Spacecraft Dynamics and Control

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Lecture 14: Interplanetary Mission Planning
Introduction

In this Lecture, you will learn:

Sphere of Influence
  • Definition

Escape and Re-insertion
  • The light and dark of the Oberth Effect

Patched Conics
  • Heliocentric Hohmann

Planetary Flyby
  • The Gravity Assist
Consider a Simple Earth-Moon Trajectory.

1. Launch
2. Establish Parking Orbit
3. Escape Trajectory
4. Arrive at Destination
5. Circularize or Depart Destination

The big difference is that now there are 3 bodies.

- We only know how to solve the 2-body problem.
- Solving the 3-body problem is beyond us.
Patched Conics

For interplanetary travel, the problem is even more complicated.

Consider the Figure

- The motion is elliptic about the sun.
- The motion is affected by the planets
  - Interference only occurs in the green bands.
  - Motion about planets is hyperbolic.

The solution is to break the mission into segments.

- During each segment we use *two-body motion*.
- The third body is a *disturbance*.
**Question:** Who is in charge??

- The Sphere of Influence of A stops when A is no longer the dominant force.
- What do we mean by dominant?

**Wrong Definition:**
The Sphere of Influence of A is the region A exerts the largest gravitational force.

This would imply the moon is not in earth’s Sphere of Influence!!!
Sun Perspective: Lets group the forces as central and disturbing. Consider motion of a spacecraft relative to the sun:

\[
\ddot{\vec{r}}_{sv} + Gm_s \frac{\vec{r}_{sv}}{||\vec{r}_{sv}||^3} = -Gm_p \left[ \frac{\vec{r}_{pv}}{||\vec{r}_{pv}||^3} + \frac{\vec{r}_{sp}}{||\vec{r}_{sp}||^3} \right]
\]

where \(p\) denotes planet, \(v\) denotes vehicles and \(s\) denotes sun.

The Central “Force” is

\[
\ddot{\vec{r}}_{central,s} = -Gm_s \frac{\vec{r}_{sv}}{||\vec{r}_{sv}||^3}
\]

The Disturbing “Force” is

\[
\ddot{\vec{r}}_{dist,s} = -Gm_p \left[ \frac{\vec{r}_{pv}}{||\vec{r}_{pv}||^3} + \frac{\vec{r}_{sp}}{||\vec{r}_{sp}||^3} \right]
\]
**Sphere of influence**

**The Planet’s Perspective**

**Planet Perspective:** The motion of the spacecraft relative to the planet is

\[
\ddot{\vec{r}}_{pv} + Gm_p \frac{\vec{r}_{pv}}{||\vec{r}_{pv}||^3} = -Gm_s \left[ \frac{\vec{r}_{sv}}{||\vec{r}_{sv}||^3} - \frac{\vec{r}_{sp}}{||\vec{r}_{sp}||^3} \right]
\]

The **Central “Force”** for the planet is

\[
\ddot{\vec{r}}_{central,p} = -Gm_p \frac{\vec{r}_{pv}}{||\vec{r}_{pv}||^3}
\]

The **Disturbing “Force”** for the planet is

\[
\ddot{\vec{r}}_{dist,p} = -Gm_s \left[ \frac{\vec{r}_{sv}}{||\vec{r}_{sv}||^3} - \frac{\vec{r}_{sp}}{||\vec{r}_{sp}||^3} \right]
\]
Definition 1.

An object is in the **Sphere of Influence** (SOI) of body 1 if

\[
\frac{\| \ddot{r}_{\text{dist},1} \|}{\| \dot{r}_{\text{central},1} \|} < \frac{\| \ddot{r}_{\text{dist},2} \|}{\| \dot{r}_{\text{central},2} \|}
\]

for any other body 2.

That is, the ratio of disturbing “force” to central “force” determines which planet is in control.

