Reformulation of a Delay System using Fundamental State: Proofs and Lemmas

Matthew M. Peet, *Member, IEEE,*

Abstract

A PIE Representation of coupled ODE-PDE systems

Index Terms

PDE Systems

I. Problem Formulation

In this note, we focus on translating the most general form of a time-delay system to the differential-difference formulation and from hence to the ODE-PDE formulation and from hance to the PIE formulation.

II. General Representation of ODEs coupled with PDEs

$$\begin{bmatrix} z(t) \\ y(t) \\ \dot{x}(t) \\ \dot{\mathbf{x}}(t,s) \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & C_1 & \mathcal{C}_{1p} \\ D_{21} & D_{22} & C_2 & \mathcal{C}_{2p} \\ B_{11} & B_{12} & A & \mathcal{E}_p \\ B_{21}(s) & B_{22}(s) & E(s) & \mathcal{A}_p \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}(t,s) \end{bmatrix} \tag{1}$$

or

$$\begin{bmatrix} z(t) \\ y(t) \\ \dot{x}(t) \\ \dot{\mathbf{x}}(t,s) \end{bmatrix} = \mathcal{P}\left[\begin{smallmatrix} \mathbf{P}, & \mathbf{Q}_1 \\ \mathbf{Q}_2, & \{\mathbf{R}_i\} \end{smallmatrix}\right] \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \Lambda_1 \mathbf{x}(t) \\ \Lambda_2 \mathbf{x}(t) \end{bmatrix} \tag{2}$$

where

$$\Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix}, \quad \Lambda_1 = \begin{bmatrix} 0 & \Delta_a & 0 \\ 0 & \Delta_b & 0 \\ 0 & 0 & \Delta_a \\ 0 & 0 & \Delta_b \\ 0 & 0 & \Delta_a \partial_s \\ 0 & 0 & \Delta_b \partial_s \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & \partial_s & 0 \\ 0 & 0 & \partial_s \\ 0 & 0 & \partial_{ss} \end{bmatrix}$$

M. Peet is with the School for the Engineering of Matter, Transport and Energy, Arizona State University, Tempe, AZ, 85298 USA. e-mail: mpeet@asu.edu

and where Δ_a is the dirac operator at point a.

$$\mathbf{P} = \begin{bmatrix} D_{11} & D_{12} & C_1 & C_{10} \\ D_{21} & D_{22} & C_2 & C_{20} \\ B_{11} & B_{12} & A & E_0 \end{bmatrix} \qquad \mathbf{Q}_1(s) = \begin{bmatrix} C_{a1}(s) & C_{b1}(s) & C_{c1}(s) \\ C_{a2}(s) & C_{b2}(s) & C_{c2}(s) \\ E_a(s) & E_b(s) & E_c(s) \end{bmatrix}$$

$$\mathbf{Q}_2(s) = \begin{bmatrix} B_{21}(s) & B_{22}(s) & E(s) & E_{???}(s) \end{bmatrix}, \quad \mathbf{R}_0(s) = \begin{bmatrix} A_0(s) & A_1(s) & A_2(s) \end{bmatrix}, \quad \mathbf{R}_1(s,\theta) = 0, \quad \mathbf{R}_2(s,\theta) = 0$$

A. Boundary Condition

$$B\Lambda_1 \mathbf{x}(t,s) = \underbrace{\begin{bmatrix} B_w & B_u & B_{xo} & \mathbf{B}_{xx} \end{bmatrix}}_{\mathcal{B}_c = \mathcal{P}\begin{bmatrix} \begin{bmatrix} B_w & B_u & B_{xo} \end{bmatrix}, & \mathbf{B}_{xx} \\ \emptyset, & \{\emptyset\} \end{bmatrix}} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t,s) \end{bmatrix} = \mathcal{B}_c \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t,s) \end{bmatrix}$$

