constants satisfying $c_i < 1$.

**Results:**

The Zeno equilibrium is $z = (0, 0)$. We searched for a 4th-order $V(x)$ and multipliers that solves Feasibility Problem 1 using SOSTOOLS. While the range of Zeno-stable parameters was complicated, we were able to show Zeno-stability on the unit ball for a range of values.

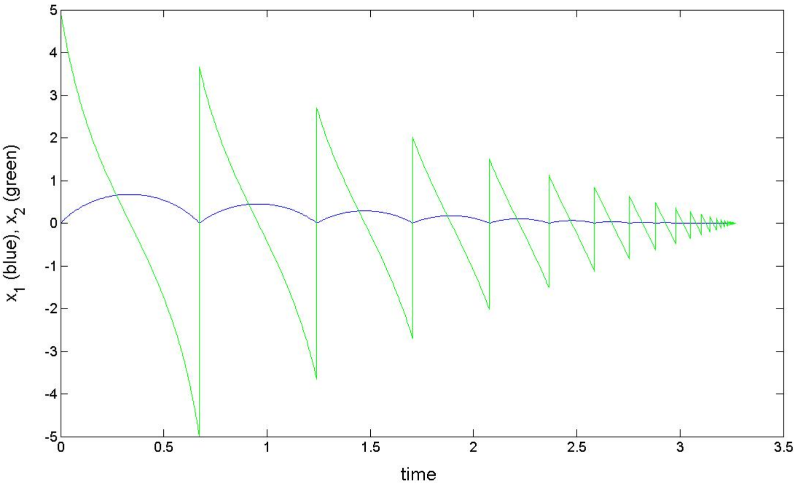


Figure 4.2. Nonlinear Hybrid System with $c_1 = 0.5$, $c_2 = 0.8$, $c_3 = 0.001$

We fix each $c_i$ at certain values, and plot the other constants. In Figure 4.3, we set $c_1 = 0.99, 0.50$, and $0.001$, and plot corresponding values of $c_1$ and $c_2$ such that **N** was stable.

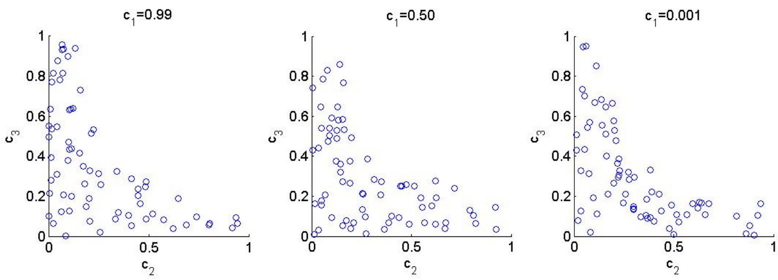


Figure 4.3. Values of $c_2$ and $c_3$ such that **N** for fixed $c_1$

We note from Figure 4.3 that the range of values of $c_1$ and $c_2$ for which **N** is stable does not depend on $c_1$. In Figure 4.4, we set $c_2$ to some constant values, and plot values of $c_1$ and $c_3$ such that **N** is stable.

Last, in Figure 4.5, we set $c_3$ equal to some constant values, and plot values of $c_1$ and $c_2$ such that **N** is stable.

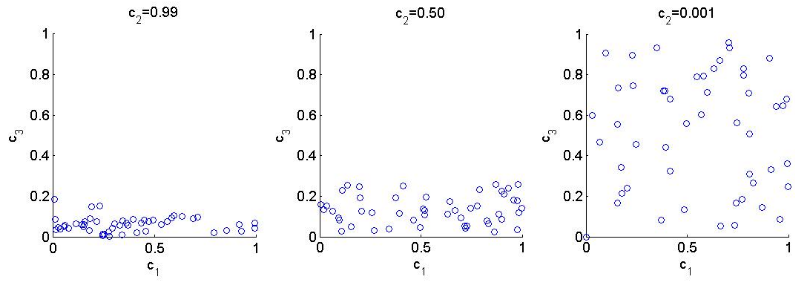


Figure 4.4. Values of $c_1$ and $c_3$ for fixed $c_2$

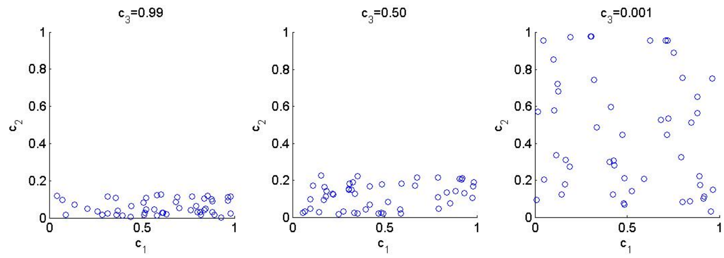


Figure 4.5. Values of $c_1$ and $c_2$ for fixed $c_3$

## 4.3 Zeno Stability for Systems with Parametric Uncertainty

We now present a method to verify Zeno stability in cyclic hybrid systems with time-invariant uncertainties in the guards, vector fields, and resets. We define the vector of parametric uncertainties $P$ to lie within a semialgebraic set

$$P := \{p \in \mathbb{R} : \tilde{p}_k(p) \geq 0, k = 1, 2, ..., K_1\}. \tag{4.19}$$

We assume implicitly that all hybrid systems in this section satisfy Assumption 2, given below.