For planets, an approximation for determining the SOI of a planet of mass \( m_p \) at distance \( d_p \) from the sun is

\[
R_{SOI} \approx \left( \frac{m_p}{m_s} \right)^{2/5} d_p
\]
### Table 7.1  Sphere of Influence Radii

<table>
<thead>
<tr>
<th>Celestial Body</th>
<th>Equatorial Radius ((km))</th>
<th>SOI Radius ((km))</th>
<th>SOI Radius (body radii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>2487</td>
<td>(1.13 \times 10^5)</td>
<td>45</td>
</tr>
<tr>
<td>Venus</td>
<td>6187</td>
<td>(6.17 \times 10^5)</td>
<td>100</td>
</tr>
<tr>
<td>Earth</td>
<td>6378</td>
<td>(9.24 \times 10^5)</td>
<td>145</td>
</tr>
<tr>
<td>Mars</td>
<td>3380</td>
<td>(5.74 \times 10^5)</td>
<td>170</td>
</tr>
<tr>
<td>Jupiter</td>
<td>71370</td>
<td>(4.83 \times 10^7)</td>
<td>677</td>
</tr>
<tr>
<td>Neptune</td>
<td>22320</td>
<td>(8.67 \times 10^7)</td>
<td>3886</td>
</tr>
<tr>
<td>Moon</td>
<td>1738</td>
<td>(6.61 \times 10^4)</td>
<td>38</td>
</tr>
</tbody>
</table>
**Example: Lunar Lander**

**Problem:** Suppose we want to plan a lunar-lander mission. Determine the spheres of influence to consider for a patched-conic approach.
- The Sphere of influence of the earth is of radius 924,000km.
- The sphere of influence of the moon is of radius 66,100km.

**Solution:** The moon orbits at a distance of 384,000km. The spacecraft will transition to the lunar sphere at distance

\[ r = 384,000 - 66,100 = 317,900 km \]

We will probably also need a plane change. A reasonable mission design is

1. Depart earth on a Hohmann transfer to radius 317,900 km.
2. Perform inclination change near apogee.
3. Enter sphere of influence of the moon.
4. Establish parking orbit.
Example: Lunar Lander

Why is a Plane Change is needed.

- Note that the lunar orbit is inclined at about $5.8^\circ$ to the ecliptic plane.
- The inclination of the lunar orbit is almost fixed with respect to the ecliptic.
- Not fixed relative to the equatorial plane (Saros cycle - Solar and $J2$).
- Inclination to equator varies by $21.3^\circ \pm 5.8^\circ$ every 18 years.
5 Stage Lunar Intercept Mission
First Stage Lunar Tug Assist
Let’s start by going through an example.

- Can serve as a template.
- Will illustrate the issues

**Problem:** Design an Earth-Venus rendez-vous. Final Venus orbit should be posigrade of altitude 500km.

**First Step:** Maneuver into a suitable parking orbit.

- Orbital plane aligned with ecliptic plane
  - \( i \approx 23^\circ \)
- Circular orbit.
  - Radius \( r \approx 6578km \)
Design a Hohmann transfer from Earth to Venus.

The perigee and apogee velocities of the transfer ellipse are

\[ v_1^+ = v_p = \sqrt{\frac{2\mu_{\text{sun}}}{r_v(r_e + r_v)}} = 37.73 \text{ km/s} \]

\[ v_2^+ = v_a = \sqrt{\frac{2\mu_{\text{sun}}}{r_e(r_e + r_v)}} = 27.29 \text{ km/s} \]

Where

- \( r_e \) is dist. from sun to earth
- \( r_v \) is dist. from sun to venus

Because Venus is an inner planet, apogee velocity occurs at Earth.

The Hohmann transfer is defined using the Sphere of Influence of the Sun

- Velocities are in the Heliocentric Frame.
Interplanetary Hohmann Transfer

We can use the Hohmann transfer (2-body, Elliptic orbits) because the voyage will take place almost exclusively in the sun’s sphere of influence.

- The earth orbits at radius \(1\, au = 1.5 \cdot 10^8\, km = 23,518\, ER\).
- The SOI of the earth is only \(145\, ER\), or .5%.
All planets in the solar system orbit the sun in the ecliptic plane.

- We need to transition to this plane.
- Transition must occur when the orbital plane and ecliptic planes intersect.

Any orbit about the earth passes through the ecliptic twice per orbit.

- But not at the ascending node (w/r to the equatorial plane).
- But not at the correct time.
To change to the ecliptic plane:

- Burn at ascending node w/r to the ecliptic plane.
- Execute a plane change.