III. De-state

$$\mathbf{x}(t,s) := \begin{bmatrix} \mathbf{x}_0(s) \\ \mathbf{x}_1(s) \\ \mathbf{x}_2(s) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0(s) \\ \mathbf{x}_1(a) + \int_a^s \mathbf{x}_{1s}(\theta) d\theta \\ \mathbf{x}_2(a) + (s-a)\mathbf{x}_{2s}(a) + \int_a^s (s-a)\mathbf{x}_{2ss}(\theta) d\theta \end{bmatrix}$$

$$= \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \emptyset, & \left\{ \{ \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (s-\theta) \end{bmatrix}, 0\} \right\} \end{bmatrix} \mathbf{x}_f(t,s) + \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & (s-a)I \end{bmatrix}, & \{\emptyset\} \end{bmatrix} \begin{bmatrix} 0 & \Delta_a & 0 \\ 0 & 0 & \Delta_a \\ 0 & 0 & \Delta_a \partial_s \end{bmatrix} \mathbf{x}(t,s)$$

$$= \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & (s-a)I \end{bmatrix}, & \left\{ \{ \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (s-\theta) \end{bmatrix}, 0\} \right\} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & \Delta_a & 0 \\ 0 & 0 & \Delta_a \\ 0 & 0 & \Delta_a \partial_s \end{bmatrix} \mathbf{x}(t,s) \\ \mathbf{x}_f(t,s) \end{bmatrix}$$

Now

$$\Lambda_1 \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & (s-a)I \end{bmatrix}, & \left\{ \{ \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (s-\theta) \end{bmatrix}, 0\} \right\} \end{bmatrix} = \begin{bmatrix} 0 & \Delta_a & 0 \\ 0 & \Delta_b & 0 \\ 0 & 0 & \Delta_a \\ 0 & 0 & \Delta_b \\ 0 & 0 & \Delta_a \partial_s \\ 0 & 0 & \Delta_b \partial_s \end{bmatrix} \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & (s-a)I \end{bmatrix}, & \left\{ \{ \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (s-\theta) \end{bmatrix}, 0\} \right\} \end{bmatrix}$$

$$= \mathcal{P}\begin{bmatrix} \overbrace{\begin{bmatrix} I & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & (b-a)I \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix}}^{T}, & \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (b-s) \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}}^{Q(s)} \\ \emptyset, & \{\emptyset\} \end{bmatrix} = \mathcal{P}\begin{bmatrix} T, & Q(s) \\ \emptyset, & \{\emptyset\} \end{bmatrix}$$

where we used

$$\partial_s \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ g(s), & \{\{0, R_1(s,\theta), 0\}\} \end{bmatrix} = \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \partial_s g(s), & \{\{R_1(s,s), \partial_s R_1(s,\theta), 0\}\} \end{bmatrix}$$

and

$$\Delta_b \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ Q_2(s), & \{\{0, R_1(s,\theta), 0\}\} \end{bmatrix} = \mathcal{P}\begin{bmatrix} Q_2(b), & R_1(b,s) \\ \emptyset, & \{\emptyset\} \end{bmatrix}$$

and

$$\Delta_a \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ Q_2(s), & \{\{0, R_1(s,\theta), 0\}\} \end{bmatrix} = \mathcal{P}\begin{bmatrix} Q_2(a), & 0 \\ \emptyset, & \{\emptyset\} \end{bmatrix}$$

Finally, we have

$$B\Lambda_1 \mathbf{x}(t,s) = \mathcal{P}\begin{bmatrix} BT, & BQ(s) \\ \emptyset, & \{\emptyset\} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & \Delta_a & 0 \\ 0 & 0 & \Delta_a \\ 0 & 0 & \Delta_a \partial_s \end{bmatrix} \mathbf{x}(t,s) \\ \mathbf{x}_f(t,s) \end{bmatrix} = \mathcal{P}\begin{bmatrix} \begin{bmatrix} B_w & B_u & B_{xo} \end{bmatrix}, & \mathbf{B}_{xx} \\ \emptyset, & \{\emptyset\} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t,s) \end{bmatrix}$$

or

$$\mathcal{P}\begin{bmatrix} BT, & \emptyset \\ \emptyset, & \{\emptyset\} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & \Delta_a & 0 \\ 0 & 0 & \Delta_a \\ 0 & 0 & \Delta_a \partial_s \end{bmatrix} \mathbf{x}(t,s) \end{bmatrix} = \mathcal{P}\begin{bmatrix} \begin{bmatrix} B_w & B_u & B_{xo} \end{bmatrix}, & \mathbf{B}_{xx} - BQ(s) \\ \emptyset, & \{\emptyset\} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t,s) \end{bmatrix}$$