*Assumption 2*

Let $P$ be defined as in (4.19). Associated with every hybrid system is a set of polynomials $g_{qi}(x,p)$, $h_{e,k}(x,p)$ for $q \in Q$, $e \in E$, $i = k = 1, \cdots, K_q$ and $k = 1, \cdots, N_q$ for some $K_q, N_q > 0$, and $p \in P$.

In this framework, the domains of the hybrid system $H$ are defined as

$$D_q = \{x \in \mathbb{R}^n \ : \ g_{qk}(x,p) \geq 0, \ k = 1, 2, \cdots, K_q\} \tag{4.20}$$

where $g_{qk} \in \mathbf{R}[x,p]$, $K_q \in \mathbb{N}$, and $p \in P$. The guard sets are defined as

$$G_e = \{x \in \mathbb{R}^n \ : \ h_{e,0}(x,p) = 0, \ h_{e,k}(x,p) \geq 0, \ k = 1, 2, \cdots, N_q\} \tag{4.21}$$

where each $h_{ek} \in \mathbf{R}[x,p]$, $N_q \in \mathbb{N}$, and $p \in P$. Lastly, for each $e = (q, q') \in E$, the reset map $\phi_e$ is given by the vector-valued polynomial function

$$\phi_e = [\phi_{e,1}(x,p), \cdots, \phi_{e,n}(x,p)]^T \tag{4.22}$$

where $\phi_{e,j} \in \mathbf{R}[x,p]$ for $j = 1, \cdots, n$, and $p \in P$.

Let $\mathbf{H} = (Q, E, D, F, G, R)$ be a hybrid system, and let $\{z_q\}_{q \in Q}$ be a collection of compact sets. Let $\{W_q\}_{q \in Q}$ be a collection of neighborhoods of $\{z_q\}_{q \in Q}$. We consider $W_q$ of the form

$$W_q := \{x \in \mathbb{R}^n : w_{qk}(x) > 0, k = 1, 2, ..., K_q\}$$

where each $w_{qk}(x) \in \mathbf{R}[x]$.

Consider feasibility problem 2:

**Feasibility Problem 2:**

For hybrid system $\mathbf{H} = (Q, E, D, F, G, R)$, find

- $a_{qk}$, $c_{qk}$, $i_{qk}$, $\in \Sigma_{x,p}$, for $k = 1, 2, ..., K_{qw}$ and $q \in Q$;
- $b_{qk}$, $d_{qk}$, $j_{qk} \in \Sigma_{x,p}$, for $k = 1, 2, ..., K_q$ and $q \in Q$.
- $\eta_{qk}$, $\beta_{qk}$, $\zeta_{qk} \in \Sigma_{x,p}$, for $k = 1, 2, ..., K_1$ and $q \in Q$.
- $m_{e,l} \in \Sigma_{x,p}$ for $e \in E$ and $l = 1, 2, ..., N_q$

- $V_q$, $m_{e,0} \in \mathbf{R}[x,p]$ for $e \in E$ and $q \in Q$.

- Constants $\alpha, \gamma > 0$, $\{r_q\}_{q \in Q} \in (0,1]$ such that $r_q < 1$ for some $q \in Q$.

such that

$$V_q - \alpha x^T x - \sum_{k=1}^{K_{qw}} a_{qk} w_{qk} - \sum_{k=1}^{K_q} b_{qk} g_{qk} - \sum_{k_1=1}^{K_1} \eta_{qk_1} \tilde{p}_{qk} \in \Sigma_{x,p} \quad \text{for all } q \in Q \tag{4.23}$$

$$V_q(z_q, p) = 0 \quad \text{for all } q \in Q \tag{4.24}$$

$$-\nabla V_q^T f_q - \gamma - \sum_{k=1}^{K_{qw}} c_{qk} w_{qk} - \sum_{k=1}^{K_q} d_{qk} g_{qk} - \sum_{k_1=1}^{K_1} \beta_{qk_1} \tilde{p}_{qk} \in \Sigma_{x,p} \quad \text{for all } q \in Q \tag{4.25}$$

$$r_q V_q - V_{q'}(\phi_e) - m_{e,0} h_{e,0} - \sum_{l=1}^{N_q} m_{e,l} h_{e,l} - \sum_{k=1}^{K_{qw}} i_{qk} w_{qk} - \sum_{k=1}^{K_q} j_{qk} g_{qk} - \sum_{k=1}^{K_1} \zeta_{qk} \tilde{p}_{qk} \in \Sigma_{x,p} \quad \text{for all } e = (q,q') \in E. \tag{4.26}$$

**Theorem 9.** *Consider a hybrid system* $\mathbf{H} = (Q, E, D, F, G, R)$ *and let* $z = \{z_q\}_{q \in Q}$ *be a compact set. If there is a solution to Feasibility Problem 2, then* $z$ *is Zeno stable for all* $p \in P$.

**Proof:**

Suppose the problem is feasible. If $p$ is in $P$, $\tilde{p}_k(p) \geq 0$. Thus, by similar logic to that employed in the proof of Theorem 6, we can show that $V_q$ satisfies Conditions (4.11)-(4.14) for all $p \in \mathcal{P}$. By Theorem 4, this implies that the Zeno equilibrium is stable for all $p \in P$ □

**4.3.1 Illustrative Example: Bouncing Ball with Uncertainy.** We use a variant of the bouncing ball model to illustrate computational analysis of robust Zeno stability using SOS as described above. Here, the coefficient of restitution is a time-invariant uncertain parameter.

