Thesis Defense

Analysis of Zeno Stability in Hybrid Dynamical Systems using Sum-of-Squares Programming

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Outline

Introduction

- Hybrid Systems are systems that exhibit different dynamics in different regions of the state space, with a logical rule governing transitions between dynamics.
- The hybrid system framework is used to model a variety of natural and man-made systems such as systems with embedded microprocessors, electrical circuits with switching, air traffic control, the bouncing ball.
- Hybrid systems also exhibit unique phenomeona, such as **Zeno behavior**.
- Zeno behavior is the phenomenon of infinite transitions occurring in finite time.
  - Similar to chattering.
  - Causes simulations to fail.
  - Arises in models of a number of system: robotic joints, communication networks, optimal control, or even simple systems (the bouncing ball)
- Our goal: To develop computational methods to analyze this behavior.

A Motivating Example: Modeling a Bouncing Ball

Model must contain the following information:

- Dynamics of the ball
- Domain of the dynamics
- Location of collisions
- Effect of collision on dynamics

A Solution: hybrid systems

Figure: A bouncing basketball (courtesy of Wikipedia)

Hybrid Systems

Definition 1: Hybrid System

A hybrid system $\mathbf{H}$ is modeled by the tuple

$$\mathbf{H} = (Q, E, D, F, G, R)$$

where

- Q is the collection of discrete states.
- $E \subset Q \times Q$ provides the set of transitions between discrete states. For each $e = (q, q')$, we say $q = s(e)$ and $q' = t(e)$.
- $D := \{D_Q\}_{q \in Q}$ is the collection of domains
- $F := \{f_q\}_{q \in Q}$ is the collection of vector fields, where for each $q \in Q$, $f_q : D_q \to \mathbb{R}^n$
- $G := \{G_e\}_{e \in E}$ is the collection of guard sets
- $R := \{\phi_e\}_{e \in E}$ is the collection of Reset Maps where for each $e \in E$, $\phi_e : G_e \to D_{t(e)}$.

Modeling a Bouncing Ball with a Hybrid System

We model the bouncing ball with the tuple

$$\mathbf{B} = (Q, E, D, F, G, R)$$

where

- $Q = \{q_0\}$
- $E = \{(q_0, q_0)\}$
- $D := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$
- $F := \{f\}$, where
$$f = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -g \end{pmatrix}$$
- $G := \{G_{(q_0,q_0)}\}$ where $G_{(q_0,q_0)} := \{x \in \mathbb{R}^2 : x_1 = 0, x_2 < 0\}$
- $R = \phi(x) = [0, -cx_2]^T$. Here, c is a coefficient of restitution.

An Important Assumption

Assumption 1

- Each $D_q \in D$ will be of the form

$$D_q := \{x \in \mathbb{R}^n : g_{qk}(x) \geq 0, q \in Q, k = 1, ..., K_q\}$$

  where each g_{qk} is a polynomial, and K_q is some positive integer.

- Each $G_e \in G$ will be of the form

$$G_e = \{x \in \mathbb{R}^n : h_{e0}(x) = 0, h_{ek}(x) \geq 0, e \in E, k = 1, 2, ..., N_q\}$$

  where each h_{e0}, h_{ek} are polynomials, and K_e is some positive integer.

- Each $\phi_e \in R : \mathbb{R}^n \to \mathbb{R}^n$ is of the form

$$\phi_e = [\phi_{e1}, ..., \phi_{en}]^T$$

  where each ϕ_{ei} is a polynomial.

Hybrid System Execution

Definition 2: Hybrid System Execution

The tuple

$$\chi = (I, T, p, C)$$

where

- $I \subseteq \mathbb{N}$ is the index set
- $T = \{T_i\}_{i \in I}$ where $T_i = (t_{i-1}, t_i)$
- $p : I \to C$ describes the evolution of the elements of Q
- $C = \{c_i(t)\}_{i \in I}$ provides a set of continuous functions

is an execution of a hybrid system $H = (Q, E, D, F, G, R)$ if

- $\dot{c}_i(t) = f_{p(i)}(c_i(t)); \quad t \in T_i$
- $c_i(t) \in D_{p(i)}, \forall t \in T_i$
- $c_i(t_{i+1}) \in G_{p(i),p(i+1)}$ for all $T_i \in T$.