or

$$\begin{bmatrix} \begin{bmatrix} 0 & \Delta_a & 0 \\ 0 & 0 & \Delta_a \\ 0 & 0 & \Delta_a \partial_s \end{bmatrix} \mathbf{x}(t,s) \end{bmatrix} = \mathcal{P}\begin{bmatrix} (BT)^{-1} \begin{bmatrix} B_w & B_u & B_{xo} \end{bmatrix}, & (BT)^{-1}(\mathbf{B}_{xx}(s) - BQ(s)) \\ \emptyset, & \{\emptyset\} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t,s) \end{bmatrix}$$

In conclusion, we have

$$
\begin{aligned}
\mathbf{x}(t,s) &= \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \emptyset, & \left\{ \{ \underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{R}_{0f}=L_0}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (s-\theta) \end{bmatrix}}_{L_1(s,\theta)}, 0\} \right\} \end{bmatrix} \mathbf{x}_f(t,s) + \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & (s-a)I \end{bmatrix}}_{K(s)}, & \{\emptyset\} \end{bmatrix} \begin{bmatrix} 0 & \Delta_a & 0 \\ 0 & 0 & \Delta_a \\ 0 & 0 & \Delta_a \partial_s \end{bmatrix} \mathbf{x}(t,s) \\
&= \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \emptyset, & \{\{L_0, L_1(s,\theta), 0\}\} \end{bmatrix} \mathbf{x}_f(t,s) + \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ K(s), & \{\emptyset\} \end{bmatrix} \mathcal{P}\begin{bmatrix} (BT)^{-1}\begin{bmatrix} B_w & B_u & B_{xo} \end{bmatrix}, & (BT)^{-1}(\mathbf{B}_{xx}(s) - BQ(s)) \\ \emptyset, & \{\emptyset\} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t,s) \end{bmatrix} \\
&= \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \underbrace{K(s)(BT)^{-1}\begin{bmatrix} B_w & B_u & B_{xo} \end{bmatrix}}_{\mathbf{Q}_{2f}}, & \left\{ \{\mathbf{R}_{0f}, \underbrace{L_1(s,\theta) + K(s)(BT)^{-1}(\mathbf{B}_{xx}(\theta) - BQ(\theta))}_{\mathbf{R}_{1f}}, \underbrace{K(s)(BT)^{-1}(\mathbf{B}_{xx}(\theta) - BQ(\theta))}_{\mathbf{R}_{2f}}\} \right\} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t,s) \end{bmatrix} \\
&= \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \mathbf{Q}_{2f}, & \left\{\{\mathbf{R}_{0f}, \mathbf{R}_{1f}, \mathbf{R}_{2f}\}\right\} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t,s) \end{bmatrix}
\end{aligned}
$$

IV. Reformulation

A. LHS

Obviously,

$$
\begin{bmatrix} z(t) \\ y(t) \\ \dot{x}(t) \\ \dot{\mathbf{x}}(t,s) \end{bmatrix} = \mathcal{P}\begin{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, & 0 \\ \begin{bmatrix} 0 & 0 & \mathbf{Q}_{2f} \end{bmatrix}, & \left\{\{\mathbf{R}_{0f}, \mathbf{R}_{1f}, \mathbf{R}_{2f}\}\right\} \end{bmatrix} \begin{bmatrix} z(t) \\ y(t) \\ \dot{w}(t) \\ \dot{u}(t) \\ \dot{x}(t) \\ \dot{\mathbf{x}}_f(t,s) \end{bmatrix}
$$