**Example 13.**

A bouncing ball with parametric uncertainties in the reset map can be described by $\mathbf{B}_p$ which is the tuple:

$$\mathbf{B}_p = (Q, E, D, F, G, R)$$

where

- $Q = \{q_0\}$, which provides the discrete state
- $E = \{(q_0, q_0)\}$, which is the single edge from $q_0$ to itself
- $D := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$ provides the domain. Thus, $g_{q_0} = x_1$.
- $G = \{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \leq 0\}$ provides the guard. Thus, $h_{(q_0,q_0),0} = x_1$, and $h_{(q_0,q_0),1} = -x_2$.
- $R = \phi(x) = [0, -px_2]^T$ provides the reset map.
- $F = f(x)$ provides a vector field mapping $D$ to itself, and where

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ -g \end{pmatrix}$$

We would like to prove stability of the Zeno equilibrium for $p \in (0, C)$ where $C \in [0, 1)$. We define $P = \{p \in \mathbb{R} : \tilde{p}(p) := p(p - C) \leq 0\}$ to describe the set of possible values of the uncertainty $p$.

**Results:**

From previous analysis, we see that the system exhibits Zeno behavior when $C < 1$. The Zeno equilibrium $z$ is (0,0). Thus we search for a parameter-dependent variables which establish this property. Specifically, we choose a maximum values of $C$ search for a 4th degree $V(x)$ along with SOS and polynomial multipliers. As a result, we were able to verify stability for $C = 0.99$, which agrees with the known analytical result to a high degree of accuracy.

# CHAPTER 5
# APPLICATIONS

In the previous chapter, we provided techniques to verify Zeno stability utilizing sum-of-squares programming. Moreover, we demonstrated the use of those techniques on simple examples of hybrid systems exhibiting Zeno stability. The true test of the techniques presented in the previous chapter is verifying Zeno stability in more complex systems. Often, Zeno behavior may occur in hybrid or variable structure systems with nonlinear vector fields and multiple domains. In this chapter, we investigate Zeno stability in three such hybrid systems.

## 5.1 Zeno Behavior in systems with nonlinear vector fields

The most obvious application of the techniques discussed in the previous chapter is Zeno behavior in hybrid systems with nonlinear vector fields. The utility of our technique in this regard is fairly obvious - not only is it difficult to solve the associated differential equations, it is difficult to ascertain whether or not executions of the hybrid systems either undergo infinite transitions, or converge to an equilibrium in finite time. Moreover, even with Lyapunov conditions to deductively prove Zeno stability, finding Lyapunov functions for nonlinear systems can be extremely difficult. We illustrate this usage with the example below:

**Example 14.**

Consider the hybrid system $\mathbf{H} = (Q, E, D, F, G, R)$, where

- $Q = \{q_1, q_2, q_3\}$
- $E = \{(q_1, q_2), (q_2, q_3), (q_3, q_1)\}$

- $D := \{D_1, D_2, D_3\}$ where

$$D_1 = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 + \frac{1}{2}x_1 \geq 0\}$$
$$D_2 = \{x \in \mathbb{R}^2 : x_2 - \frac{1}{2}x_1 \geq 0, x_2 + \frac{1}{2}x_1 < 0\}$$
$$D_3 = \{x \in \mathbb{R}^2 : x_1 < 0, x_2 + \frac{1}{2}x_1 \geq 0\}$$

- $G := \{G_{12}, G_{23}, G_{31}\}$ where

$$G_{12} := \left\{x \in \mathbb{R}^2 : x_2 \leq 0, \frac{1}{2}x_1 + x_2 = 0\right\}$$
$$G_{23} := \left\{x \in \mathbb{R}^2 : x_2 \leq 0, \frac{1}{2}x_1 - x_2 = 0\right\}$$
$$G_{31} := \left\{x \in \mathbb{R}^2 : x_2 > 0, x_1 = 0\right\}$$

- $F = \{f_1, f_2, f_3\}$, where

$$\dot{x} = f_1(x) = \begin{pmatrix} x_2 \\ \\ -5x_1^2 - x_2 \end{pmatrix}$$
$$\dot{x} = f_2(x) = \begin{pmatrix} -x_1^2 - 3 \\ \\ 2x_2^2 - \frac{1}{2}x_1^2 \end{pmatrix}$$
$$\dot{x} = f_3(x) = \begin{pmatrix} x_2^2 + x_2 \\ \\ -x_1 \end{pmatrix}$$

- $R = \{\phi_{12}(x), \phi_{23}(x), \phi_{31}(x)\}$ where each $\phi_{ij}(x) = x$.