Zeno Equilibria and Executions

Definition 3: Zeno Execution

An execution χ of a hybrid system $\mathbf{H} = (Q, E, D, F, G, R)$ is said to be Zeno if

- $I \equiv \mathbb{N}$
- $\sum_{i=1}^{\infty} t_i - t_{i-1} < \infty$

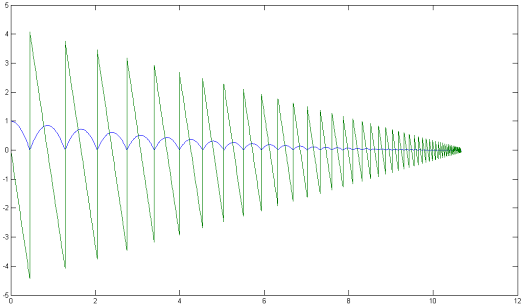


Figure: Zeno behavior in a bouncing ball

Zeno Stability

Definition 4: Zeno Equilibria

A Zeno equilibrium of a hybrid system $\mathbf{H} = (Q, E, D, F, G, R)$ is a set $z = \{z_q\}_{q \in Q}$ satisfying

- $f_q(z_q) \neq 0$
- For each edge $e = (q, q')$, $z_q \in G_e$.
- Similarly, for each edge $e = (q, q')$, $R_e(z_q) = z_{q'}$

Definition 5: Zeno Stability

Let $\mathbf{H} = (Q, E, D, F, G, R)$ be a hybrid system. The set z is Zeno stable if, for each $q \in Q$, there exist neighborhoods Z_q, where $z_q \in Z_q$, such that for any initial condition $x_0 \in \bigcup_{q \in Q} Z_q$, the execution $\chi = (I, T, p, C)$, with $c_o(t_0) = x_0$ is Zeno, and converges to z.

- We can consider Zeno stability to be a form of finite-time asymptotic stability.

Cyclic Hybrid Systems

Definition 6: Cyclic Hybrid Systems

A hybrid system $\mathbf{H} = (Q, E, D, F, G, R)$ is cyclic if the pair $\Gamma = (Q, E)$ describes a directed cycle, where Q represents the vertices, and E represents the edges.

Remark

If a hybrid system contains a Zeno equilibrium, then the graph representing the hybrid system must contain a cycle.

Necessary and Sufficient Conditions for Zeno Stability

Theorem 1 (Lamperski and Ames)

Consider a hybrid system $\mathbf{H} = (Q, E, D, F, G, R)$, with an isolated Zeno equilibrium $\{z_q\}_{q\in Q}$. Let $\{W_q\}_{q\in Q}$ be a collection of open neighborhoods of $\{z_q\}_{q\in Q}$. Suppose there exist continuously differentiable functions $V_q : \mathbb{R}^n \to \mathbb{R}$ and $B_q : \mathbb{R}^n \to \mathbb{R}$, and non-negative constants $\{r_q\}_{q\in Q}$, γ_a, and γ_b, where $r_q \in [0,1]$, and $r_q < 1$ for some q and such that

$$
\begin{aligned}
V_q(x) &> 0 \quad \text{for all } x \in W_q \backslash z_q, q \in Q && (1)\\
V_q(z_q) &= 0, \quad \text{for all } q \in Q && (2)\\
\nabla V_q^T(x) f_q(x) &\leq 0 \quad \text{for all } x \in W_q,\ q \in Q && (3)\\
B_q(x) &\geq 0 \quad \text{for all } x \in W_q,\ q \in Q && (4)\\
\nabla B_q^T(x) f_q(x) &< 0 \quad \text{for all } x \in W_q,\ q \in Q && (5)\\
V_{q'}(R_{(q,q')}(x)) &\leq r_q V_q(x), && (6)\\
& \quad \text{for all } e = (q,q') \in E \text{ and } x \in G_e \cap W_q\\
B_q(R_{(q',q)}(x)) &\leq \gamma_b \left(V_q(R_{(q,q')}(x))\right)^{\gamma_a} && (7)\\
& \text{for all } e = (q,q') \in E \text{ and } x \in G_e \cap W_q.
\end{aligned}
$$