B. $\Lambda_1 \mathbf{x}$

Now recall

$$
\left[\begin{bmatrix} 0 & \Delta_a & 0 \\ 0 & 0 & \Delta_a \\ 0 & 0 & \Delta_a \partial_s \end{bmatrix} \mathbf{x}(t,s) \right] = \mathcal{P}\begin{bmatrix} (BT)^{-1}\begin{bmatrix} B_w & B_u & B_{xo} \end{bmatrix}, & (BT)^{-1}(\mathbf{B}_{xx}(s) - BQ(s)) \\ \emptyset, & \{\emptyset\} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t,s) \end{bmatrix}
$$

and

$$\Lambda_1 \mathcal{P}\left[\begin{matrix} \emptyset, & \emptyset \\ \begin{bmatrix}0&0&0\\I&0&0\\0&I&(s-a)I\end{bmatrix}, & \left\{\left\{\begin{bmatrix}I&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&I&0\\0&0&(s-\theta)\end{bmatrix}, 0\right\}\right\}\end{matrix}\right] = \mathcal{P}\begin{bmatrix} T, & Q(s) \\ \emptyset, & \{\emptyset\}\end{bmatrix}$$

$$\begin{aligned}
\Lambda_1 \mathbf{x}(t,s) &= \Lambda_1 \mathcal{P}\left[\begin{matrix} \emptyset, & \emptyset \\ \begin{bmatrix}0&0&0\\I&0&0\\0&I&(s-a)I\end{bmatrix}, & \left\{\left\{\begin{bmatrix}I&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&I&0\\0&0&(s-\theta)\end{bmatrix}, 0\right\}\right\}\end{matrix}\right] \begin{bmatrix} \begin{bmatrix}0&\Delta_a&0\\0&0&\Delta_a\\0&0&\Delta_a\partial_s\end{bmatrix}\mathbf{x}(t,s) \\ \mathbf{x}_f(t,s)\end{bmatrix} \\
&= \mathcal{P}\begin{bmatrix} T, & Q(s) \\ \emptyset, & \{\emptyset\}\end{bmatrix} \begin{bmatrix} \begin{bmatrix}0&\Delta_a&0\\0&0&\Delta_a\\0&0&\Delta_a\partial_s\end{bmatrix}\mathbf{x}(t,s) \\ \mathbf{x}_f(t,s)\end{bmatrix} = \mathcal{P}\begin{bmatrix} T, & Q(s) \\ \emptyset, & \{\emptyset\}\end{bmatrix} \mathcal{P}\begin{bmatrix} (BT)^{-1}\begin{bmatrix}B_w & B_u & B_{xo}\end{bmatrix}, & (BT)^{-1}(\mathbf{B}_{xx}(s) - BQ(s)) \\ 0, & \{\{I,0,0\}\}\end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t,s)\end{bmatrix} \\
&= \mathcal{P}\begin{bmatrix} T, & Q(s) \\ \emptyset, & \{\emptyset\}\end{bmatrix} \begin{bmatrix} \begin{bmatrix}0&\Delta_a&0\\0&0&\Delta_a\\0&0&\Delta_a\partial_s\end{bmatrix}\mathbf{x}(t,s) \\ \mathbf{x}_f(t,s)\end{bmatrix} = \mathcal{P}\begin{bmatrix} T(BT)^{-1}\begin{bmatrix}B_w & B_u & B_{xo}\end{bmatrix}, & Q(s) + T(BT)^{-1}(\mathbf{B}_{xx}(s) - BQ(s)) \\ \emptyset, & \{\emptyset\}\end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t,s)\end{bmatrix}
\end{aligned}$$

C. $\Lambda_2 \mathbf{x}$

we have

$$\Lambda_2 \mathbf{x}(t) = \Lambda_2 \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \mathbf{Q}_{2f}, & \left\{\{\mathbf{R}_{0f}, \mathbf{R}_{1f}, \mathbf{R}_{2f}\}\right\}\end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t)\end{bmatrix}$$