We note that this hybrid system is cyclical, as the pair $(Q, E)$ forms a directed cycle, with vertices $Q$ and edges $E$. Clearly, discrete transitions only occur between

Provided below is a phase plane plot:

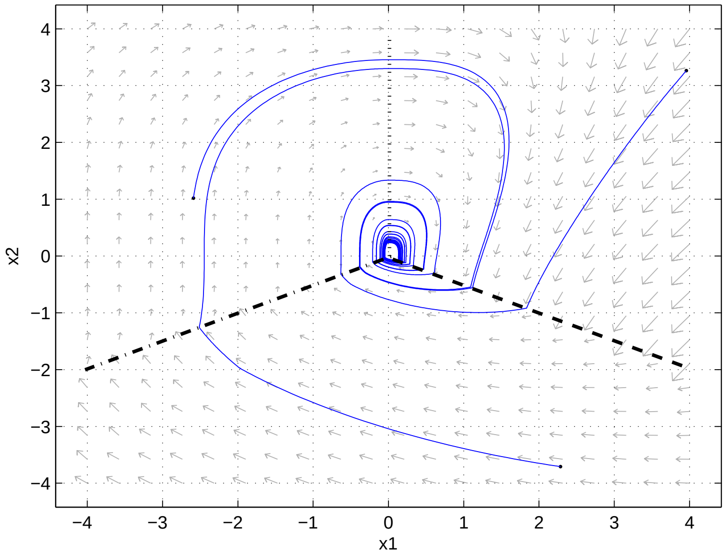


Figure 5.1. Hybrid System in Example 15. Dashed line indicates $G_{12}$, dash-dotted line indicates $G_{23}$ and dotted line indicates $G_{31}$

**Results:**

We wish to analyze Zeno stability for $z_1 = z_2 = z_3 = [0, 0]^T$. To solve Feasibility Problem 1, we consider each

$$W_q := \mathbb{B}^2 \cap D_q$$

where $\mathbb{B}^2 := \{x \in \mathbb{R}^2 : |x| \leq 1\}$.

We then search for 3 degree 8 polynomials to solve Feasibility Problem 1. Since we are able to solve Feasibility Problem 1 with such polynomials, we show via Theorem 8 that $x = [0, 0]^T$ is Zeno stable for $H$.

## 5.2 Sliding Modes and Filippov Solutions

Often, Zeno behavior can occur in systems utilizing variable structure controllers. It is well known that chattering arises in systems with sliding mode controllers [51]. In sliding mode control, a variable structure controller is used to stabilize the system, which is accomplished by forcing the system trajectories to "slide" along a predetermined manifold. This in turn is accomplished by choosing a controller which forces the vector field toward the sliding surface, which, under ideal conditions, causes that manifold to become invariant. For a more thorough treatment of the subject matter, refer to [46] or [51]. The occurrence of infinite transitions in discrete time also occurs with other variable structure controllers, such as the bang-bang optimal controllers studied by Fuller in [52].

Zeno behavior is often difficult to verify in systems with variable structure controllers, especially since many of the closed loop vector fields are nonlinear, and cannot be solved easily. Thus, our method can be utilized to demonstrate Zeno stability in variable structure control systems. In this section, we present an example of a system with a variable structure controller which exhibits Zeno behavior, and use the proposed technique to demonstrate Zeno stability.

**Example 15.**

In this example, we consider the utilization of a variable structure controller for the plant

$$\dot{x} = \begin{pmatrix} x_2 \\ x_1^2 + x_2^2 + u(x,t) \end{pmatrix}$$

With 0 input, the system is unstable. We wish system trajectories to converge to the equilibrium $(0,0)$ along the line $s(x) = x_1 + x_2 = 0$. Correspondingly, the controller $u(x,t)$ should be given by $u(x,t) = |u(x,t)|\text{sign}(s(x)) = u_0(x,t)\text{sign}(s(x))$. In this case, we can choose $u_0(x,t) = 2(x_1^2 + x_2^2)$. We can then construct the hybrid system described below. We consider the hybrid system $\mathbf{H} = (Q, E, D, F, G, R)$ where

- $Q = \{q_1, q_2\}$
- $E = \{(q_1, q_2), (q_2, q_1)$
- $D = \{D_1, D_2\}$ where

$$D_1 := \{x \in \mathbb{R}^2 : x_1 + x_2 \geq 0\}$$
$$D_2 := \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 0\}$$

- $F = \{f_1, f_2\}$ where

$$f_1 = \begin{pmatrix} x_2 \\ \\ 3(x_2^2 + x_1^2) \end{pmatrix}$$
$$f_2 = \begin{pmatrix} x_2 \\ \\ -(x_2^2 + x_1^2) \end{pmatrix}$$

- $G = \{G_{q_1q_2}, G_{q_2q_1}$ where

$$G_{q_1q_2} = G_{q_2q_1} := \{x \in \mathbb{R}^2 : x_1 + x_2 = 0\}$$

- $R = \{\phi_{12}(x), \phi_{21}(x)\}$ where each $\phi_{ij}(x) = x$.

A simulation of the closed loop system is shown below in Figure 5.2:

**Results:**

We wish to analyze the Zeno stability of $z = \{z_1, z_2\}$ is $z_1 = z_2 = (0, 0)$. It is clear that this hybrid system is cyclic, as the pair $(Q, E)$ forms a directed graph. For our analysis, we analyze Zeno stability in the set $W = W_1 \cup W_2 := \{x \in \mathbb{R}^2 : |x|< 1\}$. We are able to find degree 8 $V_1(x)$ and $V_2(x)$ to solve feasibility problem 1, which in turn demonstrates Zeno stability of $z$.

