Lecture 16: Euler’s Equations
Attitude Dynamics

In this Lecture we will cover:

**The Problem of Attitude Stabilization**
- Actuators

**Newton’s Laws**
- $\sum \vec{M}_i = \frac{d}{dt} \vec{H}$
- $\sum \vec{F}_i = m \frac{d}{dt} \vec{v}$

**Rotating Frames of Reference**
- Equations of Motion in Body-Fixed Frame
- Often Confusing
If in doubt, use the right-hand rules.

**Figure:** Positive Directions

**Figure:** Positive Rotations
There are 3 basic rotations a vehicle can make:
- Roll = Rotation about $x$-axis
- Pitch = Rotation about $y$-axis
- Yaw = Rotation about $z$-axis
Each rotation is a one-dimensional transformation.
Any two coordinate systems can be related by a sequence of 3 rotations.
These forces and moments have standard labels. The Forces are:

- **X** Axial Force  Net Force in the positive $x$-direction
- **Y** Side Force   Net Force in the positive $y$-direction
- **Z** Normal Force Net Force in the positive $z$-direction
The Moments are called, intuitively:

- $L$: Rolling Moment  
  Net Moment in the positive $\omega_x$-direction
- $M$: Pitching Moment  
  Net Moment in the positive $\omega_y$-direction
- $N$: Yawing Moment  
  Net Moment in the positive $\omega_z$-direction
Newton’s Second Law tells us that for a particle \( F = ma \). In vector form:

\[
\vec{F} = \sum_i \vec{F}_i = m \frac{d}{dt} \vec{V}
\]

That is, if \( \vec{F} = [F_x \ F_y \ F_z] \) and \( \vec{V} = [u \ v \ w] \), then

\[
F_x = m \frac{du}{dt} \quad F_y = m \frac{dv}{dt} \quad F_z = m \frac{dw}{dt}
\]

**Definition 1.**

\( m\vec{V} \) is referred to as **Linear Momentum**.

Newton’s Second Law is only valid if \( \vec{F} \) and \( \vec{V} \) are defined in an **Inertial** coordinate system.

**Definition 2.**

A coordinate system is **Inertial** if it is not accelerating or rotating.
6DOF: Newton’s Laws

Forces

Newton’s Second Law tells us that for a particle \( F = ma \). In vector form:

\[
\vec{F} = \sum \vec{F}_i = m \frac{d\vec{V}}{dt}
\]

That is, if \( \vec{F} = [F_x, F_y, F_z] \) and \( \vec{V} = [v_x, v_y, v_z] \), then

\[
F_x = m \frac{dv_x}{dt} \quad F_y = m \frac{dv_y}{dt} \quad F_z = m \frac{dv_z}{dt}
\]

Definition 1.

\( m\vec{V} \) is referred to as Linear Momentum.

Newton’s Second Law is only valid if \( \vec{F} \) and \( \vec{V} \) are defined in an Inertial coordinate system.

Definition 2.

A coordinate system is Inertial if it is not accelerating or rotating.

We are not in an inertial frame because the Earth is rotating

- ECEF vs. ECI
Newton’s Laws

Moments

Using Calculus, momentum can be extended to rigid bodies by integration over all particles.

\[ \vec{M} = \sum_i \vec{M}_i = \frac{d}{dt} \vec{H} \]

**Definition 3.**

Where \( \vec{H} = \int (\vec{r}_c \times \vec{v}_c) dm \) is the **angular momentum**.

Angular momentum of a rigid body can be found as

\[ \vec{H} = I \vec{\omega}_I \]

where \( \vec{\omega}_I = [p, q, r]^T \) is the angular rotation vector of the body about the center of mass.

- \( p = \omega_x \) is rotation about the \( x \)-axis.
- \( q = \omega_y \) is rotation about the \( y \)-axis.
- \( r = \omega_z \) is rotation about the \( z \)-axis.
- \( \vec{\omega}_I \) is defined in an **Inertial Frame**.

The matrix \( I \) is the **Moment of Inertia Matrix** (Here also in inertial frame!).
Newton’s Laws

Using Calculus, momentum can be extended to rigid bodies by integration over all particles.

\[ \vec{\mathbf{M}} = \sum_i \vec{\mathbf{M}}_i = \frac{d}{dt} \vec{\mathbf{H}} \]

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\( \vec{\mathbf{\omega}}_I \) is defined in an Inertial Frame.

The matrix \( I \) is the Moment of Inertia Matrix (Here also in inertial frame!)

