Introduction

In this Lecture, you will learn:

Motion of a satellite in time

- Given \( t \), find \( f(t) \) and vice-versa.
- New Angles
  - Mean Anomaly
  - True Anomaly
- How to convert between them
  - Kepler’s Equation

**Problem:** Let \( a = 25,512km \) and \( e = .625 \). Find \( r, v \) at \( t = 4hr \).
Recall the Conic Equation

\[ r(t) = \frac{p}{1 + e \cos f(t)} \]

Which we have shown describes elliptic, parabolic or hyperbolic motion.

**Question:** What is \( f(t) \)?

**Response:** There is no closed-form expression for \( f(t) \)!

What to do?

Start with Kepler’s Second Law: Equal Areas in Equal Time.

\[ \frac{dA}{dt} = \frac{h}{2} = constant \]

But how does \( A(t) \) relate to \( f(t) \)?
Recall the Conic Equation

\[ r(t) = p_1 + e \cos f(t) \]

Which we have shown describes elliptic, parabolic or hyperbolic motion.

Question: What is \( f(t) \)?
Response: There is no closed-form expression for \( f(t) \). What to do?
Start with Kepler’s Second Law: Equal Areas in Equal Time.
\[
\frac{dA}{dt} = \frac{h}{r} = \text{constant}
\]
But how does \( A(t) \) relate to \( f(t) \)?

---

• While semi-latus rectum and eccentricity define the conic section in polar form, \( a \) and \( b \) define the conic section in rectilinear coordinates.
A useful geometric tool is to inscribe the ellipse in a circle. The equation of a conic section is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solving for $y$,

$$y_e = \frac{b}{a} \sqrt{a^2 - x^2}$$

but for a circle of radius $a$,

$$y_c(x) = \sqrt{a^2 - x^2}.$$  Thus

$$y_e = \frac{b}{a} y_c$$

This is the ellipse scaling law.
A circle is defined by $x^2 + y^2 = a^2$. In this case $a = b$. We then solve this for $y$ to get $y = \sqrt{a^2 - x^2}$.

The ellipse scaling law states that all points on the ellipse are scaled towards the major axis by a factor of $b/a$. Obviously for an ellipse $b < a$. 
The Eccentric Anomaly is an artificial angle

- From the Center of the ellipse
- To the projection of $r$ on a fictional circular orbit of radius $a$

- Measured from center of ellipse (not focus).
- No physical interpretation.
- A mathematical convenience
• The eccentric anomaly is convenient because it gives a geometric angle which serves a substitute for time and for which we can compute based on swept area. Since the rate of area sweep is constant, this is significant.

• In the image, $u$ is the eccentric anomaly. However, we typically use $E$ to denote this angle.
The Ellipse Revisited

For convenience, suppose \( t = 0 \) at periapse. The area swept out is FVP

Kepler’s Second Law say that

\[
\frac{t_P}{T} = \frac{\text{Area of FVP}}{\text{Area of ellipse}} = \frac{A_{FVP}}{\pi ab}
\]

But what is \( A_{FVP} \)?

\[
A_{FVP} = A_{PSV} - A_{PSF}
\]

PSF is a triangle, so

\[
A_{PSF} = \frac{1}{2} \left( \frac{OF}{ae} - \frac{OS \cos E}{a\cos E} \right) \cdot \frac{QS}{a} \left( \frac{PS}{a \sin E} \right)
\]

\( E \) is the Eccentric Anomaly.

The conversion from \( E \) to \( f \) (or vice-versa) is not difficult.
The Ellipse Revisited

For convenience, suppose \( t = 0 \) at periapse. The area swept out is

\[
\text{FVP} = \frac{1}{2} \left( OF \cdot OS \cdot a \cdot \cos E - OS \cdot PS \cdot b \right)
\]

\( A_{FVP} \) is the area inside section \( FVP \)

\( t_P \) is the time at which we reach point \( P \)

\( QS \) is length \( a \sin E \). \( OS \) is length \( a \cos E \)
The Ellipse Revisited

It is easy to see by the scaling law that $A_{PSV} = \frac{b}{a}A_{QSV}$. $A_{QSV}$ is easily calculated as

$$A_{QSV} = A_{QOV} - A_{QOS}$$

$$= \frac{1}{2}a^2E - \frac{1}{2}a \cos E \cdot a \sin E$$

where $E$ is in radians. Thus we conclude

$$A_{FVP} = A_{PSV} - A_{PSF}$$

$$= \frac{1}{2}ab(E - \cos E \sin E)$$

$$- \frac{1}{2}ab(e - \cos E) \sin E$$

$$= \frac{1}{2}ba(E - e \sin E)$$
Mean Anomaly

The conclusion is that

\[
\frac{t_P}{T} = \frac{A_{FP}(t)}{\pi ab} = \frac{E(t) - e \sin E(t)}{2\pi}
\]

