Stability and Control of Functional Differential Equations

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Our Research Goal: Find ways to address fundamentally difficult problems in control. These include systems with

- Nonlinearity
- Multiple Variables
- Uncertainty
- Multiple Delays

NP hardness A problem is NP hard if it has been proven to be fundamentally difficult to compute. The following general problems in systems theory have been shown to be NP hard.

- Stability of linear systems with delay
- Stability of nonlinear systems
- Stability of linear systems with parameter uncertainty

The Stability Question: Control Theory provides many tools for stability analysis of systems with delay.

- Frequency Domain Tools
  - Padé approximations
  - Nyquist criteria
  - Bode Plots
- Time Domain Tools
  - Lyapunov Functions
  - Passivity
  - Small Gain

Our Approach:

We can combine tools from Control Theory with techniques from Mathematics and Computer Science.

- Real Algebraic Geometry
- Functional Analysis
- Convex Optimization
- Semidefinite Programming

Research Overview:

Decentralized Optimization

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Internet Congestion Control

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Convex Optimization

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Linear Time-Delay Systems

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Nonlinear Time-Delay Systems

Differential Equations with Delay are used to model systems where past events can influence present behavior.

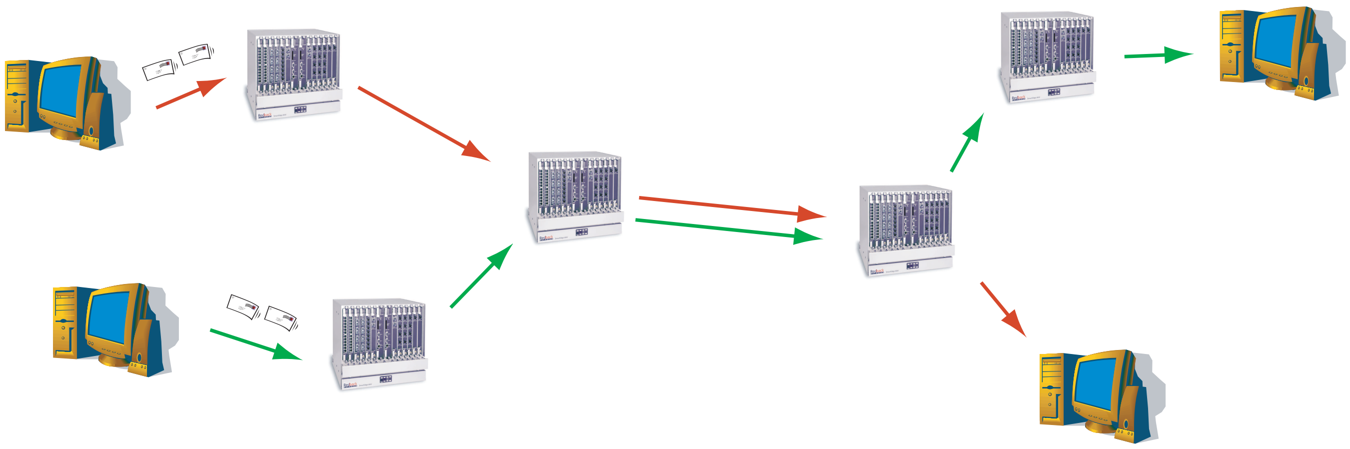
$$\dot{x}(t) = f(x(t), x(t-\tau))$$

Sources of Delay:

- Processing Time
- Communication lag
- Growth of biological organisms
- Incubation of viruses

Internet Congestion Control

How does the Internet operate? The system is like the postal service.



1. The user addresses the packet and drops it in the local router.
2. The router sends the packet to a router closer to the destination and from there to another router etc. until the packet arrives at the destination
3. The destination receives the packet and sends an acknowledgement
4. If no acknowledgement is received in a given amount of time, the user resends the packet.

Basics of Congestion Control

In response to congestion collapse, the use of a dynamic transmission control protocol(**TCP**) by the sources was proposed by Van Jacobson in 1988.

Basic idea: Have the users send fewer packets when they see congestion.

Current and Previous Versions:

TCP	Static transmission rate
TCP Tahoe	The source tries to estimate the capacity of the network. It probes the network by increasing transmission rate until a loss is detected.
TCP Reno	Faster Recovery. Damps some of the oscillations in Tahoe in TCP Tahoe
TCP Vegas	Estimates queued packets using queueing delay. Tries to keep this number below 2 or so.

Redesigning TCP from the Top Down

How to design the optimal congestion control? Consider a congested router with a fixed capacity. There are two big questions to consider when designing a TCP.

Fairness How does one distribute bandwidth to those who want to use the router?

- Clearly we want to distribute all the available bandwidth.
- But, how much to give each user?

Stability What rules can we impose that will result in the desired distribution?

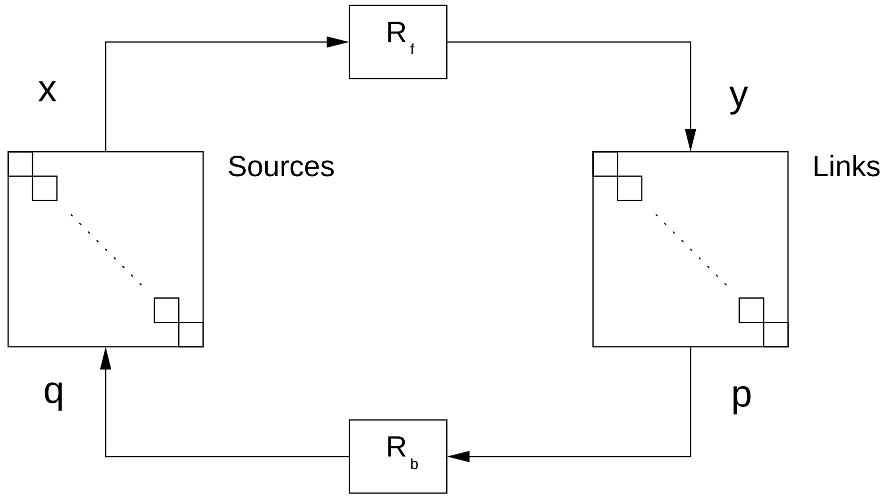
- We want the system to converge quickly, so we don't waste bandwidth
- But routers can't control source transmissions directly.

Decentralized Optimization

Kelly et al.(1998) interpreted **Internet Congestion Control** as an attempt to solve the decentralized optimization problem.

$$\max \sum_i U(x_i)$$

subject to $\quad Rx \leq c$



Where

- x_i is the rate of source i
- c_j is the capacity of link j
- R is map from users to links

Note:

- Any solution must admit a decentralized implementation

The Dual Problem

The dual to the decentralized optimization problem is given by:

$$\begin{aligned} \text{minimize} \quad & h(p) \\ \text{subject to} \quad & p \geq 0 \end{aligned}$$

Where

$$\begin{aligned} h(p) &= \sum_i U_i\big(x_{\text{opt},i}(p)\big) - p^T(Rx_{\text{opt}}(p) - c) \\ x_{\text{opt},i}(p) &= \max\{0, U_i'^{-1}(q_i(p))\} \\ q(p) &= R^T p \end{aligned}$$

Note: By convexity, if p^* solves the dual problem, then $x_{\text{opt}}(p^*)$ solves the primal problem.