We have already covered these kinds of maneuvers!
Our desired orbit has

- $i_2 = \epsilon = 23.5^\circ$ - Inclination to the ecliptic
- $\Omega_2 = 0^\circ$ - by definition: $\Omega$ is measured from FPOA (intersection of equatorial and ecliptic planes).

If our initial orbit has inclination $i_1$ and RAAN $\Omega_1$, then the angle change is

$$\cos \theta = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos(\Omega_2 - \Omega_1)$$
The position in the orbit is given by

$$\cos(\omega + f) = \frac{\cos i_1 \cos \theta - \cos i_2}{\sin i_1 \sin \theta}$$

Where recall

- \(i_2 = \epsilon = 23.5^\circ\)
The Plane Change

The $\Delta v$ required for the plane change is then

$$\Delta v = 2v \sin \frac{\theta}{2}$$
Interplanetary Hohmann Transfer
Injection \( (v_a) \)

**Problem:** How much \( \Delta v \) do we need to achieve \( v_a \)?

- \( v_a \) is in the heliocentric frame.
- We start in the earth-frame
  - The earth frame is moving with velocity
    \[
    v_e = \sqrt{\frac{\mu_s}{\|r_{se}\|}} = 29.78 \text{ km/s}
    \]
- We must find \( v_{\infty,e} \) - velocity with respect to the earth.

We have

\[
v_2^+ = v_a = v_e^- + v_{\infty,e}
\]

Thus our desired velocity with respect to the earth is

\[
v_{\infty,e} = v_a - v_e^- = 27.29 - 29.78 = -2.49 \text{ km/s}
\]
Problem: How to achieve the initial $v_\infty, e = -2.49 \text{km/s}$?
- We need to escape earth orbit.
- Must have leftover velocity (excess velocity) of $2.49 \text{km/s}$.
  - Implies the total energy after burn is
    $$ E_+ = \frac{1}{2} v_\infty^2, e = 2.067 $$
Interplanetary Hohmann Transfer

Suppose the spacecraft starts in a circular parking orbit of radius \( r_{park} = 6578 \).

- The velocity before the burn will be
  \[
  v_{park} = \sqrt{\frac{\mu_e}{r_{park}}} = 7.7843 \text{km/s}
  \]

- The velocity after burn \( v_{b,e} \) can be found by solving the energy equation.
  \[
  E = \frac{1}{2} v_{b,e}^2 - \frac{\mu_e}{r_{park}} = +2.067
  \]

Solving for \( v_{b,e} \), we get
\[
 v_{b,e} = \sqrt{2E + 2\frac{\mu_e}{r_{park}}} = \sqrt{v_{\infty,e}^2 + 2\frac{\mu_e}{r_{park}}} = 11.195 \text{km/s}
\]

- This yields a \( \Delta v_e \) of
  \[
  \Delta v_e = v_{b,e} - v_{park} = 3.4106 \text{km/s}
  \]
Light Side / Dark Side:

- The earth rotates counterclockwise about the sun.
- Vehicles typically orbit counterclockwise about the earth.

In this configuration, the burn occurs

- On the dark side for $\Delta v_\infty, e > 0$
  - Missions to outer planets.
- On the light side for $\Delta v_\infty, e < 0$
  - Missions to inner planets
**Timing:** The $\Delta v$ should occur at $\delta/2$ before midnight/noon, where the turning angle

$$\delta = 2 \sin^{-1} \frac{1}{e}$$

Eccentricity ($e$) can be found as:

- **Energy:** $E = \frac{1}{2} v_\infty^2, e = 2.067 = -\frac{\mu}{2a}$ yields

$$a = -\frac{\mu}{v_\infty^2, e} = -\frac{\mu}{2E} = -96,420 km$$

- **Perigee:** $r_{p,e} = r_c = a(1 - e) = 6578 km$ yields

$$e = 1 - \frac{r_{p,e}}{a} = 1.0682$$

This yields a turning angle of

$$\delta = 2.423 rad = 138.83^\circ$$

Thus the spacecraft should depart at $\delta/2 = 69.4^\circ$ before noon/midnight.
Arrival at Venus

At arrival, our excess velocity w/r to Venus \((v_\infty,v)\) will be

\[
v_\infty,v = v_p - v_v = v_1^- - v_1^+ = 37.81\text{km/s} - 35.09\text{km/s} = 2.71\text{km/s}
\]

where

- \(v_1^+ = v_v\) is the velocity of venus
  
  \[
v_1^+ = v_v = \sqrt{\frac{\mu_s}{r_v}}
  \]

- \(v_p\) is the periapse velocity of the Hohmann transfer

Because \(v_\infty,v > 0\), the spacecraft will approach Venus from behind.