\( \vec{\mathbf{r}}_c \) and \( \vec{\mathbf{v}}_c \) are position and velocity vectors with respect to the centroid of the body.
Newton’s Laws
Moment of Inertia

The moment of inertia matrix is defined as

\[
I = \begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{yx} & I_{yy} & -I_{yz} \\
-I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix}
\]

\[
I_{xy} = I_{yx} = \int \int \int xy \, dm \\
I_{xz} = I_{zx} = \int \int \int xz \, dm \\
I_{yz} = I_{zy} = \int \int \int yz \, dm
\]

\[
I_{xx} = \int \int \int (y^2 + z^2) \, dm \\
I_{yy} = \int \int \int (x^2 + z^2) \, dm \\
I_{zz} = \int \int \int (x^2 + y^2) \, dm
\]

So

\[
\begin{bmatrix}
H_x \\
H_y \\
H_z
\end{bmatrix} = \begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{yx} & I_{yy} & -I_{yz} \\
-I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix} \begin{bmatrix}
p_I \\
q_I \\
r_I
\end{bmatrix}
\]

where \(p_I\), \(q_I\) and \(r_I\) are the rotation vectors as expressed in the inertial frame corresponding to \(x\)-\(y\)-\(z\).
Moment of Inertia

Examples:

**Homogeneous Sphere**

\[ I_{sphere} = \frac{2}{5}mr^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

**Ring**

\[ I_{ring} = mr^2 \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
Moment of Inertia

Examples:

Homogeneous Disk

\[ I_{\text{disk}} = \frac{1}{4} mr^2 \begin{bmatrix} 1 + \frac{1}{3} \frac{h}{r^2} & 0 & 0 \\ 0 & 1 + \frac{1}{3} \frac{h}{r^2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \]

F/A-18

\[ I = \begin{bmatrix} 23 & 0 & 2.97 \\ 0 & 15.13 & 0 \\ 2.97 & 0 & 16.99 \end{bmatrix} \text{ kslug} - \text{ ft}^2 \]
### Moment of Inertia

#### Examples:

**Cube**

\[ I_{cube} = \frac{2}{3} \ell^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

**Box**

\[ I_{box} = \begin{bmatrix} \frac{b^2+c^2}{3} & 0 & 0 \\ 0 & \frac{a^2+c^2}{3} & 0 \\ 0 & 0 & \frac{a^2+b^2}{3} \end{bmatrix} \]
Moment of Inertia

Examples:

\[
\begin{bmatrix}
8655.2 & -144 & 132.1 \\
-144 & 7922.7 & 192.1 \\
132.1 & 192.1 & 4586.2
\end{bmatrix} \quad \text{kg} \cdot \text{m}^2
\]

1.5-m antenna

\[
\begin{bmatrix}
473.924 & 0 & 0 \\
0 & 494.973 & 0 \\
0 & 0 & 269.83
\end{bmatrix} \quad \text{kg} \cdot \text{m}^2
\]

Gallium arsenide solar panels

450-N thruster

Instruments

Cassini

NEAR Shoemaker
Problem:
The Body-Fixed Frame

The moment of inertia matrix, $I$, is fixed in the body-fixed frame. However, Newton’s law only applies for an inertial frame:

$$\vec{M} = \sum_i \vec{M}_i = \frac{d}{dt} \vec{H}$$

Suppose the body-fixed frame is rotating with angular velocity vector $\vec{\omega}$. Then for any vector, $\vec{a}$, $\frac{d}{dt} \vec{a}$ in the inertial frame is

$$\frac{d\vec{a}}{dt} \bigg|_I = \frac{d\vec{a}}{dt} \bigg|_B + \vec{\omega} \times \vec{a}$$

Specifically, for Newton’s Second Law

$$\vec{F} = m \frac{d\vec{V}}{dt} \bigg|_B + m\vec{\omega} \times \vec{V}$$

and

$$\vec{M} = \frac{d\vec{H}}{dt} \bigg|_B + \vec{\omega} \times \vec{H}$$
The equation for acceleration (which we will ignore) is:

\[
\begin{bmatrix}
F_x \\
F_y \\
F_z
\end{bmatrix} = m \frac{d\vec{V}}{dt} \bigg|_B + m\vec{\omega} \times \vec{V}
\]

\[
= m \begin{bmatrix}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{bmatrix} + m \det
\begin{bmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\omega_x & \omega_y & \omega_z \\
u & v & w
\end{bmatrix}
\]

\[
= m \begin{bmatrix}
\dot{u} + \omega_y w - \omega_z v \\
\dot{v} + \omega_z u - \omega_x w \\
\dot{w} + \omega_x v - \omega_y u
\end{bmatrix}
\]

As we will see, displacement and rotation in space are decoupled.

- no aerodynamic forces.
Equations of Motion

The equations for rotation are:

\[
\begin{bmatrix}
L \\
M \\
N
\end{bmatrix} = \frac{d\vec{H}}{dt} \bigg|_B + \vec{\omega} \times \vec{H}
\]

\[
\begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{yx} & I_{yy} & -I_{yz} \\
-I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix}
\begin{bmatrix}
\dot{\omega}_x \\
\dot{\omega}_y \\
\dot{\omega}_z
\end{bmatrix} + \vec{\omega} \times
\begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{yx} & I_{yy} & -I_{yz} \\
-I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix}
\begin{bmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{bmatrix}
\]

\[
\begin{bmatrix}
I_{xx} \dot{\omega}_x - I_{xy} \dot{\omega}_y - I_{xz} \dot{\omega}_z \\
-I_{xy} \dot{\omega}_x + I_{yy} \dot{\omega}_y - I_{yz} \dot{\omega}_z \\
-I_{zx} \dot{\omega}_x - I_{zy} \dot{\omega}_y + I_{zz} \dot{\omega}_z
\end{bmatrix} + \vec{\omega} \times
\begin{bmatrix}
\omega_x I_{xx} - \omega_y I_{xy} - \omega_z I_{xz} \\
-\omega_x I_{xy} + \omega_y I_{yy} - \omega_z I_{yz} \\
-\omega_x I_{xz} - \omega_y I_{yz} + \omega_z I_{zz}
\end{bmatrix}
\]