and

$$
\begin{aligned}
\Lambda_2 \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \mathbf{Q}_{2f}, & \left\{\{\mathbf{R}_{0f}, \mathbf{R}_{1f}, \mathbf{R}_{2f}\}\right\} \end{bmatrix} &= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & \partial_s & 0 \\ 0 & 0 & \partial_s \\ 0 & 0 & \partial_{ss} \end{bmatrix} \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \mathbf{Q}_{2f}, & \left\{\{\mathbf{R}_{0f}, \mathbf{R}_{1f}, \mathbf{R}_{2f}\}\right\} \end{bmatrix} \\
&= \left(\underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{I}_1} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{I}_2} \partial_s + \partial_s \partial_s \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}}_{\mathbf{I}_3} \right) \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \mathbf{Q}_{2f}, & \left\{\{\mathbf{R}_{0f}, \mathbf{R}_{1f}, \mathbf{R}_{2f}\}\right\} \end{bmatrix} \\
&= \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \mathbf{I}_1\mathbf{Q}_{2f}, & \left\{\{\mathbf{I}_1\mathbf{R}_{0f}, \mathbf{I}_1\mathbf{R}_{1f}, \mathbf{I}_1\mathbf{R}_{2f}\}\right\} \end{bmatrix} + \partial_s \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \mathbf{I}_2\mathbf{Q}_{2f}, & \left\{\{\mathbf{I}_2\mathbf{R}_{0f}, \mathbf{I}_2\mathbf{R}_{1f}, \mathbf{I}_2\mathbf{R}_{2f}\}\right\} \end{bmatrix} + \partial_s \partial_s \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \mathbf{I}_3\mathbf{Q}_{2f}, & \left\{\{\mathbf{I}_3\mathbf{R}_{0f}, \mathbf{I}_3\mathbf{R}_{1f}, \mathbf{I}_3\mathbf{R}_{2f}\}\right\} \end{bmatrix} \\
&= \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \mathbf{I}_1\mathbf{Q}_{2f}, & \left\{\{\mathbf{I}_1 L_0, \mathbf{I}_1\mathbf{R}_{1f}, \mathbf{I}_1\mathbf{R}_{2f}\}\right\} \end{bmatrix} + \partial_s \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \mathbf{I}_2\mathbf{Q}_{2f}, & \left\{\{0, \mathbf{I}_2\mathbf{R}_{1f}, \mathbf{I}_2\mathbf{R}_{2f}\}\right\} \end{bmatrix} + \partial_s \partial_s \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \mathbf{I}_3\mathbf{Q}_{2f}, & \left\{\{0, \mathbf{I}_3\mathbf{R}_{1f}, \mathbf{I}_3\mathbf{R}_{2f}\}\right\} \end{bmatrix} \\
&= \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ (\mathbf{I}_1 + \mathbf{I}_2\partial_s)\mathbf{Q}_{2f}, & \left\{\{\mathbf{I}_1 L_0 + \mathbf{I}_2 L_1(s,s), (\mathbf{I}_1 + \mathbf{I}_2\partial_s)\mathbf{R}_{1f}, (\mathbf{I}_1 + \mathbf{I}_2\partial_s)\mathbf{R}_{2f}\}\right\} \end{bmatrix} + \partial_s \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \partial_s\mathbf{I}_3\mathbf{Q}_{2f}, & \left\{\{0, \partial_s\mathbf{I}_3\mathbf{R}_{1f}, \partial_s\mathbf{I}_3\mathbf{R}_{2f}\}\right\} \end{bmatrix}
\end{aligned}
$$

Where

$$
\mathbf{R}_{1f}(s,s) - \mathbf{R}_{2f}(s,s) = L_1(s,s) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and hence} \quad \mathbf{I}_3(\mathbf{R}_{1f}(s,s) - \mathbf{R}_{2f}(s,s)) = \mathbf{I}_3 L_1(s,s) = 0
$$

Finally,

$$
\begin{aligned}
&\Lambda_2 \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \mathbf{Q}_{2f}, & \left\{\{\mathbf{R}_{0f}, \mathbf{R}_{1f}, \mathbf{R}_{2f}\}\right\} \end{bmatrix} \\
&= \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ (\mathbf{I}_1 + \mathbf{I}_2\partial_s)\mathbf{Q}_{2f}, & \left\{\{\mathbf{I}_1 L_0 + \mathbf{I}_2 L_1(s,s), (\mathbf{I}_1 + \mathbf{I}_2\partial_s)\mathbf{R}_{1f}, (\mathbf{I}_1 + \mathbf{I}_2\partial_s)\mathbf{R}_{2f}\}\right\} \end{bmatrix} + \partial_s \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \partial_s\mathbf{I}_3\mathbf{Q}_{2f}, & \left\{\{0, \partial_s\mathbf{I}_3\mathbf{R}_{1f}, \partial_s\mathbf{I}_3\mathbf{R}_{2f}\}\right\} \end{bmatrix} \\
&= \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \left(\mathbf{I}_1 + \mathbf{I}_2\partial_s + \mathbf{I}_3\partial_s^2\right)\mathbf{Q}_{2f}, & \left\{\{\mathbf{I}_1 L_0 + \mathbf{I}_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathbf{I}_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \left(\mathbf{I}_1 + \mathbf{I}_2\partial_s + \mathbf{I}_3\partial_s^2\right)\mathbf{R}_{1f}, \left(\mathbf{I}_1 + \mathbf{I}_2\partial_s + \mathbf{I}_3\partial_s^2\right)\mathbf{R}_{2f}\}\right\} \end{bmatrix} \\
&= \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \hat{\mathbf{Q}}_{2f}, & \left\{\{\hat{\mathbf{R}}_{0f}, \hat{\mathbf{R}}_{1f}, \hat{\mathbf{R}}_{2f}\}\right\} \end{bmatrix}
\end{aligned}
$$