The Gradient Projection Algorithm

The gradient projection algorithm applied to the dual problems is

$$p_j(t+1) = \max\{0, p_j(t) - \gamma_j D_j h(p(t))\},$$

Where

$$D_j h(p) = c_j - y_{\text{opt},j}(p)$$
$$y_{\text{opt}}(p) = R x_{\text{opt}}(p).$$

In continuous-time, this corresponds to control laws

Link:

$$\dot{p}_j(t) = F^+(\gamma_j(y_j(t) - c_j), p(t))_0$$

Source:

$$x_i(t) = x_{\text{opt},i}(p) = (U_i')^{-1}(q_i(t))$$

Where

$$F^+(x,y)_k := \begin{cases} x & y \geq k \text{ or } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- q_i is the aggregate price seen by source i
- y_j is the aggregate rate seen by link j

Delay

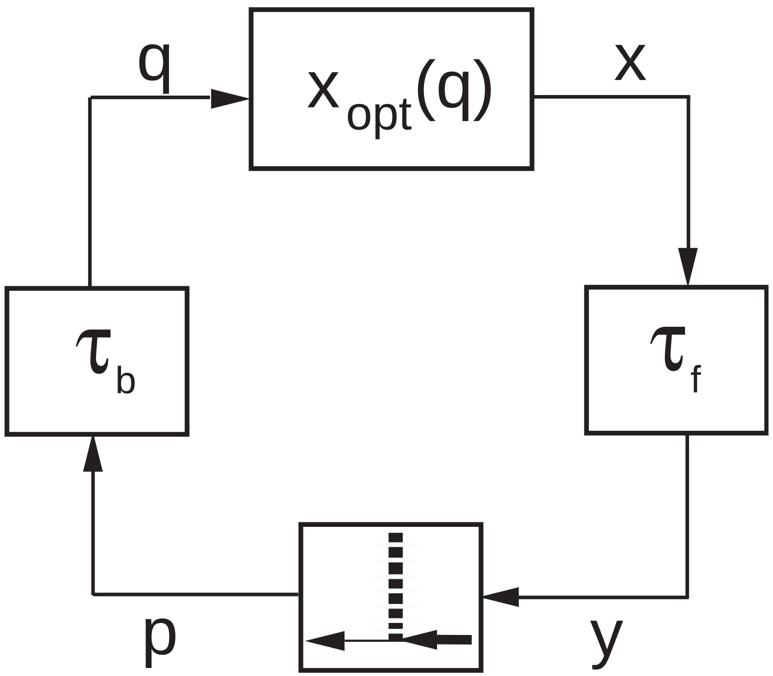
Due to the global reach of the internet, there is delay in the feedback.

Link:

$$\dot{p}_j(t) = F^+(\gamma_j(y_j(t-\tau_{f,j}) - c_j), p(t))_0$$

Source:

$$x_i(t) = x_{\text{opt},i}(q_i(t-\tau_{b,i}))$$



Linear Stability

A. Choose

$$U_i(x_i) = \frac{M_i \tau_i}{\alpha_i}\left(1 - \log \frac{x_i}{x_{m,i}}\right) \qquad \text{and} \qquad \gamma_j = \frac{1}{c_j}$$

- M_i is the number of congested links seen by source i
- $R x_m = c$

B. Linearize the dynamics about the equilibrium

Link:

$$\dot{p}_j(t) = -\frac{y_j(t)}{c_j}$$

Source:

$$x_i(t) = \frac{\alpha_i x_{m,i}}{\tau_i M_i} q_i(t)$$

C. Then the linear system is stable for $\alpha_i < \frac{\pi}{2}$.

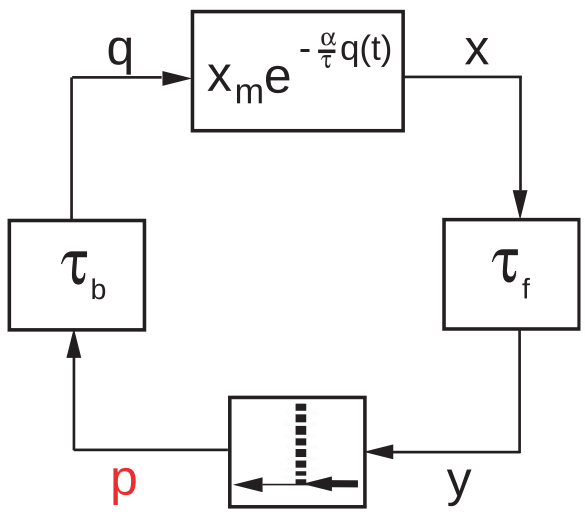
A Single Link with a Single Source

For a single source with a single link, the dynamics for queue price p are given by:

$$\dot{p}(t) = F^{+}\left(\frac{x_m}{c}e^{-\frac{\alpha}{\tau}p(t-\tau)} - 1, p(t)\right)_0$$

Main Point:

Because the dynamics are nonlinear, discontinuous and have delay, few analysis tools are powerful enough to directly address the stability question.

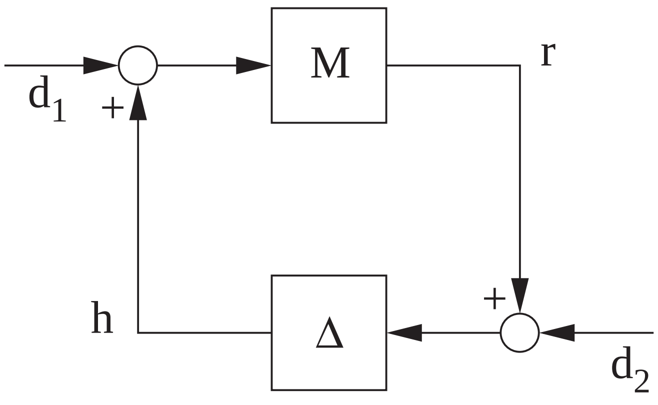


Passivity Theory

Consider the Interconnection of Operators M and Δ.

Assume

- M and Δ are bounded on L_2
- The interconnection of M and Δ is well-posed



Definition 1 *An operator M is passive if for any $u \in L_2$,*

$$\langle Mu, u \rangle \leq 0.$$

Where $\langle \cdot, \cdot \rangle$ is the inner product on L_2

Theorem 1 *The interconnection of two passive operators M and $-\Delta$ is L_2-stable.*

Theory of Integral Quadratic Constraints

For any bounded linear transformation Π, we define the following functional,

$$\langle u, w\rangle_\Pi := \langle \begin{bmatrix} u \\ w \end{bmatrix}, \Pi \begin{bmatrix} u \\ w \end{bmatrix} \rangle$$

Theorem 2 (Megretski and Rantzer,1997) *Modulo technical conditions, the interconnection of M and Δ is stable if for some $\epsilon > 0$, we have that for $u \in L_2$,*

$$\langle u, \Delta u\rangle_\Pi \geq 0 \qquad \textit{and} \qquad \langle Mu, u\rangle_\Pi \leq -\epsilon \|u\|^2.$$

Example: Small Gain

- Small-gain :

$$\|\Delta u\|^2 \leq k\|u\|^2$$
$$\|Mu\|^2 \leq 1/k\|u\|^2$$

- follows from

$$\Pi = \begin{bmatrix} kI & \\ & -I \end{bmatrix}$$

$$\langle u, w\rangle_{\Pi} := \langle \begin{bmatrix} u \\ w \end{bmatrix}, \begin{bmatrix} kI & \\ & -I \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} \rangle = k\|u\|^2 - \|w\|^2$$

Dynamics of a Single-Source/Single-Link

Replace $p(t) = p(t) - p_0$, where p_0 is the equilibrium.