Since by Kepler’s third law,

\[
T = \sqrt{\frac{4\pi^2 a^3}{\mu}}
\]

we have

\[
\frac{E(t_P) - e \sin E(t_P)}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{\mu}{a^3 t_P}}
\]

- Thus we have an expression for \( t_P \) in terms of \( E(t_P) \).
- What we really want is an expression for \( E \) in terms of \( t_P \).
- Unfortunately no such solution exists.
  - Equation must be solved numerically for each value of \( t \).
  - Prompted invention of first known numerical algorithm, Newton’s Method.
Mean Anomaly

We define some terms

**Definition 1.**
The mean motion, \( n \) is defined as

\[
n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}
\]

**Definition 2.**
The mean anomaly, \( M(t) \) is defined as

\[
M(t) = nt = \sqrt{\frac{\mu}{a^3}} t
\]

Neither of these have good physical interpretations.

\[
M(t) = E(t) - e \sin E(t)
\]
Mean Anomaly

We define some terms

**Definition 1.**
The mean motion, \( n \), is defined as
\[
n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}
\]

**Definition 2.**
The mean anomaly, \( M(t) \), is defined as
\[
M(t) = nt = \sqrt{\frac{\mu}{a^3}} t
\]

Neither of these have good physical interpretations.

\[
M(t) = E(t) - e \sin E(t)
\]

- Mean Anomaly can be thought of as the fraction of the period of the orbit which has elapsed, but put into radians.
- However, it simplifies the expression for \( E \)

\[
E(t) - e \sin E(t) = nt = M(t)
\]

- We will use Newton’s algorithm to solve this equation.
Converting Between $E$ and $f$

Once we get $E$ from solving Kepler’s equation, we still need to find the angle $f$ in order to recover position. Going back to the ellipse, We express the line OS using both $E$ and $f$.

\[ OS = a \cos E \]
\[ = ae + r \cos f \]

But $r = \frac{a(1-e^2)}{1+e \cos f}$, so

\[ \cos E = (1 - e^2) \frac{\cos f}{1 + e \cos f} \]

Using the half-angle formula, we can get the expression

\[ \tan \frac{E}{2} = \sqrt{\frac{1 - e}{1 + e}} \tan \frac{f}{2} \]

Given $f$, we can find $E$. 
Converting Between $E$ and $f$

Alternatively, given $E$, we can find $f$.

$$\tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}$$

We can also now directly express the orbit equation using $E$,

$$r(t) = a(1 - e \cos E(t))$$
Example: Use geometry to get time (The Easy Problem)

**Problem:** Given an orbit with \( a = 10,000 \text{km} \) and \( e = .5 \), determine the times at which \( r = 14,147 \text{km} \).

**Solution:** First solve for the true anomaly, \( f \). we have

\[
r(t) = \frac{a(1 - e^2)}{1 + e \cos f(t)}
\]

which yields

\[
\cos f(t) = \frac{a(1 - e^2) - r(t)}{er(t)} = - .9397
\]

Solving for \( f \) yields two solutions \( f = 160 \text{ deg}, 200 \text{ deg} \).

Now we want to find \( E(t) \).

\[
\tan \frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{f}{2} = \pm 3.27
\]

This yields

\[
E = \pm 146.0337 \text{ deg},
\]
Example: Going from $E$ to $M$ is easy

Solving for mean anomaly (in radians!!),

$$M(t) = E(t) - e \sin E(t) = 2.2694 \text{rad}, 4.0138 \text{rad}$$

Now the mean motion is

$$n = \sqrt{\frac{\mu}{a^3}} = 6.3135 E - 4$$

So finally, the times of arrival are

$$t = \frac{M(t)}{n} = 3594 \text{s}, 6357 \text{s}$$

**Note:** In this way, it is easy to find the time between any 2 points in the orbit. e.g. from $f = 160 \text{deg}$ to $f = 200 \text{deg}$ takes time $\Delta t = 6357 - 3594 = 2763 \text{s}$. 
Problem 2: Prediction (The Harder One)

Given $t$, find $r$ and $v$

Generally speaking we can follow the previous steps in reverse.

1. Given time, $t$, solve for Mean Anomaly

   \[ M(t) = nt \]

2. Given Mean Anomaly, solve for Eccentric Anomaly
   
   ▶ How???

3. Given eccentric anomaly, solve for true anomaly

   \[ \tan \frac{f}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2} \]

4. Given true anomaly, solve for $r$

   \[ r(t) = \frac{a(1 - e^2)}{1 + e \cos f(t)} \]

The Missing Piece is how to solve for Eccentric Anomaly, $E$ given Mean Anomaly, $M$. 
Solving the Kepler Equation

Given $M$, find $E$

$$M = E - e \sin E$$

- A Transcendental Equation
- No Closed-Form Solution
- However, for any $M$, there is a unique $E$.

To Solve Kepler’s Equation, Newton had to redefine the meaning of a solution.