- Spacecraft is catching up to planet (not vice-versa)
- The back door
Arrival at Venus

Venus Data:

\[ R_v = 6187 \text{km}, \quad \mu_v = 324859, \quad a_{\text{Venus}} = 1.08 \cdot 10^8 \]

Desired Orbit: Circular, posigrade grade (counterclockwise) with

\[ r_c = 6187 + 500 = 6687 \text{km} \]

For a counterclockwise orbit, we want to approach Venus on the Dark Side
Arrival at Venus

For orbital insertion, we want to perform a retrograde burn at periapse of the incoming hyperbola.

To achieve a circular orbit of radius \( r_c = 6687 \text{ km} \), we need the periapse of our incoming hyperbola to occur at

\[
r_{p,v} = a(1 - e) = 6687 \text{ km}
\]

The energy of the incoming hyperbola is given by the excess velocity as

\[
E = \frac{1}{2} v_{\infty,v}^2 = 3.67
\]

This fixes the semimajor axis at

\[
a = -\frac{\mu v}{v_{\infty,f,v}^2} = -44,232 \text{ km}
\]

Thus to achieve \( r_p = a(1 - e) \), we need

\[
e = 1 - \frac{r_p}{a} = 1.15
\]
Arrival at Venus

To achieve the desired $e = 1.15$, we control the conditions at the *Patch Point*.

- We do this through the angular momentum, $h$.

We can control the **Target Radius**, $\Delta$ through small adjustments far from the planet. Angular momentum can be exactly controlled through target radius, $\Delta$.

\[ h_v = v_\infty v \Delta \]
Arrival at Venus

**Solution:** For a given $a$, $e$ is determined by $p = a(1 - e^2)$.

- But $p$ is defined by angular momentum (and thus target radius).

$$p = \frac{h^2}{\mu_v} = \frac{\Delta^2 v^2_{\infty,v}}{\mu_v}$$

- For $a = -44,232 km$ and $e = 1.15$, we get $p = 14,265 km$.

Given a desired $p$ we solve for target radius, $\Delta$,

$$\Delta = \sqrt{\frac{p\mu_v}{v^2_{\infty,v}}} = \sqrt{\frac{a(1 - e^2)\mu_v}{v^2_{\infty,v}}} = 25,120 km$$
Injection into Circular Orbit

Finally, we need to slow down to achieve circular orbit.

- The velocity at periapse (6687km) is given by the vis-viva equation.
  \[ v = \sqrt{\frac{2\mu v}{r_{p,v}} - \frac{\mu v}{a}} = 10.223\, \text{km/s} \]
- The velocity of a circular orbit is
  \[ v_c = \sqrt{\mu v r_{p,v}} = 6.97\, \text{km/s} \]

Thus the \( \Delta v \) required to circularize the orbit is
\[ \Delta v = 6.97 - 10.223 = -3.253\, \text{km/s} \]

**Figure:** Aerobraking can also assist with \( \Delta v \)
Messenger Probe to Mercury
Gravity Assist Trajectories

**Concept:** Planets rotate the relative velocity vector.
- The relative motion changes as
  \[ \vec{v}_f - \vec{v}_{\text{planet}} = R_1(\delta) (\vec{v}_i - \vec{v}_{\text{planet}}) \]
- In the inertial frame (2 dimensions) this means
  \[ \vec{v}_f = \begin{bmatrix} \cos \delta & -\sin \delta \\ \sin \delta & \cos \delta \end{bmatrix} (\vec{v}_i - \vec{v}_{\text{planet}}) + \vec{v}_{\text{planet}} \]