where we used

$$
\partial_s \mathbf{I}_3 \mathbf{R}_{1f}(s,\theta) - \partial_s \mathbf{I}_3 \mathbf{R}_{2f}(s,\theta) = \partial_s \mathbf{I}_3 L_1(s,\theta) = \mathbf{I}_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}
$$

For reference:

$$\hat{\mathbf{Q}}_{2f} = \left(\mathbf{I}_1 + \mathbf{I}_2\partial_s + \mathbf{I}_3\partial_s^2\right)\mathbf{Q}_{2f}$$

$$\hat{\mathbf{R}}_{0f} = \mathbf{I}_1 \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathbf{I}_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathbf{I}_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$\hat{\mathbf{R}}_{1f} = \left(\mathbf{I}_1 + \mathbf{I}_2\partial_s + \mathbf{I}_3\partial_s^2\right)\mathbf{R}_{1f}$$

$$\hat{\mathbf{R}}_{2f} = \left(\mathbf{I}_1 + \mathbf{I}_2\partial_s + \mathbf{I}_3\partial_s^2\right)\mathbf{R}_{2f}$$

D. In Conclusion

we have

$$\Lambda_1 \mathbf{x}(t,s) = \mathcal{P}\begin{bmatrix} T(BT)^{-1}\begin{bmatrix} B_w & B_u & B_{xo} \end{bmatrix}, & Q(s) + T(BT)^{-1}(\mathbf{B}_{xx}(s) - BQ(s)) \\ \emptyset, & \{\emptyset\} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t,s) \end{bmatrix}$$

and

$$\Lambda_2 \mathbf{x}(t) = \Lambda_2 \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \mathbf{Q}_{2f}, & \{\{\mathbf{R}_{0f}, \mathbf{R}_{1f}, \mathbf{R}_{2f}\}\} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t) \end{bmatrix} = \mathcal{P}\begin{bmatrix} \emptyset, & \emptyset \\ \hat{\mathbf{Q}}_{2f}, & \{\{\hat{\mathbf{R}}_{0f}, \hat{\mathbf{R}}_{1f}, \hat{\mathbf{R}}_{2f}\}\} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t) \end{bmatrix}$$

hence

$$\begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \Lambda_1 \mathbf{x}(t) \\ \Lambda_2 \mathbf{x}(t) \end{bmatrix} = \mathcal{P}\begin{bmatrix} \overbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ T(BT)^{-1}B_w & T(BT)^{-1}B_u & T(BT)^{-1}B_{xo} \end{bmatrix}}^{\hat{\mathbf{P}}_f}, & \overbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ Q(s) + T(BT)^{-1}(\mathbf{B}_{xx}(s) - BQ(s)) \end{bmatrix}}^{\hat{\mathbf{Q}}_{1f}(s)} \\ \hat{\mathbf{Q}}_{2f}, & \left\{\{\hat{\mathbf{R}}_{0f}, \hat{\mathbf{R}}_{1f}, \hat{\mathbf{R}}_{2f}\}\right\} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t,s) \end{bmatrix}$$

$$= \mathcal{P}\begin{bmatrix} \hat{\mathbf{P}}_f, & \hat{\mathbf{Q}}_{1f}(s) \\ \hat{\mathbf{Q}}_{2f}, & \left\{\{\hat{\mathbf{R}}_{0f}, \hat{\mathbf{R}}_{1f}, \hat{\mathbf{R}}_{2f}\}\right\} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ x(t) \\ \mathbf{x}_f(t,s) \end{bmatrix}$$