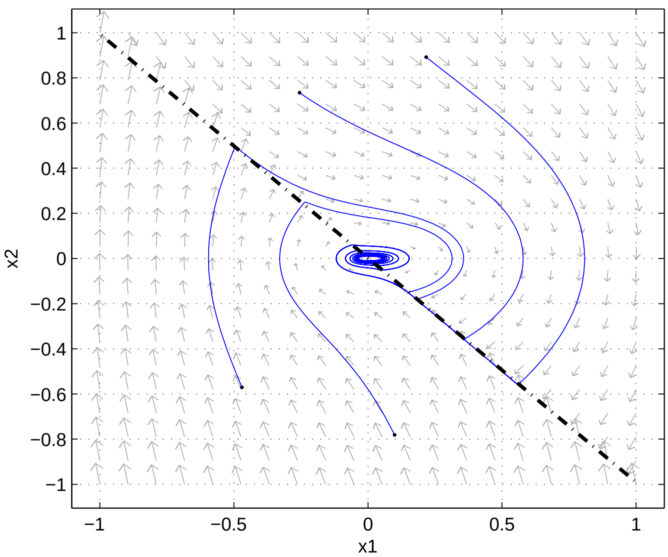


Figure 5.2. Closed loop system of Example 16. The dashed line indicates $s(x)$.

## 5.3 Uncertain Switching

There are often instances when perfect measurements and information regarding a system may not be available. For continuous time dynamical systems, this is usually due to incomplete information in the vector field. However, for hybrid systems, incomplete information on the state may translate into incomplete information regarding the transition rules. Here, we provide an example where exact information regarding the switching rule may be absent.

Note that all hybrid systems discussed in this section satisfy Assumption 2 of Chapter 4.

**Example 16.**

Consider the hybrid system $H = (Q, E, D, F, G, R)$ with uncertain parameter

$p \in (c_1, c_2)$ where

- $Q = \{q_1, q_2\}$
- $E = \{(q_1, q_2), (q_2, q_1)$
- $D = \{D_1, D_2\}$ where

$$D_1 := \{x \in \mathbb{R}^2 : x_1 + x_2 \geq 0, p * x_1 - x_2 \geq 0\}$$
$$D_2 := \{x \in \mathbb{R}^2 : -p * x_1 + x_2 \geq 0\} \cup \{x \in \mathbb{R}^2 : p * x_1 - x_2 \geq 0, -x_1 - x_2 \geq 0\}$$

- $F = \{f_1, f_2\}$ where

$$f_1 = \begin{pmatrix} -0.1 \\ \\ 2 \end{pmatrix}$$
$$f_2 = \begin{pmatrix} -x_2 - x_1^3 \\ \\ x_1 \end{pmatrix}$$

- $G = \{G_{q_1q_2}, G_{q_2q_1}$ where

$$G_{q_1q_2} = x2 - p * x1 = 0$$
$$G_{q_2q_1} := \{x \in \mathbb{R}^2 : x_1 + x_2 = 0\}$$

- $R = \{\phi_{12}(x), \phi_{21}(x)\}$ where each $\phi_{ij}(x) = x$.

In this example, the uncertain parameter affects the switching rule. Provided below are simulations with 3 different fixed values of $p$. First, we consider the case when $p = 1$, in figure 5.3:

We note that the origin is Zeno stable. Furthermore, if we consider figure 5.4, we see that even if we increase $p$ (thereby increasing the slope of $G_{21}$), we notice that the system remains Zeno stable.

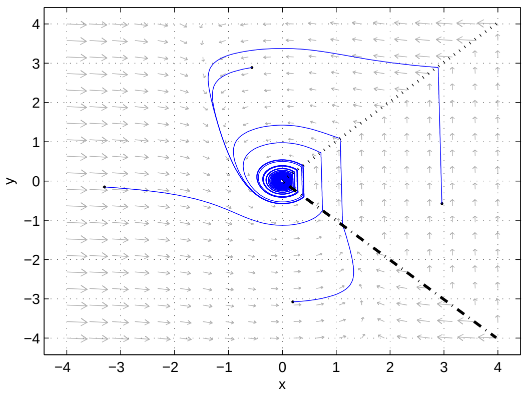


Figure 5.3. Trajectories of Hybrid System in Example 17 with p=1. Dotted line indicates $G_{12}$ and dash-dotted line indicates $G_{21}$

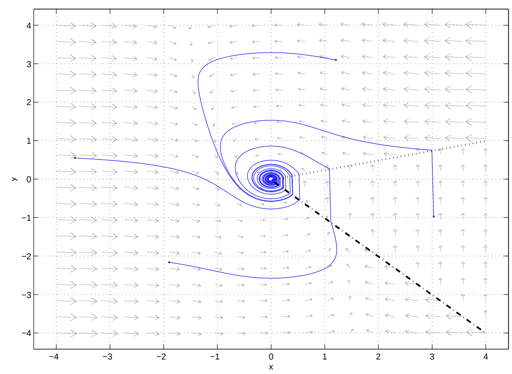


Figure 5.4. Trajectories of Hybrid System in Example 17 with p=4. Dotted line indicates $G_{12}$ and dash-dotted line indicates $G_{21}$