A Simplification of Theorem 1

We can simplify Theorem 1 as follows:

Theorem 2:
Let $\mathbf{H} = (Q, E, D, F, G, R)$ be a cyclic hybrid system with Zeno equilibrium $z = \{z_q\}_{q \in Q}$. Let $\{W_q \subset D_q\}_{q \in Q}$, be a collection of neighborhoods of the $\{z_q\}_{q \in Q}$. Suppose that there exist continuously differentiable functions $V_q : W_q \to \mathbb{R}$, and positive constants $\{r_q\}_{q \in Q}$ and γ, where $r_q \in (0, 1]$, and $r_q < 1$ for some q and such that

$$
\begin{aligned}
V_q(x) &> 0 \quad \text{for all } x \in W_q \backslash z_q, q \in Q && (8)\\
V_q(z_q) &= 0, \quad \text{for all } q \in Q && (9)\\
\nabla V_q^T(x) f_q(x) &\leq -\gamma \quad \text{for all } x \in W_q,\ q \in Q && (10)\\
r_q V_q(x) &\geq V_{q'}(\phi_e(x)) && (11)\\
&\text{for all } e = (q, q') \in E \text{ and } x \in G_e \cap W_q.
\end{aligned}
$$

then z is Zeno stable.

Advantages of Method

- Theorems 1 and 2 provide necessary and sufficient conditions for Zeno stability.
- They allow for the verification of Zeno stability even for systems where calculating total time of executions analytically is difficult.
- Theorem 1 has also been used to develop Lyapunov conditions for the existence of Zeno executions in Lagrangian Hybrid systems.
- More importantly, each V_q can be constructed algorithmically. Here, we use Sum-of-Squares programming.
- We construct our SOS conditions using the Positivstellensatz.

Sum of Squares Polynomials

- A polynomial p is said to be Sum of Squares if it can be expressed as

$$p = \sum_i f_i^2$$

where each $f_i \in \mathbf{R}[x]$. The set of all SOS polynomials is denoted by Σ_x.

Theorem 3

Consider a polynomial p of degree $2d$. Then, the following two statements are equivalent:

- p is SOS.
- There exists a positive semidefinite matrix Q and a vector Z of all monomials of degree upto d such that

$$p = Z^T Q Z$$

Use of Sum-of-Squares polynomials

- Verifying polynomial nonnegtivity cannot be accomplished in polynomial time.
- However, verifying whether a polynomial is SOS has been shown to be decidable in polynomial time.
- We use SOSTOOLS to solve such problems.

A simple example

Consider $p(x) = x^2 + 2x + 1 = (x+1)^2$:
If we choose $Z = [1, x]^T$, we search for a 2×2 matrix Q. We can then find

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Verifying Zeno Stability with Sum of Squares Programming

- We apply the Positivstellensatz to Theorem 2:

Let $\mathbf{H} = (Q, E, D, F, G, R)$ be a hybrid system with a Zeno equilibrium $\{z_q\}_{q \in Q}$. Let $\{W_q\}_{q \in Q}$ be a collection of neighborhoods of $\{z_q\}_{q \in Q}$. Moreover, suppose that each W_q is a semialgebraic set defined as

$$W_q := \{x \in \mathbb{R}^n : w_{qk}(x) > 0, k = 1, 2, ..., K_{qw}\}$$

where $w_{qk} \in \mathbf{R}[x]$.