**Example:** If \( \delta = 180^\circ \) and \( \vec{v}_i = \begin{bmatrix} -20 \\ 0 \end{bmatrix} \) km/s and \( \vec{v}_p = \begin{bmatrix} 20 \\ 0 \end{bmatrix} \) km/s, then
  \[ v_f = R(180^\circ) \begin{bmatrix} -40 \\ 0 \end{bmatrix} \text{ km/s} + \begin{bmatrix} 20 \\ 0 \end{bmatrix} \text{ km/s} = \begin{bmatrix} 40 \\ 0 \end{bmatrix} \text{ km/s} + \begin{bmatrix} 20 \\ 0 \end{bmatrix} \text{ km/s} = \begin{bmatrix} 60 \\ 0 \end{bmatrix} \text{ km/s} \]

Thus a probe can potentially *triple* its velocity!
Gravity Assist Trajectories

To achieve the desired turning angle, we must control the geometry.

The turning angle $\delta$ is given by

$$\delta = 2 \sin^{-1} \frac{1}{e}$$

Recall that energy of the orbit is fixed. Thus we can solve for

$$a = -\mu_{\text{planet}} / \| \vec{v}_i - \vec{v}_{\text{planet}} \|^2$$

Then the eccentricity can be fixed by the target radius as

$$\Delta = \sqrt{\frac{a(1 - e^2)\mu_{\text{planet}}}{\| \vec{v}_i - \vec{v}_{\text{planet}} \|^2}} = 25,120 \text{km}$$

In 3 dimensions, the calculations are more complex.
Gravity Assist Trajectories

Example: Jupiter flyby

**Problem:** Suppose we perform a Hohman transfer from Earth to Jupiter. What is the best-case gravity assist we can expect?

**Solution:** The velocity of arrival at apogee (Jupiter) in the Heliocentric frame is:

\[ v_a = \sqrt{\frac{2\mu_{\text{sun}}}{r_j(r_j + r_e)}} = 7.414 \text{ km/s} \]

The velocity of jupiter itself is

\[ v_j = \sqrt{\frac{\mu_s}{d_j}} = 13.0573 \text{ km/s} \]

Since this is an outer planet, we approach from the front door. In the jupiter \( R - T - N \) frame we have

\[ \vec{v}_s = \begin{bmatrix} 7.414 \\ 0 \end{bmatrix}, \quad \vec{v}_j = \begin{bmatrix} 13.0573 \\ 0 \end{bmatrix} \]

The velocity of the spacecraft relative to jupiter is

\[ \vec{v}_s - \vec{v}_j = \begin{bmatrix} -5.6429 \\ 0 \end{bmatrix} \text{ km/s} \]
**Example: Jupiter flyby**

**Jupiter Data:** Radius $r_j = 11.209ER$; Distance $d_j = 5.2028AU$; $\mu_j = 317.938\mu_e$.

The velocity of the spacecraft relative to Jupiter is

$$\vec{v}_s - \vec{v}_j = \begin{bmatrix} -5.6429 \\ 0 \end{bmatrix} \text{ km/s}$$

Thus we can calculate the energy of the hyperbolic approach as

$$a = -\frac{\mu_s}{||\vec{v}_s - \vec{v}_j||^2} = -3.98E6\text{km}$$

The closest we can approach Jupiter is its radius. If we use this for periapse, we get

$$e = 1 - \frac{r_j}{a} = 1.018$$

The eccentricity yields the maximum turning angle as

$$\delta = 2 \sin^{-1} \left( \frac{1}{e} \right) = 158.44^\circ$$
Example: Jupiter flyby

Applying this rotation (light-side approach), we get

\[
\vec{v}_f = \begin{bmatrix} \cos \delta & -\sin \delta \\ \sin \delta & \cos \delta \end{bmatrix} (\vec{v}_i - \vec{v}_{planet}) + \vec{v}_{planet} = \begin{bmatrix} 18.305 \\ 2.076 \end{bmatrix}
\]

The magnitude of the $\Delta v$ from this flyby is 11.01km/s. A factor of 2.5.

Note that if we could have reversed our direction of flight (clockwise approach), we could achieve a $\Delta v = 20.05\text{km/s}$. 
Trajectories for Voyager 1 and Voyager 2 Spacecraft
Trajectories for Voyager 1, Voyager 2, and Pioneer Spacecraft

Pioneer-10: 3 March 1972
Pioneer 11: 6 April, 1973
Voyager 2: 20 August 1977
Voyager 1: 5 September 1977
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