However, when we reduce the value of $p$, we notice that the system exhibits different asymptotic behavior. First, it is evident that if $p \leq -0.1$, the system will no longer be Zeno stable. Indeed, in that circumstance, the system would not display any form of stable behavior. This is because the trajectories in $D_1$ would never reach the guard set (since the direction of the vector field would be parallel to the guard set). But even if $p \in (-0.1, 1)$, we notice that the system asymptotically converges to limit cycles, as seen in figure 5.5:

**Results:**

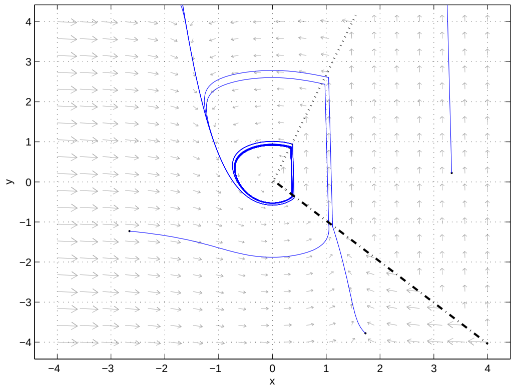


Figure 5.5. Trajectories of Hybrid System in Example 17 with p=0.4. Dotted line indicates $G_{12}$ and dash-dotted line indicates $G_{21}$

For our computational analysis, we first divide $D_2$ into $D_{21}$ and $D_{22}$. We then search for a common Lyapunov function for both domains. The set of uncertain parameters is given by the inequality $P := \{p \in \mathbb{R} : p - C > 0\}$, where $C$ is determined a priori. We use $W = W_1 \cup W_2 = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 5\}$. We then search for Lyapunov functions of varying degrees for different values of $C$. We note that as we increase the degree of $V_1$ and $V_2$, we obtain a tighter bound. These results are given below in table 5.1:

Table 5.1. Bound on $C$ obtained for different degrees of feasible $V_1, V_2$.

| **Degree of $V_1, V_2$** | **Bound on $C$** |
|---|---|
| 8 | 2.11 |
| 10 | 1.87 |
| 12 | 1.73 |

We were unable to find a feasible $V_1$ and $V_2$ of degree less than 8. Unfortunately, we were unable to search for polynomials of degree greater than 12 owing to computational limitations.

# CHAPTER 6
# CONCLUSIONS

In this thesis, we considered Zeno behavior, a phenomenon unique to hybrid systems. Zeno behavior is the phenomenon of infinite transitions between discrete modes occurring in finite time. This behavior is undesirable, and is most often caused by abstractions in modeling. Zeno behavior may result in simulation failures, since finite time is required to simulate each transition. Moreover, if a hybrid system exhibiting Zeno behavior is physically implemented, it can cause equipment failure (as is the case with the related chattering phenomenon).

Determining whether a hybrid system exhibits Zeno behavior is difficult. The principal characteristic of Zeno behavior is that infinite discrete transitions occur in finite time. While it is simple to show that infinite transitions occur in finite time for simple hybrid systems, such as the bouncing ball, by solving the associated differential equations, this is not possible for more complicated systems. To remedy this, Zeno behavior was framed as a form of asymptotic stability, aptly named Zeno stability. Accordingly, Lyapunov based techniques were developed to prove Zeno stability for hybrid systems.

The goal of our research was to answer the question, "how can we use computational techniques to determine whether a hybrid system is Zeno stable?" The technique we used was to construct Lyapunov functions using sum-of-squares optimization. To accomplish this, we chose to model the invariant domains and guard sets of hybrid systems using semialgebraic sets, and chose hybrid systems with polynomial vector fields and reset maps. We then used Stengle's positivstellensatz to construct constraints for our sum-of-squares program. We also developed a technique to verify Zeno stability for systems with time-invariant parametric uncertainties. We modeled the set of uncertain parameters as a semialgebraic set, which allowed us to use the positivstellensatz to construct a sum-of-squares program for the uncertain hybrid system.

We then applied our technique to various hybrid systems. We first demonstrated the use of our technique on hybrid systems that were known to be Zeno, such as the bouncing ball and the two-tank system. We then used our technique to verify Zeno stability to more complex systems, such as a nonlinear hybrid system and a system with a sliding mode controller. Lastly, we also used our technique for verifying robust Zeno stability to determine the range of parametric values for which a hybrid system with an uncertain guard set was Zeno stable.

There are numerous avenues for future work. Hybrid systems are used extensively in robotics; in fact, Zeno behavior was observed in models of knee joints bipedal robots. Our technique could be used to analyze Zeno stability in such systems, even in the presence of uncertain parameters. Moreover, given the growing prevalence of hybrid systems in modeling of complex systems, determining whether models exhibit Zeno stability will become a necessity.

# BIBLIOGRAPHY

[1] A. van der Schaaft and H. Schumacher, *An Introduction to Hybrid Dynamical Systems*, ser. Lecture Notes in Control and Information Sciences. Springer-Verlag, 2000.

[2] R. Goebel, J. Hespanha, A. Teel, C. Cai, and R. Sanfelice, "Hybrid systems: generalized solutions and robust stability."