Feasibility Problem 1

For hybrid system $\mathbf{H} = (Q, E, D, F, G, R)$, find

- a_{qk}, c_{qk}, i_{qk}, $\in \Sigma_x$, for $k = 1, 2, ..., K_{qw}$ and $q \in Q$;
- b_{qk}, d_{qk}, $j_{qk} \in \Sigma_x$, for $k = 1, 2, ..., K_q$ and $q \in Q$.
- $m_{e,l} \in \Sigma_x$ for $e \in E$ and $l = 1, 2, ..., N_q$
- V_q, $m_{e,0} \in \mathbf{R}[x]$ for $e \in E$ and $q \in Q$.
- Constants $\alpha, \gamma > 0$, $\{r_q\}_{q \in Q} \in (0, 1]$ such that $r_q < 1$ for some $q \in Q$.

such that (continued on next slide)

Feasibility Problem 1 (continued)

$$V_q - \alpha x^T x - \sum_{k=1}^{K_{qw}} a_{qk} w_{qk} - \sum_{k=1}^{K_q} b_{qk} g_{qk} \in \Sigma_x \quad \text{for all } q \in Q \tag{12}$$

$$V_q(z_q) = 0 \quad \text{for all } q \in Q \tag{13}$$

$$-\nabla V_q^T f_q - \gamma - \sum_{k=1}^{K_{qw}} c_{qk} w_{qk} - \sum_{k=1}^{K_q} d_{qk} g_{qk} \in \Sigma_x \quad \text{for all } q \in Q \tag{14}$$

$$r_q V_q - V_{q'}(\phi_e) - m_{e,0} h_{e,0} - \sum_{l=1}^{N_q} m_{e,l} h_{e,l}$$

$$- \sum_{k=1}^{K_{qw}} i_{qk} w_{qk} - \sum_{k=1}^{K_q} j_{qk} g_{qk} \in \Sigma_x \quad \text{for all } e = (q, q') \in E \tag{15}$$

A Theorem for Verifying Zeno Stability

Using the notation defined previously, and Feasibility Problem 1, we state the main theorem:

Theorem 4

Let $z = \{z_q\}_{q \in Q}$ be an isolated Zeno equilibrium of a hybrid system $\mathbf{H} = (Q, E, D, F, G, R)$. If Feasibility Problem 1 has a solution, then z is Zeno stable.

Nonlinear Numerical Example (1)

Nonlinear Bouncing Ball

The nonlinear bouncing ball hybrid system can be represented by the tuple:

$$H = (Q, E, D, F, G, R)$$

where

- $Q = \{q_0\}$
- $E = \{(q_0, q_0)\}$
- $D := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$
- $G := \{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \leq 0\}$
- $F = \{f\}$, where

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ -g + c_1 x_2^2 \end{pmatrix}$$

- $R = \phi(x) = [0, -c_2 x_2 (1 - c_3 x_2^2)]^T$. Here, c_1, c_2, and c_3 are positive constants satisfying $c_i < 1$.

Numerical Example (2)

Simulation Results

- The Zeno equilibrium is $z = (0,0)^T$.
- We choose $W_q := \{x \in \mathbb{R}^n : x_1 \geq 0, 1 - x_1^2 - x_2^2 > 0\}$ for Feasibility Problem 1
- We search for a 4-th order $V(x)$ that solves Feasibility Problem 1. We were unable to obtain an explicit range of values of c_1, c_2, and c_3 such that **N** was stable.
- We fix each c_i at certain values, and plot values of the other constants such that **N** was Zeno stable (next slide).

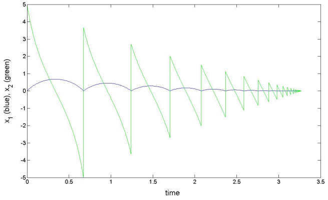


Figure: Execution of nonlinear hybrid system

Numerical Example (3)