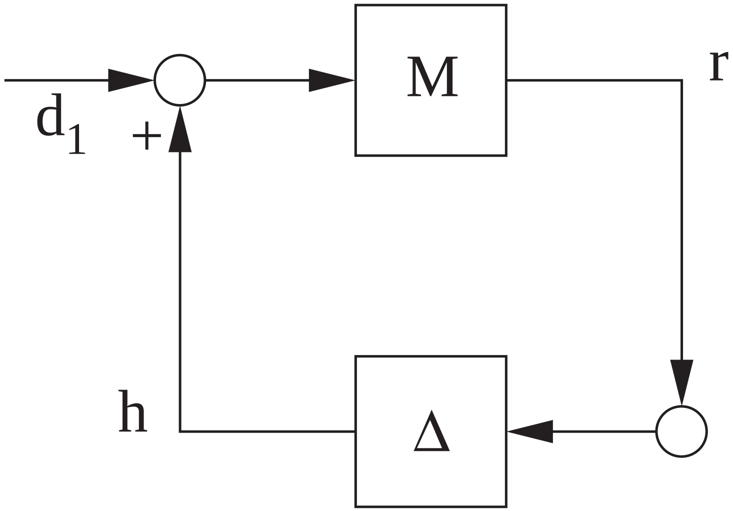
$$\dot{p}(t) = F^{+}\left(f\left(p(t-\tau)\right), p(t)\right)_{-p_0}$$

$$\text{where} \quad f(x) = e^{-\frac{\alpha}{\tau}x} - 1$$

Q: Is the system stable for any initial condition?

We can separate the delayed and nonlinear elements.

Decomposition: Decompose the system into the interconnection of a linear system, M, and a memoryless system, Δ.



Define:

- $\mathbf{M}$: $r = Mh$ if

$$\dot{r}(t) = h(t) - h(t-\tau) \qquad r(0) = 0$$

- $\boldsymbol{\Delta}$: $h = \Delta r$ if, for some z,

$$h(t) = \dot{z}(t) = F^{+}\left(f\left(z(t) - r(t)\right), z(t)\right)_{-p_0} \qquad z(0) = 0$$

Generalized Passivity We use the following separating functional:

$$\langle u, w\rangle_{\Pi} = \langle u, \frac{2}{\pi}(\dot{w} - u) + \beta w - u\rangle$$

Here $\beta = \alpha/(\alpha_{\max}\tau)$ where $\alpha_{\max}$ is a certain constant.

Proof Technique:

- $\langle u, \Delta u\rangle_{\Pi} \geq 0$ for all $u \in L_2$
- $\langle Mu, u\rangle_{\Pi} \leq -\epsilon\|u\|^2$ for all $u \in L_2$

Result:

- Although this Π-transformation is unbounded due to the presence of $\dot{w}$, the transformation is still valid under stricter conditions which are satisfied for this problem

Part 1: Time-Domain Approach

Recall: $w = \Delta u$ if, for some z,

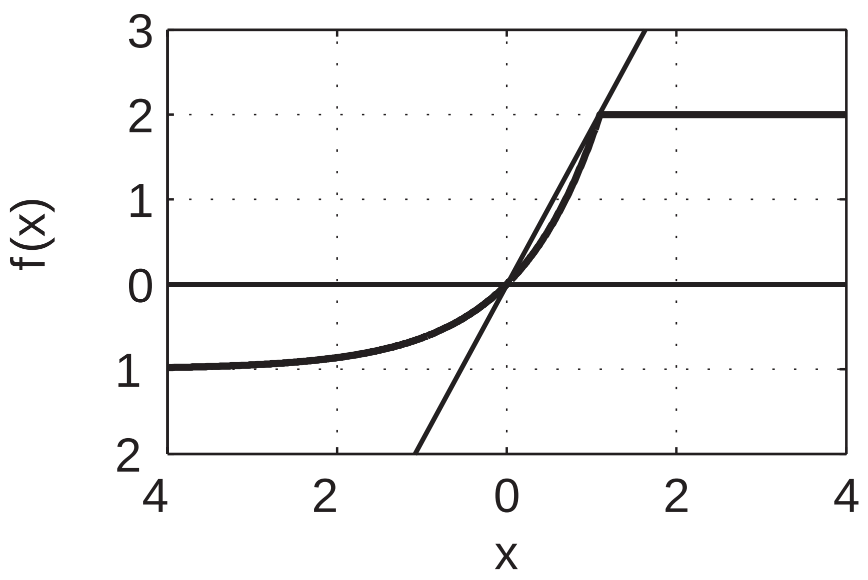
$$w(t) = \dot{z}(t) = F^{+}\left(f\left(z(t) - u(t)\right), z(t)\right)_{-p_0} \qquad z(0) = 0$$

Result:

- For all $w = \Delta u$,

$$\langle u, w\rangle_{\Pi} = \langle u, \frac{2}{\pi}(\dot{w} - u) + \beta w - u\rangle \geq 0$$

Note: The proof utilizes properties of solutions and a sector-bound of β on the nonlinearity.



Part 2: Frequency-Domain Approach

Recall: $r = Mh$ if

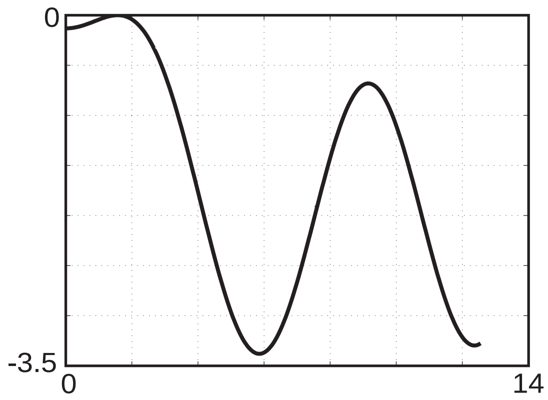
$$\dot{r}(t) = h(t) - h(t-\tau) \qquad r(0) = 0$$

Results:

- For $\beta\tau < \frac{\pi}{2}$,

$$\langle Mu, u\rangle_{\Pi} = \int_{-\infty}^{\infty} \hat{u}(j\omega)^* \left(\beta\tau\frac{\sin(\omega\tau)}{\omega\tau} - \frac{2}{\pi}\cos(\omega\tau) - 1\right)\hat{u}(j\omega)d\omega$$
$$\leq -\epsilon\|u\|^2$$

- See figure of $\frac{\pi}{2}\frac{\sin(\omega)}{\omega} - \frac{2}{\pi}\cos(\omega) - 1$ vs. ω



Main Result:

Theorem 3 *For $\alpha < \frac{\pi}{2}$, the interconnection is stable.*

This follows since $\alpha < \frac{\pi}{2}$ means $\beta\tau < \frac{\pi}{2}$ and so

$$\langle Mu, u\rangle_{\Pi} \leq -\epsilon\|u\|^2 \qquad \text{and} \qquad \langle u, \Delta u\rangle_{\Pi} \geq 0$$

Our Results:

- A global bound of $\alpha < \frac{\pi}{2}$ proves stability of nonlinear TCP(CDC, 2004).
- Exactly defines region of stability.

Practical Impact:

- Best previous bound was $\alpha < 1$
- 57% increase in gain parameter α
- 37% decrease in queues at equilibrium.