Combining with

$$\begin{bmatrix} z(t) \\ y(t) \\ \dot{x}(t) \\ \dot{\mathbf{x}}(t,s) \end{bmatrix} = \mathcal{P}\begin{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, & 0 \\ \begin{bmatrix} 0 & 0 & \mathbf{Q}_{2f} \end{bmatrix}, & \left\{\{\mathbf{R}_{0f}, \mathbf{R}_{1f}, \mathbf{R}_{2f}\}\right\} \end{bmatrix} \begin{bmatrix} z(t) \\ y(t) \\ \dot{w}(t) \\ \dot{u}(t) \\ \dot{x}(t) \\ \dot{\mathbf{x}}_f(t,s) \end{bmatrix}$$

we have

$$\mathcal{P}\begin{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I\end{bmatrix}, & 0\\ \begin{bmatrix} 0 & 0 & \mathbf{Q}_{2f}\end{bmatrix}, & \left\{\{\mathbf{R}_{0f},\mathbf{R}_{1f},\mathbf{R}_{2f}\}\right\}\end{bmatrix} \begin{bmatrix} z(t)\\ y(t)\\ \dot{w}(t)\\ \dot{u}(t)\\ \dot{x}(t)\\ \dot{\mathbf{x}}_f(t,s)\end{bmatrix} = \underbrace{\mathcal{P}\begin{bmatrix} \mathbf{P}, & \mathbf{Q}_1\\ \mathbf{Q}_2, & \{\mathbf{R}_i\}\end{bmatrix}\mathcal{P}\begin{bmatrix} \hat{\mathbf{P}}_f, & \hat{\mathbf{Q}}_{1f}(s)\\ \hat{\mathbf{Q}}_{2f}, & \left\{\{\hat{\mathbf{R}}_{0f},\hat{\mathbf{R}}_{1f},\hat{\mathbf{R}}_{2f}\}\right\}\end{bmatrix}}_{\mathcal{P}\begin{bmatrix} \tilde{\mathbf{P}}, & \tilde{\mathbf{Q}}_1\\ \tilde{\mathbf{Q}}_2, & \left\{\{\tilde{\mathbf{R}}_i\}\right\}\end{bmatrix}} \begin{bmatrix} w(t)\\ u(t)\\ x(t)\\ \mathbf{x}_f(t,s)\end{bmatrix} \tag{3}$$

V. Old Format

$$\begin{bmatrix} I & & & & \\ & I & & & \\ & & \mathcal{T}_{Bw} & \mathcal{T}_{Bu} & \mathcal{T}\end{bmatrix} \begin{bmatrix} z(t)\\ y(t)\\ \dot{w}(t)\\ \dot{u}(t)\\ \begin{bmatrix}\dot{x}(t)\\ \dot{\mathbf{x}}_f(t,s)\end{bmatrix}\end{bmatrix} = \begin{bmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} & \mathcal{C}_1\\ \mathcal{D}_{21} & \mathcal{D}_{22} & \mathcal{C}_2\\ \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{A}\end{bmatrix} \begin{bmatrix} w(t)\\ u(t)\\ \begin{bmatrix} x(t)\\ \mathbf{x}_f(t,s)\end{bmatrix}\end{bmatrix} \tag{4}$$

$$\begin{bmatrix}\mathcal{T}_{Bw} & \mathcal{T}_{Bu} & \mathcal{T}\end{bmatrix} = \mathcal{P}\begin{bmatrix} \begin{bmatrix} 0 & 0 & I\end{bmatrix}, & 0\\ \begin{bmatrix}\mathbf{Q}_{2f}\end{bmatrix}, & \left\{\{0,\mathbf{R}_{1f},\mathbf{R}_{2f}\}\right\}\end{bmatrix}, \qquad \begin{bmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} & \mathcal{C}_1\\ \mathcal{D}_{21} & \mathcal{D}_{22} & \mathcal{C}_2\\ \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{A}\end{bmatrix} = \mathcal{P}\begin{bmatrix} \tilde{\mathbf{P}}, & \tilde{\mathbf{Q}}_1\\ \tilde{\mathbf{Q}}_2, & \left\{\{\tilde{\mathbf{R}}_i\}\right\}\end{bmatrix}$$