Simulation Results: Plots of stable values of c_1, c_2, and c_3

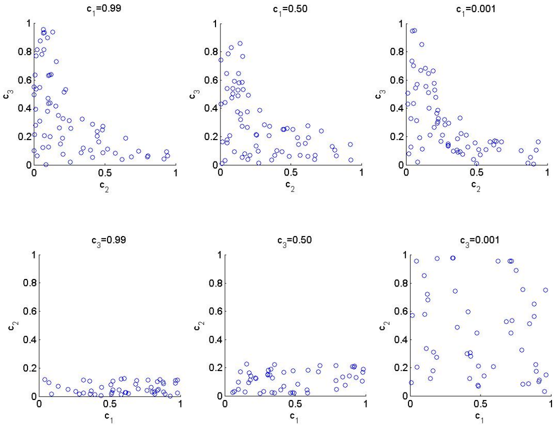


Figure: Stable values of c_1 and c_2 with c_3 fixed

Zeno Stability for Systems with Uncertainty

- We now present a method to verify Zeno stability in systems with time-invariant parametric uncertainty.
- Treat the set of uncertain parameters as a semialgebraic set.
- Apply the positivstellensatz to Theorem 2 and the uncertain set.

Another Important Assumption

Define the vector of uncertain parameters to lie in the semialgebraic set

$$P := \{p \in \mathbb{R} : \tilde{p}_k(p) \geq 0, k = 1, 2, ..., K_1\}$$

where $\tilde{p}_k(p)$ are polynomials.

Assumption 2

- Each $D_q \in D$ will be of the form

$$D_q := \{x \in \mathbb{R}^n : g_{qk}(x, p) \geq 0, q \in Q, k = 1, ..., K_q\}$$

  where each $g_{qk} \in \mathbf{R}[x, p]$, $K_q \in \mathbb{N}$.

- Each $G_e \in G$ will be of the form

$$G_e = \{x \in \mathbb{R}^n : h_{e0}(x, p) = 0, h_{ek}(x, p) \geq 0, e \in E, k = 1, 2, ..., N_q\}$$

  where each $h_{e0}, h_{ek} \in \mathbf{R}[x, p]$, and $K_e \in \mathbb{N}$.

- Each $\phi_e \in R : \mathbb{R}^n \to \mathbb{R}^n$ is of the form

$$\phi_e = [\phi_{e1}, ..., \phi_{en}]^T$$

  where each $\phi_{ei} \in \mathbf{R}[x, p]$.

Feasibility Problem 2

For hybrid system $\mathbf{H} = (Q, E, D, F, G, R)$, find

- $a_{qk},\ c_{qk},\ i_{qk},\ \in \Sigma_{x,p}$, for $k = 1, 2, ..., K_{qw}$ and $q \in Q$;
- $b_{qk},\ d_{qk},\ j_{qk} \in \Sigma_{x,p}$, for $k = 1, 2, ..., K_q$ and $q \in Q$.
- $\eta_{qk},\ \beta_{qk},\ \zeta_{qk} \in \Sigma_{x,p}$, for $k = 1, 2, ..., K_1$ and $q \in Q$.
- $m_{e,l} \in \Sigma_{x,p}$ for $e \in E$ and $l = 1, 2, ..., N_q$
- $V_q,\ m_{e,0} \in \mathbf{R}[x, p]$ for $e \in E$ and $q \in Q$.
- Constants $\alpha, \gamma > 0$, $\{r_q\}_{q \in Q} \in (0, 1]$ such that $r_q < 1$ for some $q \in Q$.

such that (continued on next slide)

Feasibility Problem 2 (continued)

$$V_q - \alpha x^T x - \sum_{k=1}^{K_{qw}} a_{qk} w_{qk} - \sum_{k=1}^{K_q} b_{qk} g_{qk} - \sum_{k_1=1}^{K_1} \eta_{qk_1} \tilde{p}_{qk} \in \Sigma_{x,p} \quad \text{for all } q \in Q \tag{16}$$

$$V_q(z_q, p) = 0 \quad \text{for all } q \in Q \tag{17}$$

$$-\nabla V_q^T f_q - \gamma - \sum_{k=1}^{K_{qw}} c_{qk} w_{qk} - \sum_{k=1}^{K_q} d_{qk} g_{qk} - \sum_{k_1=1}^{K_1} \beta_{qk_1} \tilde{p}_{qk} \in \Sigma_{x,p} \quad \text{for all } q \in Q \tag{18}$$

$$r_q V_q - V_{q'}(\phi_e) - m_{e,0} h_{e,0} - \sum_{l=1}^{N_q} m_{e,l} h_{e,l} - \sum_{k=1}^{K_{qw}} i_{qk} w_{qk} - \sum_{k=1}^{K_q} j_{qk} g_{qk} - \sum_{k=1}^{K_1} \zeta_{qk} \tilde{p}_{qk} \in \Sigma_{x,p} \quad \text{for all } e = (q, q') \in E. \tag{19}$$