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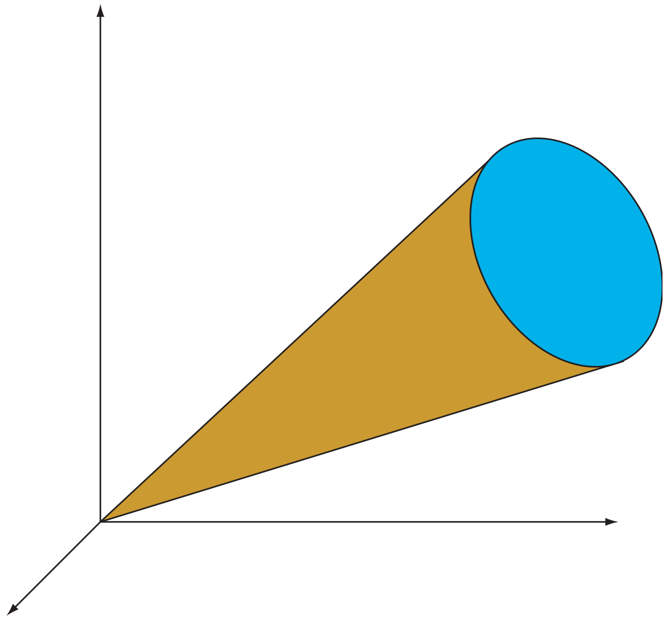
Next Topic:

- We use convex optimization to prove stability of general time-delay systems.

Convex Optimization



Problem:

$$\max \; c^T x$$
$$\text{subject to} \quad Ax + b \in C$$

The problem is a *convex optimization problem* if

- C is a convex cone.
- c and A are affine.

Computational Tractability: Convex Optimization over C is, in general, tractable if

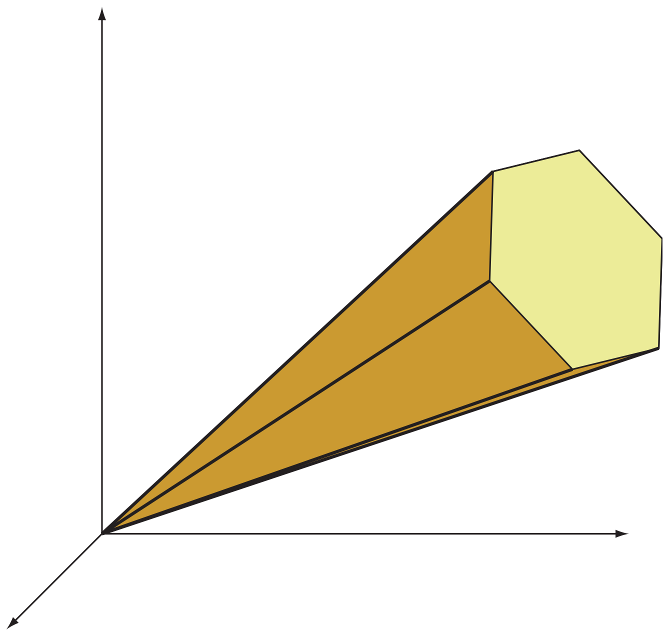
- There is an efficient set membership test for $x \in C$

Semidefinite Programming(SDP)

Problem:

$$\max\ c^T x$$

subject to $\quad A_0 + \sum_{i=1}^{m} A_i x_i \succeq 0$



Here

- $x \in \mathbb{R}^m$ and the A_i are symmetric matrices.
- The inequality $\succeq 0$ denotes membership in the convex cone of positive semidefinite matrices.

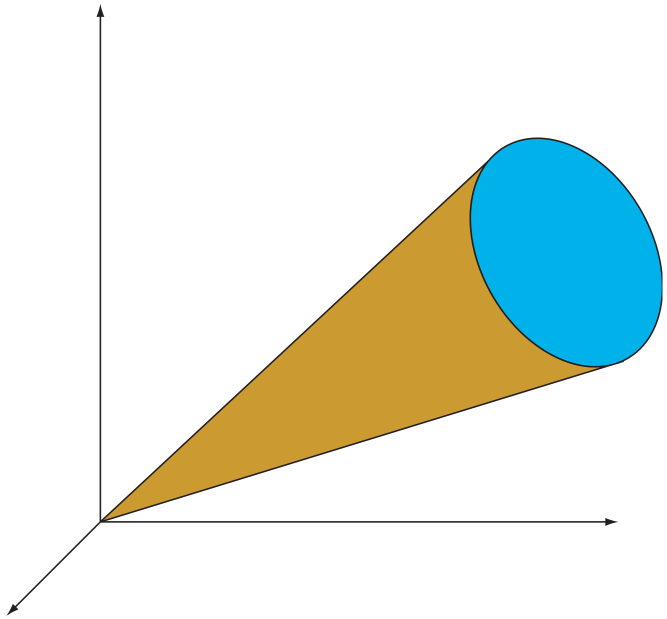
Computationally Tractable: Semidefinite programming problems can be solved efficiently using interior-point algorithms.

Polynomial Programming

Problem:

$$\max \ c^T x$$

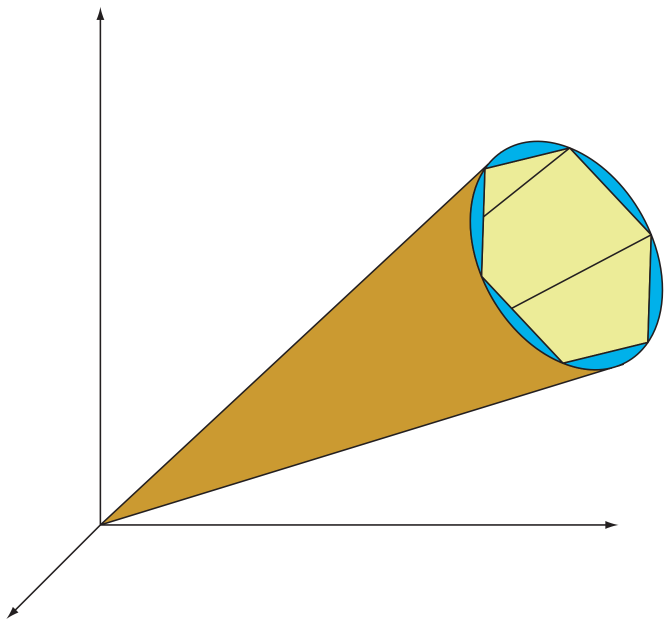
subject to $\quad A(y,x) \succeq 0 \quad \forall y$



Computationally Intractable:

- Testing whether $A(y,x) \succeq 0$ for all y is NP-hard.
- Many NP-hard problems can be recast as polynomial programming problems.

Sum-of-Squares(SOS) Programming



Problem:

$$\max\ c^T x$$

subject to $A(x,y) \in \Sigma_s$

- Σ_s is the convex cone of matrices of polynomials which can be represented as a *sum-of-squares* of some matrix polynomials G_i.

$$s(y) = \sum_{i=1}^{r} G_i(y)^T G_i(y)$$

Computationally Tractable: We can use SDP to test $M \in \Sigma_s$.

SOS Programming: Testing $M \in \Sigma_s$

Lemma 1 *Suppose M is polynomial of degree $2d$. Then $M \in \Sigma_s$ if and only if there exists some matrix $Q \succeq 0$ such that*

$$M(x) = Z(x)^T Q Z(x).$$

$Z(x)$ is the vector of monomial bases of degree d or less.