VI. Variable References

$$\mathcal{P}\begin{bmatrix} \tilde{\mathbf{P}}, & \tilde{\mathbf{Q}}_1\\ \tilde{\mathbf{Q}}_2, & \left\{\{\tilde{\mathbf{R}}_i\}\right\}\end{bmatrix} = \mathcal{P}\begin{bmatrix} \mathbf{P}, & \mathbf{Q}_1\\ \mathbf{Q}_2, & \{\mathbf{R}_i\}\end{bmatrix}\mathcal{P}\begin{bmatrix} \hat{\mathbf{P}}_f, & \hat{\mathbf{Q}}_{1f}(s)\\ \hat{\mathbf{Q}}_{2f}, & \left\{\{\hat{\mathbf{R}}_{0f},\hat{\mathbf{R}}_{1f},\hat{\mathbf{R}}_{2f}\}\right\}\end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} D_{11} & D_{12} & C_1 & C_{10}\\ D_{21} & D_{22} & C_2 & C_{20}\\ B_{11} & B_{12} & A & E_0\end{bmatrix} \qquad \mathbf{Q}_1(s) = \begin{bmatrix} C_{a1}(s) & C_{b1}(s) & C_{c1}(s)\\ C_{a2}(s) & C_{b2}(s) & C_{c2}(s)\\ E_a(s) & E_b(s) & E_c(s)\end{bmatrix}$$

$$\mathbf{Q}_2(s) = \begin{bmatrix} B_{21}(s) & B_{22}(s) & E(s) & E_{???}(s)\end{bmatrix}, \quad \mathbf{R}_0(s) = \begin{bmatrix} A_0(s) & A_1(s) & A_2(s)\end{bmatrix}, \quad \mathbf{R}_1(s,\theta) = 0, \quad \mathbf{R}_2(s,\theta) = 0$$

$$K(s) = \begin{bmatrix} 0 & 0 & 0\\ I & 0 & 0\\ 0 & I & (s-a)I\end{bmatrix}, \quad L_0 = \begin{bmatrix} I & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad L_1(s,\theta) = \begin{bmatrix} 0 & 0 & 0\\ 0 & I & 0\\ 0 & 0 & (s-\theta)\end{bmatrix}$$

$$\mathbf{R}_{0f}(s) = L_0, \qquad \mathbf{R}_{1f}(s,\theta) = L_1(s,\theta) + \mathbf{R}_{2f}(s,\theta), \quad \mathbf{R}_{2f}(s,\theta) = K(s)(BT)^{-1}(\mathbf{B}_{xx}(\theta) - BQ(\theta))$$

$$\mathbf{Q}_{2f} = K(s)(BT)^{-1}\begin{bmatrix} B_w & B_u & B_{xo}\end{bmatrix}$$

$$\hat{\mathbf{P}}_f = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ T(BT)^{-1}B_w & T(BT)^{-1}B_u & T(BT)^{-1}B_{xo} \end{bmatrix}, \quad \hat{\mathbf{Q}}_{1f} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Q(s) + T(BT)^{-1}(\mathbf{B}_{xx}(s) - BQ(s)) \end{bmatrix}$$

$$\hat{\mathbf{Q}}_{2f} = \left(\mathbf{I}_1 + \mathbf{I}_2\partial_s + \mathbf{I}_3\partial_s^2\right)\mathbf{Q}_{2f}$$

$$\hat{\mathbf{R}}_{0f} = \mathbf{I}_1 \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathbf{I}_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathbf{I}_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$\hat{\mathbf{R}}_{1f} = \left(\mathbf{I}_1 + \mathbf{I}_2\partial_s + \mathbf{I}_3\partial_s^2\right)\mathbf{R}_{1f}$$

$$\hat{\mathbf{R}}_{2f} = \left(\mathbf{I}_1 + \mathbf{I}_2\partial_s + \mathbf{I}_3\partial_s^2\right)\mathbf{R}_{2f}$$

$$\mathbf{I}_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{I}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$T = \begin{bmatrix} I & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & (b-a)I \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix}, \qquad Q(s) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (b-s) \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$