A Theorem for Zeno Stability in Systems with Uncertainties

Theorem 5

Let $z = \{z_q\}_{q \in Q}$ be a Zeno equilibrium of a hybrid system $\mathbf{H} = (Q, E, D, F, G, R)$. If there is a solution to Feasibility Problem 2, then z is Zeno stable for all $p \in P$.

Numerical Example for uncertain hybrid systems

Bouncing ball with uncertain coefficient of restitution

We now consider a bouncing ball with an uncertain coefficient of restitution: A bouncing ball with parametric uncertainties in the reset map can be described by $\mathbf{B}_p$ which is the tuple:

$$\mathbf{B}_p = (Q, E, D, F, G, R)$$

where

- $Q = \{q_0\}$, which provides the discrete state
- $E = \{(q_0, q_0)\}$, which is the single edge from q_0 to itself
- $D := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$ provides the domain. Thus, $g_{q_0} = x_1$.
- $G = \{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \leq 0\}$ provides the guard. Thus, $h_{(q_0,q_0),0} = x_1$, and $h_{(q_0,q_0),1} = -x_2$.
- $R = \phi(x) = [0, -px_2]^T$ provides the reset map.
- $F = f(x)$ provides a vector field mapping D to itself, and where

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ -g \end{pmatrix}$$

Numerical Example: Bouncing Ball with uncertainty (2)

Simulation Results

- We consider $p \in (0, C)$.
- Construct semialgebraic set:

$$\mathcal{P} := \{p \in \mathbb{R} : p(C - p) \leq 0\}$$

- Analytically, we know that for Zeno stability, the largest C that allows for Zeno stability is 1.
- We search for a 4-th order $V(x)$ and sos multipliers that solves Feasibility Problem 2.
- 4th order $V(x)$ allows us verify Zeno stability for uncertain parameters on the set $(0.001, 0.999)$.

Hybrid System with Multiple Modes and Nonlinear Vector Fields (1)

- Often, we need to analyze Zeno stability of hybrid systems with multiple modes and nonlinear vector fields.
- Very difficult to accomplish this analytically.

We considered a nonliear hybrid system with 3 discrete modes:

Nonlinear Hybrid Systems with 3 discrete modes

Consider the hybrid system $\mathbf{H} = (Q, E, D, F, G, R)$, where

- $Q = \{q_1, q_2, q_3\}$
- $E = \{(q_1, q_2), (q_2, q_3), (q_3, q_1)\}$
- $D := \{D_1, D_2, D_3\}$ where

$$D_1 = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 + \frac{1}{2}x_1 \geq 0\} \tag{20}$$

$$D_2 = \{x \in \mathbb{R}^2 : x_2 - \frac{1}{2}x_1 \geq 0, x_2 + \frac{1}{2}x_1 < 0\} \tag{21}$$

$$D_3 = \{x \in \mathbb{R}^2 : x_1 < 0, x_2 + \frac{1}{2}x_1 \geq 0\} \tag{22}$$

Hybrid System with Multiple Modes and Nonlinear Vector Fields (2)

Nonlinear hybrid system with 3 discrete modes

- $G := \{G_{12}, G_{23}, G_{31}\}$ where

$$G_{12} := \left\{x \in \mathbb{R}^2 : x_2 \leq 0, \frac{1}{2}x_1 + x_2 = 0\right\} \tag{23}$$

$$G_{23} := \left\{x \in \mathbb{R}^2 : x_2 \leq 0, \frac{1}{2}x_1 - x_2 = 0\right\} \tag{24}$$

$$G_{31} := \left\{x \in \mathbb{R}^2 : x_2 > 0, x_1 = 0\right\} \tag{25}$$

- $F = \{f_1, f_2, f_3\}$, where

$$\dot{x} = f_1(x) = (x_2, -5x_1^2 - x_2)^T \tag{26}$$

$$\dot{x} = f_2(x) = (-x_1^2 - 3, 2x_2^2 - \frac{1}{2}x_1^2)^T \tag{27}$$

$$\dot{x} = f_3(x) = (x_2^2 + x_2, -x_1)^T \tag{28}$$

- $R = \{\phi_{12}(x), \phi_{23}(x), \phi_{31}(x)\}$ where each $\phi_{ij}(x) = x$.