Example:

$$\begin{bmatrix}(y^2+1)z^2 & yz \\ yz & y^4+y^2-2y+1\end{bmatrix} = \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix}$$

$$= \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix} = \begin{bmatrix} yz & 1-y \\ z & y^2 \end{bmatrix}^T \begin{bmatrix} yz & 1-y \\ z & y^2 \end{bmatrix} \in \Sigma_s$$

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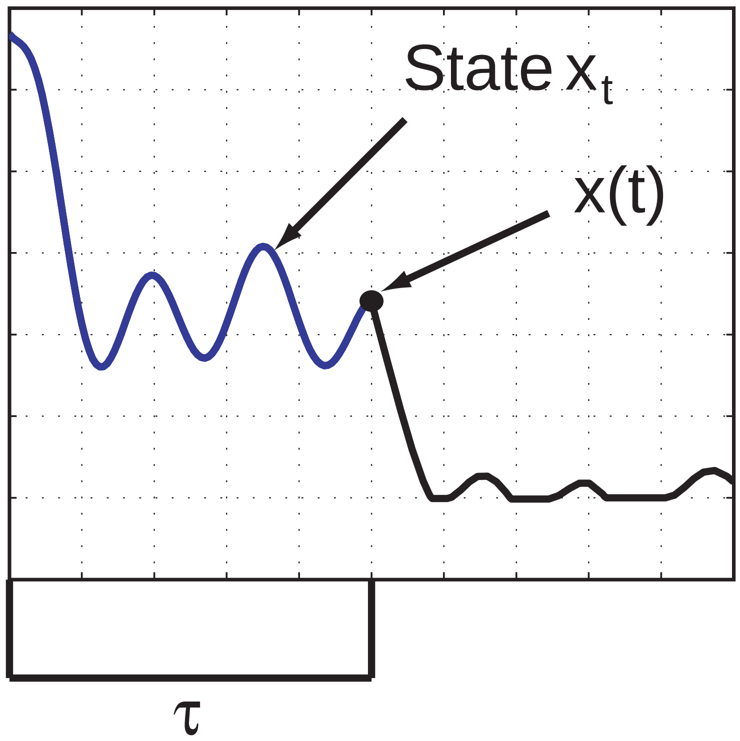
Next Topic:

- We can use Lyapunov theory to prove stability.

Functional Differential Equations

$$\dot{x}(t) = f(x_t)$$

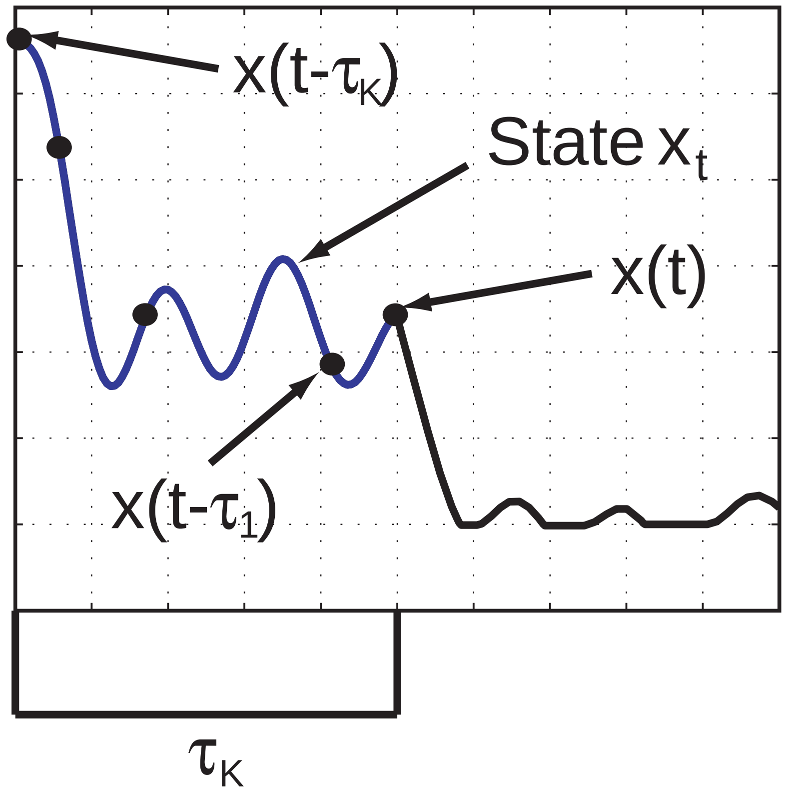
$$x_t(\theta) := x(t+\theta) \qquad \theta \in [-\tau, 0]$$



- Here $x(t) \in \mathbb{R}^n$ and $f : \mathcal{C}_\tau \to \mathbb{R}^n$.
- $x_t \in \mathcal{C}_\tau$ is the full state of the system at time t.
- $x(t) \in \mathbb{R}^n$ is the present state of the system at time t.
- $\mathcal{C}_\tau$ is the space of continuous functions defined in the interval $[-\tau, 0]$.

Time-Delay Systems

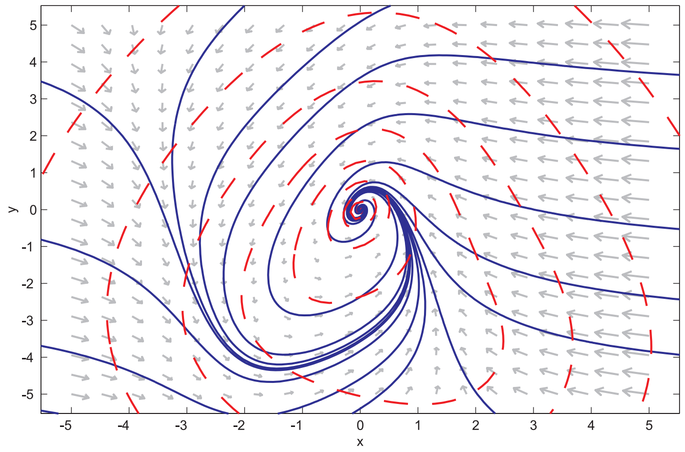
$$\dot{x}(t) = f(x(t), x(t-\tau_1), \ldots, x(t-\tau_K))$$



- Assume f is a polynomial.

Question: Is the System Stable?

Lyapunov-Krasovskii Functionals



Consider the functional differential equation

$$\dot{x}(t) = f(x_t) \tag{1}$$

Lyapunov Theory: System 1 is stable if there exists some function $V : \mathcal{C}_\tau \rightarrow \mathbb{R}$ for which the following holds for all $\phi \in \mathcal{C}_\tau$.

$$V(\phi) \geq \epsilon \|\phi(0)\|_2$$
$$\dot{V}(\phi) \leq 0$$

Here $\dot{V}(x)$ is the derivative of the functional along trajectories of the system.

Linear time-delay systems

$$\dot{x}(t) = \sum_{i=1}^{m} A_i x(t - \tau_i)$$

- Here $x(t) \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n \times n}$.
- We say the system has K delays, $\tau_i > \tau_{i-1}$ for $i = 1, \ldots, K$ and $\tau_0 = 0$

Question: Is the System Stable?

