Spacecraft Dynamics and Control

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Lecture 4: Position and Velocity
Introduction

In this Lecture, you will learn:

Motion of a satellite in time

- How to predict position given time.
- New Angles
  - Mean Anomaly
  - True Anomaly
- How to convert between them
  - Kepler’s Equation

Problem: Let $a = 25,512$ km and $e = .625$. Find $r, v$ at $t = 4$ hr.
Recall the Conic Equation

\[ r(t) = \frac{p}{1 + e \cos f(t)} \]

Which we have shown describes elliptic, parabolic or hyperbolic motion.

**Question:** What is \( f(t) \)?

**Response:** There is no closed-form expression for \( f(t) \)!

What to do?

Start with Kepler’s Second Law: Equal Areas in Equal Time.

\[ \frac{dA}{dt} = \frac{h}{2} = \text{constant} \]

But how does \( A(t) \) relate to \( f(t) \)?
A useful geometric tool is to inscribe the ellipse in a circle.

The equation of a conic section is

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

Solving for \(y\),

\[
y_e = \frac{b}{a} \sqrt{a^2 - x^2}
\]

but for a circle of radius \(a\),

\[
y_c(x) = \sqrt{a^2 - x^2}. \text{ Thus}
\]

\[
y_e = \frac{b}{a} y_c
\]

This is the ellipse scaling law.
The **Eccentric Anomaly** is an artificial angle

- From the *Center* of the ellipse
- To the projection of $r$ on a fictional circular orbit of radius $a$

- Measured from center of ellipse (not focus).
- No physical interpretation.
- A mathematical convenience
For convenience, suppose \( t = 0 \) at periapse. The area swept out is FVP. Kepler’s Second Law say that

\[
\frac{t}{T} = \frac{\text{Area of FVP}}{\text{Area of ellipse}} = \frac{A_{FVP}}{\pi ab}
\]

But what is \( A_{FVP} \)?

\[
A_{FVP} = A_{PSV} - A_{PSF}
\]

PSF is a triangle, so

\[
A_{PSF} = \frac{1}{2} (ae - a \cos E) \cdot \frac{b}{a} (a \sin E)
\]

\( E \) is the **Eccentric Anomaly**.

The conversion from \( E \) to \( f \) (or vice-versa) is not difficult.
The Ellipse Revisited

It is easy to see by the scaling law that \( A_{PSV} = \frac{b}{a} A_{QSV} \). \( A_{QSV} \) is easily calculated as

\[
A_{QSV} = A_{QOV} - A_{QOS} = \frac{1}{2} a^2 E - \frac{1}{2} a \cos E \cdot a \sin E
\]

where \( E \) is in radians. Thus we conclude

\[
A_{FVP} = A_{PSV} - A_{PSF} = \frac{1}{2} ab(E - \cos E \sin E) - \frac{1}{2} ab(e - \cos E) \sin E = \frac{1}{2} ba(E - e \sin E)
\]
Mean Anomaly

The conclusion is that

$$\frac{t}{T} = \frac{A_{FVP}(t)}{\pi ab} = \frac{E(t) - e \sin E(t)}{2\pi}$$

Since by Kepler’s third law,

$$T = \sqrt{\frac{4\pi^2 a^3}{\mu}}$$

we have

$$\frac{E(t) - e \sin E(t)}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{\mu}{a^3}} t$$

- Thus we have an expression for $t$ in terms of $E(t)$.
- What we really want is an expression for $E$ in terms of $t$.
- Unfortunately no such solution exists.
  - Equation must be solved numerically for each value of $t$.
  - Prompted invention of first known numerical algorithm, Newton’s Method.
Mean Anomaly

We define some terms

**Definition 1.**
The mean motion, $n$ is defined as

$$n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$$

**Definition 2.**
The mean anomaly, $M(t)$ is defined as

$$M(t) = nt = \sqrt{\frac{\mu}{a^3}} t$$

Neither of these have good physical interpretations.

$$M(t) = E(t) - e \sin E(t)$$
Converting Between $E$ and $f$

Once we get $E$ from solving Kepler’s equation, we still need to find the angle $f$ in order to recover position. Going back to the ellipse, we express the line OS using both $E$ and $f$.

$$OS = a \cos E$$

$$= ae + r \cos f$$

But $r = \frac{a(1-e^2)}{1+e \cos f}$, so

$$\cos E = (1 - e^2) \frac{\cos f}{1 + e \cos f}$$

Using the half-angle formula, we can get the expression

$$\tan \frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{f}{2}$$

Given $f$, we can find $E$. 
Converting Between $E$ and $f$

Alternatively, given $E$, we can find $f$.

$$\tan \frac{f}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2}$$

We can also now directly express the orbit equation using $E$,

$$r(t) = a(1 - e \cos E(t))$$
Example

Problem: Given an orbit with \( a = 10,000 km \) and \( e = .5 \), determine the times at which \( r = 14,147 km \).

Solution: First solve for the true anomaly, \( f \). we have

\[
    r(t) = \frac{a(1 - e^2)}{1 + e \cos f(t)}
\]

which yields

\[
    \cos f(t) = \frac{a(1 - e^2) - r(t)}{er(t)} = - .9397
\]

Solving for \( f \) yields two solutions \( f = 160 \text{ deg}, 200 \text{ deg} \).