Hybrid System with Multiple Modes and Nonlinear Vector Fields (3)

Simulation Results

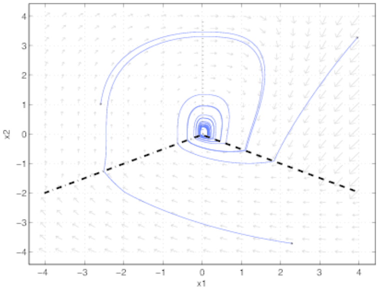


Figure: Phase Portrait of a hybrid system with 3 modes and nonlinear vector fields

- We analyzed Zeno stability of $z = \{z_1, z_2, z_3\}$, $z_1 = z_2 = z_3 = 0$.
- Analyzed Zeno stability of z in the unit ball
- Found degree 8 V_1, V_2, and V_3 to solve feasibility problem 2.

Systems with Variable Structure Controllers (1)

- Often, variable structure controllers (such as sliding mode, bang-bang, and gain scheduling controllers) are required for stabilization.
- However, chattering and Zeno behavior can occur in the closed loop systems.
- This can be difficult to verify analytically.

A system with a variable structure controller

- We consider the plant

$$\dot{x} = (x_2, x_1^2 + x_2^2 + u(x,t))^T.$$

- Suppose the chosen controller is

$$u(x,t) = -2(x_1^2 + x_2^2)\mathrm{sgn}(s(x))$$

where $s(x) = x_1 + x_2$.

Systems with Variable Structure Controllers (2)

A system with a variable structure controller

We can model the closed loop system with a hybrid system $H = (Q, E, D, F, G, R)$, where

- $Q = \{q_1, q_2\}$
- $E = \{(q_1, q_2), (q_2, q_1)\}$
- $D = \{D_1, D_2\}$ where

$$D_1 := \{x \in \mathbb{R}^2 : x_1 + x_2 \geq 0\}; \quad D_2 := \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 0\} \tag{29}$$

- $F = \{f_1, f_2\}$ where

$$f_1 = (x_2, -(x_2^2 + x_1^2))^T, \quad f_2 = (x_2, 3(x_2^2 + x_1^2))^T \tag{30}$$

- $G = \{G_{12}, G_{21}\}$ where

$$G_{12} = G_{21} := \{x \in \mathbb{R}^2 : x_1 + x_2 = 0\} \tag{31}$$

- $R = \{\phi_{12}(x), \phi_{21}(x)\}$ where each $\phi_{ij}(x) = x$.

Systems with Variable Structure Controllers (3)

Simulation Results

A phase portrait of the closed loop vector field is shown below:

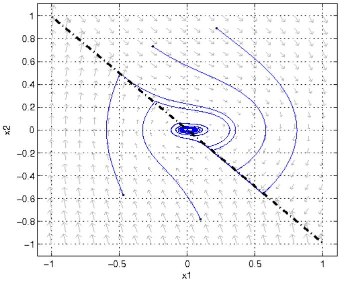


Figure: Phase Portrait of a system with a variable structure controller exhibiting Zeno behavior.

- We analyzed Zeno stability of $z = \{z_1, z_2\} = (0, 0)$.
- We studied Zeno stability in the unit ball around z.
- We were able to find degree 8 V_1 and V_2 to solve feasibility problem 2.

Hybrid System with Uncertain Switching (1)

- Complete information regarding system parameters may be unavailable to us - this is parametric uncertainty.
- In the case of hybrid systems, this results in uncertainty in the vector fields, as well as the transition rules: uncertainties may also be present in the guard set and the reset map.
- We now consider a system with an uncertain guard set.