Converse Lyapunov Theorem:

Definition 2 *We say that V is a complete quadratic functional if can be represented as:*

$$V(\phi) = \int_{-\tau_K}^{0} \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix} d\theta + \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} \phi(\theta) R(\theta, \omega) \phi(\omega) d\theta d\omega$$

Theorem 4 *If a linear time-delay system is asymptotically stable, then there exists a complete quadratic functional, V, and $\eta > 0$ such that for all $\phi \in \mathcal{C}_\tau$*

$$V(\phi) \geq \eta \|\phi(0)\|^2 \qquad \textit{and} \qquad \dot{V}(\phi) \leq -\eta \|\phi(0)\|^2$$

Note: Furthermore, M and R can be taken to be continuous everywhere except possibly at points $\theta, \eta = -\tau_i$ for $i = 1, \ldots, K-1$.

Problem Statement

We would like to construct polynomials M and R such that

$$V(\phi) = \int_{-\tau_K}^{0} \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix} d\theta + \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} \phi(\theta) R(\theta,\omega) \phi(\omega) d\theta d\omega \geq \epsilon \|\phi(0)\|^2$$

and

$$\dot{V}(\phi) = \int_{-\tau_K}^{0} \begin{bmatrix} \phi(-\tau_0) \\ \vdots \\ \phi(-\tau_K) \\ \phi(\theta) \end{bmatrix}^T D(\theta) \begin{bmatrix} \phi(-\tau_0) \\ \vdots \\ \phi(-\tau_K) \\ \phi(\theta) \end{bmatrix} d\theta + \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} \phi(\theta) L(\theta,\omega) \phi(\omega) d\theta d\omega \leq 0$$

Where D and L are polynomials defined by the derivative.

Positive Quadratic Functionals

Consider the complete quadratic functional.

$$V(\phi) = \int_{-\tau_K}^{0} \begin{bmatrix}\phi(0)\\ \phi(\theta)\end{bmatrix}^T M(\theta) \begin{bmatrix}\phi(0)\\ \phi(\theta)\end{bmatrix} d\theta + \int_{-\tau_K}^{0}\int_{-\tau_K}^{0} \phi(\theta) R(\theta,\omega)\phi(\omega) d\theta d\omega$$

The complete quadratic Lyapunov functional is **positive** if

- $M \geq_{\mathbf{1}} 0$,
- $R \geq_{\mathbf{2}} 0$.

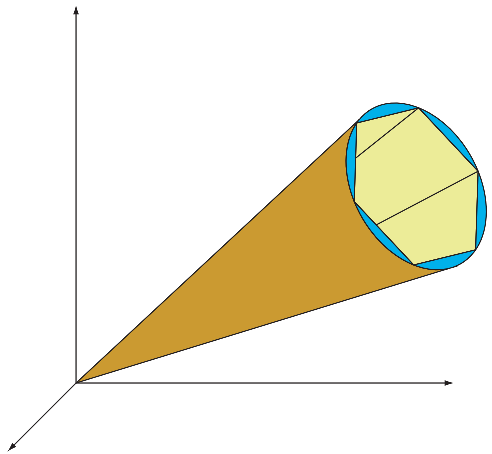
Definition 3 $M \geq_{\mathbf{1}} 0$ *if for all* $\phi \in \mathcal{C}_\tau$

$$\int_{-\tau_K}^{0} \begin{bmatrix}\phi(0)\\ \phi(\theta)\end{bmatrix}^T M(\theta) \begin{bmatrix}\phi(0)\\ \phi(\theta)\end{bmatrix} d\theta \geq 0$$

Definition 4 $R \geq_{\mathbf{2}} 0$ *if for all* $\phi \in \mathcal{C}_\tau$

$$\int_{-\tau_K}^{0}\int_{-\tau_K}^{0} \phi(\theta) R(\theta,\omega)\phi(\omega) d\theta d\omega \geq 0$$

Searching for Positive Quadratic Functionals



- $\geq_1$ and $\geq_2$ define convex cones.
- Q: How can we represent $\geq_1$ and $\geq_2$ for polynomials using SDP?

Note: Even for matrices, determining positivity on a subset is difficult. e.g. Matrix Copositivity

Result: Representing the Cone $\geq_1$

Theorem 5 *For a given M, the following are equivalent*

1. $M \geq_1 \epsilon I$ *for some* $\epsilon > 0$.
2. *There exists a function* T *and* $\epsilon' > 0$ *such that*

$$\int_{-\tau_K}^{0} T(\theta)d\theta = 0 \quad \textit{and} \quad M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & 0 \end{bmatrix} \succeq \epsilon' I$$

Computationally Tractable:

- Assume M and T are polynomials
- The constraint $\int_{-\tau_K}^{0} T(\theta)d\theta = 0$ is linear
- For the 1-D case, Σ_s is exact.

$$\geq_1 \longrightarrow \Sigma_s \longrightarrow \text{SDP}$$

Example: Positive Multipliers

$$M(\theta) = \begin{bmatrix} -2\theta^2+2 & \theta^3-\theta \\ \theta^3-\theta & \theta^4+\theta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \theta & 0 \\ 0 & \theta \\ 0 & \theta^2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \theta & 0 \\ 0 & \theta \\ 0 & \theta^2 \end{bmatrix} + \begin{bmatrix} 3\theta^2-1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \theta & 0 \\ 0 & \theta \\ 0 & \theta^2 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \theta & 0 \\ 0 & \theta \\ 0 & \theta^2 \end{bmatrix} + \begin{bmatrix} 3\theta^2-1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \theta & \theta^2 \\ 1 & -\theta \end{bmatrix}^T \begin{bmatrix} \theta & \theta^2 \\ 1 & -\theta \end{bmatrix} + \begin{bmatrix} 3\theta^2-1 & 0 \\ 0 & 0 \end{bmatrix} \geq_{\mathbf{1}} 0$$

Since

$$\int_{-1}^{0} (3\theta^2 - 1) d\theta = 0$$

Result: Positive Integral Operators

Theorem 6 $R \geq_2 0$ *if there exists a* $Q \succeq 0$ *such that*

$$R(\theta, \omega) = Z(\theta)^T Q Z(\omega).$$

$Z(\theta)$ *is the* $(d+1)$*-dimensional vector of powers of* θ *of degree* d *or less.*

Results:

- We can construct operators with finite rank and with polynomial eigenvectors.
- We can also construct operators with piecewise-polynomial eigenvectors

Computationally Tractable:

- Map is affine
- Can use SDP.

$$\geq_2 \rightarrow \text{SDP}$$

Example: Positive Integral Operators

If

$$R(\theta,\omega)=\begin{bmatrix}1-\omega-\theta+2\theta\omega & 1-\theta-\theta\omega^2\\ 1-\omega-\theta^2\omega & 1+\theta^2\omega^2\end{bmatrix}=\begin{bmatrix}1&0\\ \theta&0\\0&1\\0&\theta^2\end{bmatrix}^T\begin{bmatrix}1&-1&1&0\\-1&2&-1&-1\\1&-1&1&0\\0&-1&0&1\end{bmatrix}\begin{bmatrix}1&0\\ \omega&0\\0&1\\0&\omega^2\end{bmatrix}$$

$$=\begin{bmatrix}1&0\\ \theta&0\\0&1\\0&\theta^2\end{bmatrix}^T\begin{bmatrix}1&-1&1&0\\0&-1&0&1\end{bmatrix}^T\begin{bmatrix}1&-1&1&0\\0&-1&0&1\end{bmatrix}\begin{bmatrix}1&0\\ \omega&0\\0&1\\0&\omega^2\end{bmatrix}=\begin{bmatrix}1-\theta&1\\-\theta&\theta^2\end{bmatrix}^T\begin{bmatrix}1-\omega&1\\-\omega&\omega^2\end{bmatrix}\geq_{\mathbf{2}}0$$