[3] R. Alur, T. Dang, J. Esposito, R. Fierro, Y. Hur, F. Ivani, V. Kumar, I. Lee, P. Mishra, G. Pappas, and O. Sokolsky, "Hierarchical hybrid modeling of embedded systems," in *Embedded Software*, ser. Lecture Notes in Computer Science, T. Henzinger and C. Kirsch, Eds. Springer Berlin / Heidelberg, 2001, vol. 2211, pp. 14–31, 10.1007/3-540-45449-7˙2. [Online]. Available: http://dx.doi.org/10.1007/3-540-45449-7˙2

[4] M. Hejri and H. Mokhtari, "Global hybrid modeling and control of a buck converter: A novel concept," *International Journal of Circuit Theory and Applications*, vol. 37, no. 9, pp. 968–986, 2009. [Online]. Available: http://dx.doi.org/10.1002/cta.521

[5] C. Rivetta and G. Williamson, "Large-signal analysis and control of buck converters loaded by dc-dc converters," in *Power Electronics Specialists Conference, 2004. PESC 04. 2004 IEEE 35th Annual*, vol. 5. IEEE, 2004, pp. 3675–3680.

[6] J. Hespanha, "Stochastic hybrid systems: Application to communication networks," in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, R. Alur and G. Pappas, Eds. Springer Berlin / Heidelberg, 2004, vol. 2993, pp. 47–56, 10.1007/978-3-540-24743-2˙26. [Online]. Available: http://dx.doi.org/10.1007/978-3-540-24743-2˙26

[7] ——, "A model for stochastic hybrid systems with application to communication networks," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 62, no. 8, pp. 1353–1383, 2005.

[8] C. Tomlin, G. Pappas, and S. Sastry, "Conflict resolution for air traffic management: a study in multiagent hybrid systems," *Automatic Control, IEEE Transactions on*, vol. 43, no. 4, pp. 509 –521, apr 1998.

[9] A. Fillipov, *Differential Equations with Discontinuous Right-Hand Sides*, F. Arscott, Ed. Kluwer Academic Press, 1988.

[10] F. Clarke, "Generalized gradients and applications," *Trans. Amer. Math. Soc*, vol. 205, no. 2, pp. 247–262, 1975.

[11] A. Bemporad, G. Ferrari-Trecate, and M. Morari, "Observability and controllability of piecewise affine and hybrid systems," *Automatic Control, IEEE Transactions on*, vol. 45, no. 10, pp. 1864 – 1876, oct. 2000.

[12] S. Pettersson and B. Lennartson, "Stability and robustness for hybrid systems," in *Decision and Control, 1996., Proceedings of the 35th IEEE*, vol. 2, Dec 1996, pp. 1202 –1207.

[13] M. Branicky, "Stability of hybrid systems: state of the art," in *Decision and Control, 1997., Proceedings of the 36th IEEE Conference on*, vol. 1, dec 1997, pp. 120 –125.

[14] ——, "Stability of switched and hybrid systems," in *Decision and Control, 1994., Proceedings of the 33rd IEEE Conference on*, vol. 4, dec 1994, pp. 3498 –3503.

[15] M. Johansson and A. Rantzer, "Computation of piecewise quadratic lyapunov functions for hybrid systems," *Automatic Control, IEEE Transactions on*, vol. 43, no. 4, pp. 555 –559, apr 1998.

[16] A. Papachristodoulou and S. Prajna, "Robust stability analysis of nonlinear hybrid systems," *Automatic Control, IEEE Transactions on*, vol. 54, no. 5, pp. 1035 –1041, may 2009.

[17] S. Prajna and A. Papachristodoulou, "Analysis of switched hybrid systems - beyond piecewise quadratic methods," in *Proceedings of the American Control Conference*, June 2003.

[18] J. Zhang, K. Johansson, J. Lygeros, and S. Sastry, "Dynamical systems revisited: Hybrid systems with zeno executions," in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, N. Lynch and B. Krogh, Eds. Springer Berlin / Heidelberg, 2000, vol. 1790, pp. 451–464, 10.1007/3-540-46430-1˙37. [Online]. Available: http://dx.doi.org/10.1007/3-540-46430-1˙37

[19] K. H. Johansson, M. Egerstedt, J. Lygeros, and S. Sastry, "On the regularization of zeno hybrid automata," *Systems & Control Letters*, vol. 38, no. 3, pp. 141 – 150, 1999. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0167691199000596

[20] A. Ames, H. Zheng, R. Gregg, and S. Sastry, "Is there life after zeno? taking executions past the breaking (zeno) point," in *American Control Conference, 2006*, june 2006, p. 6 pp.

[21] A. Ames, A. Abate, and S. Sastry, "Sufficient conditions for the existence of zeno behavior," in *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05. 44th IEEE Conference on*, dec. 2005, pp. 696 – 701.

[22] ——, "Sufficient conditions for the existence of zeno behavior in a class of nonlinear hybrid systems via constant approximations," in *Decision and Control, 2007 46th IEEE Conference on*, dec. 2007, pp. 4033 –4038.

[23] A. Lamperski and A. D. Ames, "Lyapunov-like conditions for the existence of zeno behavior in hybrid and lagrangian hybrid systems," in *Proceedings of the 46th IEEE Conference on Decision and Control*, December 2007.

[24] R. Goebel and A. Teel, "Lyapunov characterization of zeno behavior in hybrid systems," in *Proceedings of the 47th Conference on Decision and Control*, 2008.

[25] A. Lamperski and A. Ames, "Lyapunov-like conditions for the existence of zeno behavior in hybrid and lagrangian hybrid systems," in *Decision and Control, 2007 46th IEEE Conference on.* IEEE, 2007, pp. 115–120.