Now we want to find \( E(t) \).

\[
    \tan \frac{E}{2} = \sqrt{\frac{1 - e}{1 + e}} \tan \frac{f}{2} = \pm 3.27
\]

This yields

\[
    E = \pm 146.0337 \text{ deg},
\]
Solving for mean anomaly (in radians!!),
\[ M(t) = E(t) - e \sin E(t) = 2.2694\text{rad}, 4.0138\text{rad} \]

Now the mean motion is
\[ n = \sqrt{\frac{\mu}{a^3}} = 6.3135 E - 4 \]

So finally, the times of arrival are
\[ t = \frac{M(t)}{n} = 3594s, 6357s \]

**Note:** In this way, it is easy to find the time between any 2 points in the orbit. e.g. from \( f = 160 \text{deg} \) to \( f = 200 \text{deg} \) takes time \( \Delta t = 6357 - 3594 = 2763s \).
Problem 2
Given \( t \), find \( r \) and \( v \)

Generally speaking we can follow the previous steps in reverse.

1. Given time, \( t \), solve for Mean Anomaly

\[
M(t) = nt
\]

2. Given Mean Anomaly, solve for Eccentric Anomaly

\[
\tan \frac{f}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2}
\]

3. Given eccentric anomaly, solve for true anomaly

4. Given true anomaly, solve for \( r \)

\[
r(t) = \frac{a(1 - e^2)}{1 + e \cos f(t)}
\]

The Missing Piece is how to solve for Eccentric Anomaly, \( E \) given Mean Anomaly, \( M \).
Solving the Kepler Equation

Given $M$, find $E$

\[ M = E - e \sin E \]

- A Transcendental Equation
- No Closed-Form Solution
- However, for any $M$, there is a unique $E$.

To Solve Kepler’s Equation, Newton had to redefine the meaning of a solution.

**Iterative Methods (Algorithms):**

Instead of solving a single equation, we solve a sequence of equations until a stopping criterion (usually error tolerance) is met.

- The solution is never exact.
- Perfect for implementation on computers
- Dramatically increased the set of solvable problems.
- Today, most problems are solved via Algorithms.
Newton-Raphson Iteration

An Algorithm for solving equations

\[ f(x) = 0 \]

Start by guessing the solution \( x_k \).

- Approximate \( f(x) = f(x_k) + f'(x_k)(x - x_k) \).
- Solve \( f(x_k) + f'(x_k)(x - x_k) = 0 \)

\[ x = x_k - \frac{f(x_k)}{f'(x_k)} \]

- Update your guess, \( x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \)
- Repeat until \( \| f(x_k) \| \) is sufficiently small.
Newton’s Method
Illustration
A scaled Newton iteration always converges for *convex functions*, \( (f''(x) > 0) \)
Given $M$, we want to solve

$$f(E) = M - E + e \sin E = 0$$

then,

$$f'(E) = -1 + e \cos E$$

**Algorithm:** Choose $E_1 = M$.

- Update

$$E_{k+1} = E_k - \frac{M - E_k + e \sin E_k}{e \cos E_k - 1}$$

- If $\|M - E_k + e \sin E_k\| < .001$ or whatever, quit.

- Otherwise repeat.
**Example**

**Problem:** Let $a = 25,512 km$ and $e = .625$. Find $r$, $v$ at $t = 4 hr$.

**Solution:** First, solve for Mean Anomaly.

$$ n = \sqrt{\frac{\mu}{a^3}} = 1.549E - 4s^{-1} $$

Thus

$$ M(t) = nt = 1.549 \cdot 10^{-4} \times 4 \times 3600 = 2.231rad $$

**Newton Iteration:** Now to solve for $E$, we set $E_1 = M$ and iterate

$$ E_2 = E_1 - \frac{2.231 - E_1 + .625 \sin E_1}{.625 \cos E_1 - 1} = 2.588 $$

$$ f(E_2) = 2.231 - E_2 + .625 \sin E_2 = -.0284 $$

We verify that $\|f(E_2)\| = .0284 > .001$, so continue:

$$ E_3 = E_2 - \frac{2.231 - E_2 + .625 \sin E_2}{.625 \cos E_2 - 1} = 2.570 $$

$$ f(E_3) = 2.231 - E_3 + .625 \sin E_3 = -.000892 $$
Example

Now $\| f(E_3) \| < .001$, so quit. $E = E_3 = 2.570$. Now Solve for true anomaly

$$f = 2 \tan^{-1} \left( \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \right) = 2.861 \text{rad}$$

$$r(t) = \frac{a(1-e^2)}{1+e \cos f(t)} = 38920 \text{km}$$

Now via vis-viva,

$$v = \sqrt{\mu \left( \frac{2}{r} - \frac{1}{a} \right)} = 2.2043 \text{km/s}$$
Conclusion

In this Lecture, you learned:

- How to predict position given time.
- New Angles
  - Mean Anomaly
  - Eccentric Anomaly
  - True Anomaly
- How to convert between them
  - How to Solve Kepler’s Equation

Key Equations:

\[ n = \sqrt{\frac{\mu}{a^3}} \]

\[ M(t) = nt \]

\[ M(t) = E(t) - e \sin E(t) \]

\[ \tan \frac{E}{2} = \sqrt{\frac{1 - e}{1 + e}} \tan \frac{f}{2} \]

Newton Iteration:

\[ E_0 = M \]

\[ E_{k+1} = E_k - \frac{M - E_k + e \sin E_k}{e \cos E_k - 1} \]