Then

$$\int_{-\tau}^0\int_{-\tau}^0 x(\theta)^T R(\theta,\omega)x(\omega)d\theta d\omega=\int_{-\tau}^0\int_{-\tau}^0 x(\theta)^T G(\theta)^T G(\omega)x(\omega)d\theta d\omega$$

$$=\int_{-\tau}^0 x(\theta)^T G(\theta)^T d\theta\int_{-\tau}^0 G(\omega)x(\omega)d\omega=K^TK\geq 0$$

The Derivative of Positive Quadratic Functionals

If $M \geq_1 0$ and $R \geq_2 0$, then $V(\phi) \geq 0$. However, the derivative of V is given by

$$\dot{V}(\phi) = \int_{-\tau_K}^{0} \begin{bmatrix} \phi(-\tau_0) \\ \vdots \\ \phi(-\tau_K) \\ \phi \end{bmatrix}^T D(\theta) \begin{bmatrix} \phi(-\tau_0) \\ \vdots \\ \phi(-\tau_K) \\ \phi \end{bmatrix} d\theta + \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} \phi(\theta) L(\theta, \omega) \phi(\omega) d\theta d\omega \leq 0$$

The derivative is **negative** if

- $L \geq_2 0$
- $D \geq_3 0$

Definition 5 $D \geq_3 0$ *if for all* $\phi \in \mathcal{C}_\tau$

$$\int_{-\tau_K}^{0} \begin{bmatrix} \phi(-\tau_0) \\ \vdots \\ \phi(-\tau_K) \\ \phi \end{bmatrix}^T D(\theta) \begin{bmatrix} \phi(-\tau_0) \\ \vdots \\ \phi(-\tau_K) \\ \phi \end{bmatrix} d\theta \geq 0$$

Result: We can use a generalization of Theorem 5 for $D \geq_3 0$

A Lyapunov Inequality

Theorem 7 *The linear time-delay system is asymptotically stable if there exist polynomials M and R and constant $\eta > 0$ such that*

Positive Functional:

- $M \geq_1 \eta I$
- $R \geq_2 0$

Negative Derivative:

- $D \leq_3 -\eta I$
- $L \leq_2 0$

Where for a single delay,

$$D(\theta) = \begin{bmatrix} D_{11} & PB - Q(-\tau) & \tau(A^T Q(\theta) - \dot{Q}(\theta) + R(0,\theta)) \\ *^T & -S(-\tau) & \tau(B^T Q(\theta) - R(-\tau,\theta)) \\ *^T & *^T & -\tau \dot{S}(\theta) \end{bmatrix}$$

$$L(\theta,\omega) = \frac{d}{d\theta} R(\theta,\omega) + \frac{d}{d\omega} R(\theta,\omega)$$

$$D_{11} = PA + A^T P + Q(0) + Q(0)^T + S(0)$$

where we represent M as $M(\theta) = \begin{bmatrix} P & \tau Q(\theta) \\ \tau Q(\theta)^T & \tau S(\theta) \end{bmatrix}$

Example: Standard Test Case 1 - Single Delay

We apply the algorithm to a standard test problem.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - \tau)$$

We use a bisection method to determine the minimum and maximum stable τ. The results are compared to a piecewise-linear method by Gu et al. and the SDP is solved using SeDuMi.

Our results			Piecewise Functional		
d	τ_{min}	τ_{max}	N_2	τ_{min}	τ_{max}
1	.10017	1.6249	1	.1006	1.4272
2	.10017	1.7172	2	.1003	1.6921
3	.10017	1.71785	3	.1003	1.7161
Analytic	.10017	1.71785			

Table 1: τ_{max} and τ_{min} for discretization level N_2 and for degree d and compared to the analytical limit

Example: Standard Test Case 2 - Multiple Delays

We now consider a system with multiple delays.

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{9}{10} \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \left[\frac{1}{20} x(t - \frac{\tau}{2}) + \frac{19}{20} x(t-\tau) \right]$$

The delays are commensurate so one can more easily compare results. Again a bisection method was used and results are listed below.

Our Approach			Piecewise Functional		
d	$\tau_{\min}$	$\tau_{\max}$	N_2	$\tau_{\min}$	$\tau_{\max}$
1	.20247	1.354	1	.204	1.35
2	.20247	1.3722	2	.203	1.372
Analytic	.20246	1.3723			

Table 2: τ_{max} and τ_{min} using a piecewise-linear functional and our approach and compared to the analytical limit.

Parametric Uncertainty

Result: We can construct parameter-dependent Lyapunov functionals.

Approach: We replace the semidefinite programming constraint

$$Q \succeq 0$$

with the SOS programming constraint

$$Q(\alpha) \in \Sigma_s.$$

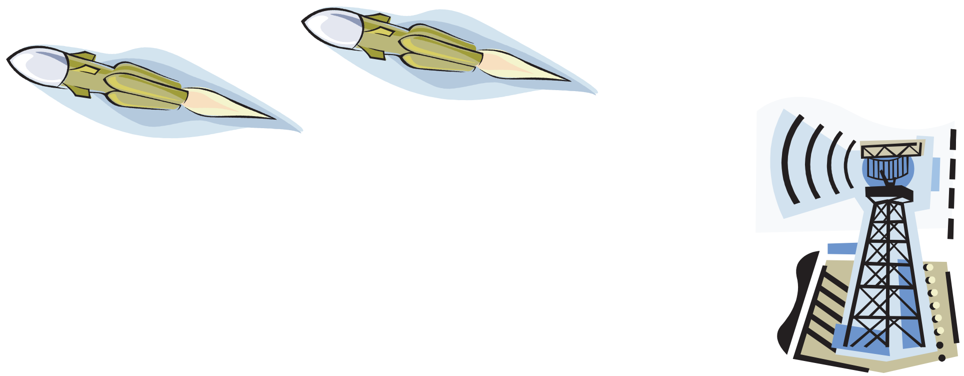
Example: Standard Test Case 1 Revisited

By including τ as an uncertain parameter in the Lyapunov functionals, we can prove stability over the interval $[\tau_{\min}, \tau_{\max}]$ directly.

d in τ	d in θ	$\tau_{\min}$	$\tau_{\max}$
1	1	.1002	1.6246
1	2	.1002	1.717
Analytic		.10017	1.71785

Table 3: Stability on the interval $[\tau_{\min}, \tau_{\max}]$ vs. degree using a parameter-dependent functional

Example: Remote Control



A Simple Inertial System: Suppose we are given a specific type of PD controller that we want to implement.

$$\ddot{x}(t) = -ax(t) - \frac{a}{2}\dot{x}(t)$$

The controller is stable for all positive a. Now suppose we want to maintain control from a remote location. When we include the **communication delay**, the equation becomes.

$$\ddot{x}(t) = -ax(t-\tau) - \frac{a}{2}\dot{x}(t-\tau)$$

Question: For what range of a and τ will the controller be stable. The model is linear, but contains a parameter and an uncertain delay.