**Iterative Methods (Algorithms):**

Instead of solving a single equation, we solve a sequence of equations until a stopping criterion (usually error tolerance) is met.

- The solution is never exact.
- Perfect for implementation on computers
- Dramatically increased the set of solvable problems.
- Today, most problems are solved via Algorithms.
Newton-Raphson Iteration

An Algorithm for solving equations

\[ f(x) = 0 \]

Start by guessing the solution \( x_k \).

- Approximate \( f(x) \approx f(x_k) + f'(x_k)(x - x_k) \).
- Solve \( f(x_k) + f'(x_k)(x - x_k) = 0 \)

\[ x = x_k - \frac{f(x_k)}{f'(x_k)} \]

- Update your guess, \( x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \)
- Repeat until \( \|f(x_k)\| \) is sufficiently small.
Newton’s Method

Illustration
Failure of Newton-Raphson Iteration

When Newton’s Method Works, it works well

A scaled Newton iteration always converges for convex functions, \((f''(x) > 0)\)
Failure of Newton-Raphson Iteration

When Newton’s Method Works, it works well

A scaled Newton iteration always converges for convex functions, ($f''(x) > 0$)

• By scaled, we mean we use the linear approximation, but only update the guess as

$$x_{k+1} = x_k - \alpha \frac{f(x_k)}{f'(x_k)}$$

where $0 < \alpha < 1$ is some sufficiently small step size.

• For Kepler’s equation, we can use $\alpha = 1$. 

Given $M$, we want to solve

$$f(E) = M - E + e \sin E = 0$$

then,

$$f'(E) = -1 + e \cos E$$

**Algorithm:** Choose $E_1 = M$.

- Update

$$E_{k+1} = E_k - \frac{M - E_k + e \sin E_k}{e \cos E_k - 1}$$

- If $\|M - E_k + e \sin E_k\| < .001$ or whatever, quit.
- Otherwise repeat.
Given $M$, we want to solve

$$f(E) = M - E + e \sin E = 0$$

then,

$$f'(E) = -1 + e \cos E$$

Algorithm:

1. Choose $E_1 = M$.
2. Update $E_{k+1} = E_k - \frac{M - E_k + e \sin E_k}{e \cos E_k - 1}$
3. If $\|M - E_k + e \sin E_k\| < .001$ or whatever, quit.
4. Otherwise repeat.

- If N-R converges, it usually only takes 2 or 3 iterations.
- Good for you, as you will do this by hand.
- Balance between convergence rate and stability.
- Scaled N-R always converges but is much slower.
Example: Prediction (The Hard Problem)

**Problem:** Let \( a = 25,512 \text{km} \) and \( e = .625 \). Find \( r, v \) at \( t = 4 \text{hr} \).

**Solution:** First, solve for Mean Anomaly.

\[
n = \sqrt{\frac{\mu}{a^3}} = 1.549 E - 4 s^{-1}
\]

Thus

\[
M(t) = nt = 1.549 \cdot 10^{-4} \cdot 4 \cdot 3600 = 2.231 \text{rad}
\]

**Newton Iteration:** Now to solve for \( E \), we set \( E_1 = M \) and iterate

\[
E_2 = E_1 - \frac{2.231 - E_1 + .625 \sin E_1}{.625 \cos E_1 - 1} = 2.588
\]

\[
f(E_2) = 2.231 - E_2 + .625 \sin E_2 = -0.0284
\]

We verify that \( \|f(E_2)\| = .0284 > .001 \), so continue:

\[
E_3 = E_2 - \frac{2.231 - E_2 + .625 \sin E_2}{.625 \cos E_2 - 1} = 2.570
\]

\[
f(E_3) = 2.231 - E_3 + .625 \sin E_3 = -0.000892
\]
Example

Now \( \|f(E_3)\| < .001 \), so quit. \( E = E_3 = 2.570 \). Now Solve for true anomaly

\[
f = 2 \tan^{-1}\left( \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \right) = 2.861 \text{ rad}
\]

\[
r(t) = \frac{a(1-e^2)}{1+e \cos f(t)} = 38920 \text{ km}
\]

Now via vis-viva,

\[
v = \sqrt{\mu \left( \frac{2}{r} - \frac{1}{a} \right)} = 2.2043 \text{ km/s}
\]
Conclusion

In this Lecture, you learned:

- How to predict position given time.
- New Angles
  - Mean Anomaly
  - Eccentric Anomaly
  - True Anomaly
- How to convert between them
  - How to Solve Kepler’s Equation

Key Equations:

\[
n = \sqrt{\frac{\mu}{a^3}}
\]

\[
M(t) = nt
\]

\[
M(t) = E(t) - e \sin E(t)
\]

\[
\tan \frac{E}{2} = \sqrt{\frac{1 - e}{1 + e}} \tan \frac{f}{2}
\]

Newton Iteration:

\[
E_0 = M
\]

\[
E_{k+1} = E_k - \frac{M - E_k + e \sin E_k}{e \cos E_k - 1}
\]