A hybrid system with an uncertain guard set

Let the uncertain parameters be defined by

$$P := \{p \in \mathbb{R} : p - C > 0\}.$$

We then consider the hybrid system $H = (Q, E, D, F, G, R)$ which satisfies assumption 2, where

- $Q = \{q_1, q_2\}$
- $E = \{(q_1, q_2), (q_2, q_1)\}$

Hybrid System with Uncertain Switching (2)

A hybrid system with an uncertain guard set

- $D = \{D_1, D_2\}$ where

$$D_1 := \{x \in \mathbb{R}^2 : x_1 + x_2 \geq 0, px_1 - x_2 \geq 0\} \quad (32)$$

$$D_2 := \{x \in \mathbb{R}^2 : -px_1 + x_2 \geq 0\} \cup \{x \in \mathbb{R}^2 : px_1 - x_2 \geq 0, -x_1 - x_2 \geq 0\} \quad (33)$$

- $F = \{f_1, f_2\}$ where

$$f_1 = (-0.1, 2), \quad f_2 = (-x_2 - x_1^3, x_1) \quad (34)$$

- $G = \{G_{12}, G_{21}\}$ where

$$G_{12} = \{x_2 - px_1\} = 0 \quad (35)$$

$$G_{21} := \{x \in \mathbb{R}^2 : x_1 + x_2 = 0\} \quad (36)$$

- $R = \{\phi_{12}(x), \phi_{21}(x)\}$ where each $\phi_{ij}(x) = x$.

Hybrid System with Uncertain Switching (3)

Simulation Results

Phase planes for different values of p are given below:

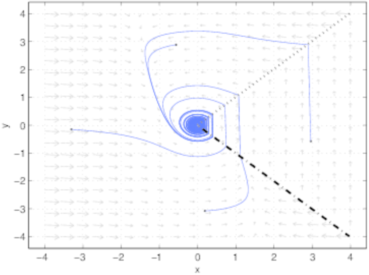
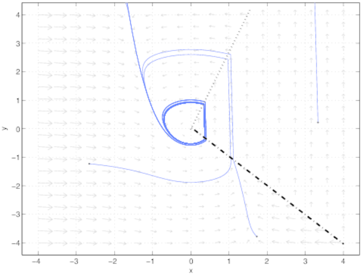


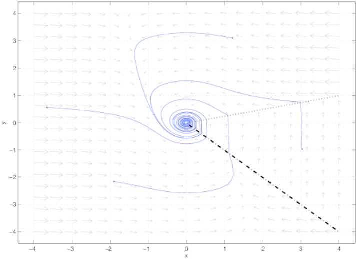
Figure: Phase plane for $p = 1$

Hybrid System with Uncertain Switching (4)

Simulation Results (continued)



(a) Phase plane for $p = 0.4$



(b) Phase plane for $p = 4$

Hybrid System with Uncertain Switching (5)

Simulation Results (continued)

- We analyzed Zeno stability of $z = \{z_1, z_2\}$, $z_1 = z_2 = (0, 0)$ in the unit ball around z.
- We searched for V_1 and V_2 of increasing degree in order to obtain lower bounds on C. This is shown in the table below:

Table: Bound on C obtained for different degrees of feasible V_1, V_2.

Degree of V_1, V_2	**Bound on C**
8	2.11
10	1.87
12	1.73

Conclusions

Conclusions

- Hybrid systems are dynamical systems that exhibit both continuous and discrete behavior. Zeno behavior is a phenomenon unique to hybrid systems.
- Necessary and sufficient conditions for Zeno stability were provided.
- Verification of Zeno stability is accomplished by solving Feasibility Problem 1.
- It is possible to verify Zeno stability for a hybrid system with uncertainties by solving Feasibility Problem 2.

Future Work

- Determine methods to decrease or alleviate the computational cost of the method.
- Apply the technique to verification of Zeno stability in network congestion problems.
- Possible applications of convex optimization and sum-of-squares programming to regularization of Zeno hybrid systems.

END

Thank you for listening!