[26] M. Ahmadi, H. Mojallali, R. Wisniewski, and R. Izadi-Zamanabadi, "Robust stability analysis of nonlinear switched systems with filippov solutions," in *7th IFAC Symposium on Robust Control Design*, 2011.

[27] G. Stengle, "A nullstellensatz and positivstellensatz in semialgebraic geometry," *Math. Annals*, vol. 207, 1974.

[28] M. Putinar, "Positive polynomials on compact semi-algebraic sets," *Indiana University Mathematics Journal*, vol. 42, no. 3, pp. 969–984, 1993.

[29] K. Schmüdgen, "Thek-moment problem for compact semi-algebraic sets," *Mathematische Annalen*, vol. 289, no. 1, pp. 203–206, 1991.

[30] B. Reznick, "Some concrete aspects of hilbert's 17th problem," *Contemporary Mathematics*, vol. 253, pp. 251–272, 2000.

[31] L. Blum, M. Shub, and S. Smale, "On a theory of computation over the real numbers; np completeness, recursive functions and universal machines," in *Foundations of Computer Science, 1988., 29th Annual Symposium on.* IEEE, 1988, pp. 387–397.

[32] P. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Ph.D. dissertation, California Institute of Technology, 2002.

[33] S. Prajna, A. Papachristodoulou, P. Seiler, and P. A. Parrilo, *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*, Available from `http://www.cds.caltech.edu/sostools` and `http://www.mit.edu/~parrilo/sostools`, 2004.

[34] A. Papachristodoulou and S. Prajna, "A tutorial on sum of squares techniques for systems analysis," in *American Control Conference, 2005. Proceedings of the 2005.* IEEE, 2005, pp. 2686–2700.

[35] ——, "On the construction of lyapunov functions using the sum of squares decomposition," in *Decision and Control, 2002, Proceedings of the 41st IEEE Conference on*, vol. 3, dec. 2002, pp. 3482 – 3487.

[36] A. Papachristodoulou, M. Peet, and S. Lall, "Constructing lyapunov-krasovskii functionals for linear time delay systems," in *American Control Conference, 2005. Proceedings of the 2005*, june 2005, pp. 2845 – 2850 vol. 4.

[37] S. Prajna, P. Parrilo, and A. Rantzer, "Nonlinear control synthesis by convex optimization," *Automatic Control, IEEE Transactions on*, vol. 49, no. 2, pp. 310 – 314, feb. 2004.

[38] S. Prajna, A. Papachristodoulou, and F. Wu, "Nonlinear control synthesis by sum of squares optimization: A lyapunov-based approach," in *Control Conference, 2004. 5th Asian*, vol. 1. IEEE, 2004, pp. 157–165.

[39] S. Glavaski, D. Subramanian, K. Ariyur, R. Ghosh, N. Lamba, and A. Papachristodoulou, "A nonlinear hybrid life support system: Dynamic modeling, control design, and safety verification," *Control Systems Technology, IEEE Transactions on*, vol. 15, no. 6, pp. 1003–1017, 2007.

[40] A. Ataei-Esfahani and Q. Wang, "Nonlinear control design of a hypersonic aircraft using sum-of-squares method," in *American Control Conference, 2007. ACC'07.* IEEE, 2007, pp. 5278–5283.

[41] A. Julius and G. Pappas, "Approximations of stochastic hybrid systems," *Automatic Control, IEEE Transactions on*, vol. 54, no. 6, pp. 1193–1203, 2009.

[42] S. Prajna, A. Jadbabaie, and G. Pappas, "A framework for worst-case and stochastic safety verification using barrier certificates," *Automatic Control, IEEE Transactions on*, vol. 52, no. 8, pp. 1415–1428, 2007.

[43] A. Julius, "Approximate abstraction of stochastic hybrid automata," *Hybrid Systems: Computation and Control*, pp. 318–332, 2006.

[44] E. F. Stephen Boyd, Laurent El Ghaoui and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control*. Society for Industrial and Applied Mathematics, 1994.

[45] W. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*. Princeton University Press, 2008.

[46] H. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2001.

[47] G. Dullerud and F. Paganini, *A Course in Robust Control Theory: A Convex Approach*. Springer, 2000.

[48] J. Lygeros, K. Johansson, S. Simic, J. Zhang, and S. Sastry, "Dynamical properties of hybrid automata," *Automatic Control, IEEE Transactions on*, vol. 48, no. 1, pp. 2–17, 2003.

[49] M. Branicky, "Multiple lyapunov functions and other analysis tools for switched and hybrid systems," *Automatic Control, IEEE Transactions on*, vol. 43, no. 4, pp. 475 –482, apr 1998.

[50] R. Alur and T. Henzinger, "Modularity for timed and hybrid systems," in *CONCUR '97: Concurrency Theory*, ser. Lecture Notes in Computer Science, A. Mazurkiewicz and Winkowski, Eds. Springer Berlin/Heidelberg, vol. 1243, pp. 74–88.

[51] J. J. Slotine and W. Li, *Applied Nonlinear Control*. Prentice Hall, 1991.

[52] P. Grensted and A. Fuller, "Minimization of integral-square-error for non-linear control systems of third and higher order," *International Journal of Control*, vol. 2, no. 1, pp. 33–73, 1965.