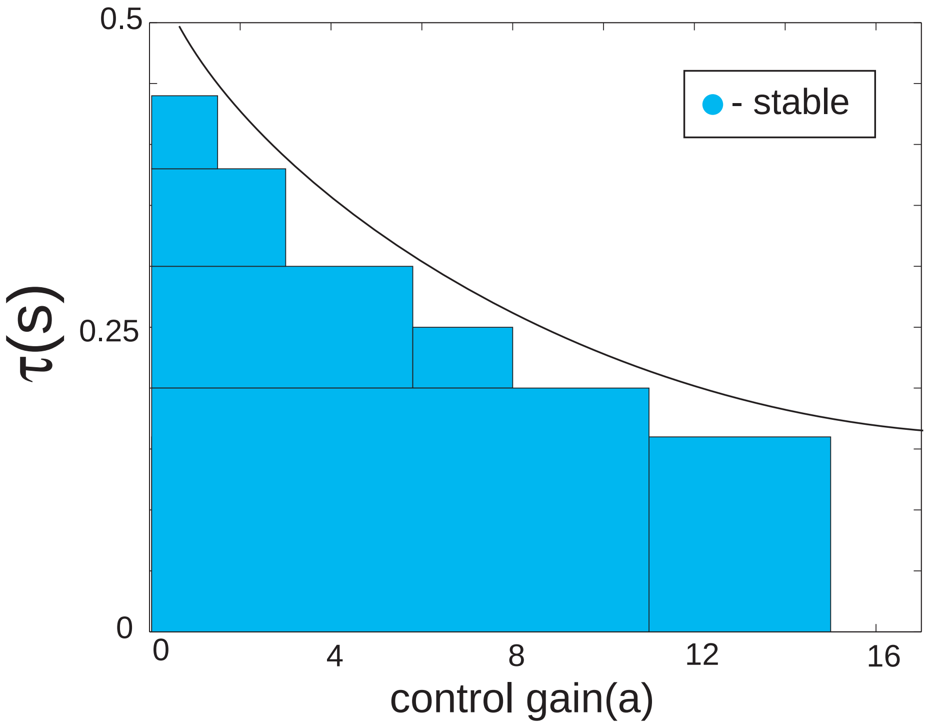
Example: Remote Control

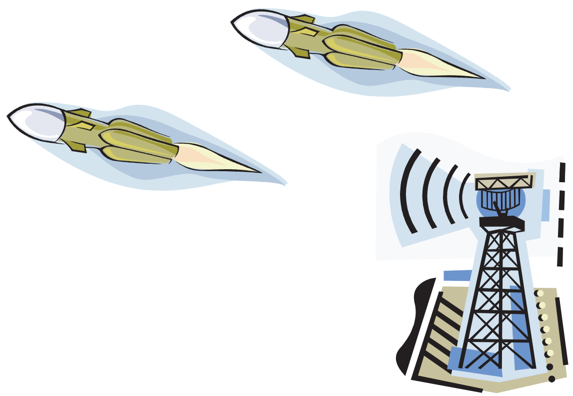
Recall that we considered an inertial system controlled remotely using PD control

$$\ddot{x}(t) = -ax(t-\tau) - \frac{a}{2}\dot{x}(t-\tau)$$

Question: For what range of a and τ will the controller be stable?

- We use parameter-dependent functionals.





Research Overview:

Decentralized Optimization

$\Downarrow$

Internet Congestion Control

$\Downarrow$

Convex Optimization

$\Downarrow$

Linear Time-Delay Systems

$\Downarrow$

Nonlinear Time-Delay Systems

Next Topic:

- We generalize the complete quadratic functional.

Nonlinear Time-delay systems

Consider nonlinear systems which have a single delay.

$$\dot{x}(t) = f(x(t), x(t-\tau_1), \dots, x(t-\tau_K))$$

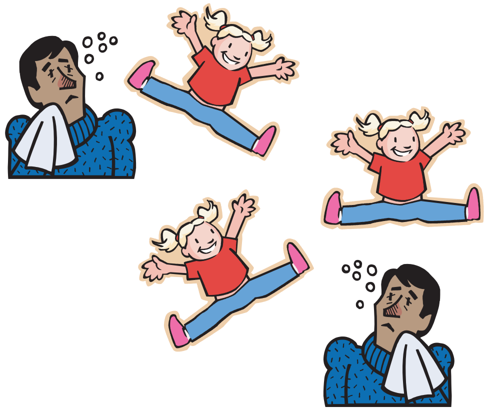
Here we assume $x(t) \in \mathbb{R}^n$ and f is polynomial.

We use a generalization of the compete quadratic functional of the following form.

$$\begin{aligned} V(\phi) :&= \int_{-\tau_K}^{0} f_1(\phi(0), \phi(\theta), \theta) d\theta + \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} f_2(\phi(\theta), \phi(\omega), \theta, \omega) d\theta d\omega \\ &= \int_{-\tau_K}^{0} Z(\phi(0), \phi(\theta))^T M(\theta) Z(\phi(0), \phi(\theta)) d\theta \\ &+ \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} Z(\phi(\theta))^T R(\theta, \omega) Z(\phi(\omega)) d\theta d\omega \end{aligned}$$

Computation: We represent M and R using results generalized from the linear case.

Example: Epidemiological Model of Infection



Consider a human population subject to non-lethal infection by a cold virus. The disease has **incubation period** (τ). Cooke(1978) models the percentage of infected humans(y) using the following equation.

$$\dot{y}(t) = -ay(t) + by(t-\tau)\left[1 - y(t)\right]$$

Where

- a is the rate of recovery for infected persons
- b is the rate of infection for exposed people

The model is nonlinear and contains delay. Equilibria exist at $y^* = 0$ and $y^* = (b-a)/b$.

Example: Epidemiological Model

Recall the dynamics of infection are given by

$$\dot{y}(t) = -ay(t) + by(t-\tau)\left[1 - y(t)\right]$$

Cooke used the following Lyapunov functional to prove delay-independent stability of the 0 equilibrium for $a > b > 0$.

$$V(\phi) = \frac{1}{2}\phi(0)^2 + \frac{1}{2}\int_{-\tau}^{0} a\phi(\theta)^2 d\theta$$

Using semidefinite programming, we were also able to prove delay-independent stability for $a > b > 0$ using the following functional.

$$V(\phi) = 1.75\phi(0)^2 + \int_{-\tau}^{0} (1.47a + .28b)\phi(\theta)^2 d\theta$$

Conclusion: When the rate of recovery is greater than the rate of infection, the epidemic will die out.

Our Results:

- An entirely new approach to solving the Lyapunov inequality
- Provides best stability conditions available
- Only algorithm to address general parametric uncertainty
- Uses the most general form of non-quadratic Lyapunov functional available

Practical Impact:

- Linear with Time-Delay

  Numerically well-conditioned and convergent

  We can show that relatively large linear time-delay systems are stable
- Uncertain with Time-Delay

  We can prove stability over ranges of operating conditions
- Nonlinear with Time-Delay

  Provides an easy way of testing stability of very complicated systems

Conclusion

Topics

- Internet Congestion Control
- Global Stability
- Linear Time-Delay Systems
- Nonlinear Time-Delay Systems

Results

A proof of stability of Internet Congestion Control

- Extends previous results to the delayed nonlinear case(CDC, 2004)

An algorithm for solving Lyapunov's operator inequality(ACC, 2005)

- Extended to systems with parametric uncertainty(CDC, 2006)
- Extended to nonlinear systems(NOLCOS, 2004)

Research Directions

Theory

- Stabilizing Controllers
- Partial Differential Equations
- Optimal Controller Synthesis
- The KYP lemma

Applications

Industrial and Electrical:

- Communication Systems
- Manufacturing

Biological:

- Cancer Therapy
- HIV Therapy

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