Which is too much for any mortal. We simplify as:

- For spacecraft, we have \(I_{yz} = I_{xy} = I_{xz} = 0\) (two planes of symmetry).
- For aircraft, we have \(I_{yz} = I_{xy} = 0\) (one plane of symmetry).
If we use the matrix version of the cross-product, we can write

\[
\vec{M} = I\dot{\omega}(t) + [\omega(t)] \times I\omega(t)
\]

Which is a much-simplified version of the dynamics!

Recall

\[
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} \times \begin{bmatrix}
0 & -z & y \\
z & 0 & -x \\
-y & x & 0 \\
\end{bmatrix}
\]
With $I_{xy} = I_{yz} = I_{xz} = 0$, we get: Euler’s Equations

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + \omega_y\omega_z(I_{zz} - I_{yy}) \\ I_{yy}\dot{\omega}_y + \omega_x\omega_z(I_{xx} - I_{zz}) \\ I_{zz}\dot{\omega}_z + \omega_x\omega_y(I_{yy} - I_{xx}) \end{bmatrix}$$

Thus:

- Rotational variables $(\omega_x, \omega_y, \omega_z)$ do not depend on translational variables $(u, v, w)$.
  - For spacecraft, Moment forces $(L, M, N)$ do not depend on rotational and translational variables.
  - Can be decoupled
- However, translational variables $(u, v, w)$ depend on rotation $(\omega_x, \omega_y, \omega_z)$.
  - But we don’t care.
Euler Equations
Torque-Free Motion

Notice that even in the absence of external moments, the dynamics are still active:

\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = 
\begin{bmatrix}
I_x \dot{\omega}_x + \omega_y \omega_z (I_z - I_y) \\
I_y \dot{\omega}_y + \omega_x \omega_z (I_x - I_z) \\
I_z \dot{\omega}_z + \omega_x \omega_y (I_y - I_x)
\end{bmatrix}
\]

which yield the 3-state nonlinear ODE:

\[
\begin{align*}
\dot{\omega}_x &= -\frac{I_z - I_y}{I_x} \omega_y(t) \omega_z(t) \\
\dot{\omega}_y &= -\frac{I_x - I_z}{I_y} \omega_x(t) \omega_z(t) \\
\dot{\omega}_z &= -\frac{I_y - I_x}{I_z} \omega_x(t) \omega_y(t)
\end{align*}
\]

Thus even in the absence of external moments

- The axis of rotation \( \bar{\omega} \) will evolve
- Although the angular momentum vector \( \bar{\mathbf{h}} \) will NOT.
  - occurs because tensor \( I \) changes in inertial frame.
- This can be problematic for spin-stabilization!
We can use Euler's equation to study Spin Stabilization.

There are two important cases:

**Axisymmetric:** $I_x = I_y$

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_x & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

**Non-Axisymmetric:** $I_x \neq I_y$

$$I = \begin{bmatrix} 473.924 & 0 & 0 \\ 0 & 494.973 & 0 \\ 0 & 0 & 269.83 \end{bmatrix} \text{ kg}\cdot\text{m}^2$$
We can use Euler’s equation to study Spin Stabilization. There are two important cases:

1. Introduce products of inertia in the spacecraft inertia tensor.
2. Angular Momentum coincides with Nominal Spin Axis.

Figure 3: Spin Stabilized Spacecraft

Angular Momentum Vector
Nominal Spin Axis
Nutation Angle

Figure 4: Spacecraft Nutational Motion

In the absence of energy dissipation, nutational motion is stable about the axis of either the maximum or minimum moment of inertia. This implies that the amplitude of motion is bounded by initial conditions. However, all real spacecraft experience some form of energy dissipation. In this case, nutational motion is only stable about the axis of maximum moment of inertia. The axes of minimum and maximum moments of inertia are referred to as minor and major axes, respectively. Thus, if a spacecraft is spinning about its minor axis, nutational motion will grow until the spacecraft tumbles and eventually reorients itself spinning about its major axis. Reorientation of the spin axis is illustrated in Figure 5. Conversely, if a spacecraft is spinning about its major axis, any nutational motion will simply decay.

ASMOS can be used to investigate stability and energy dissipation effects. With ASMOS, the user can introduce various rates of internal energy dissipation into the rigid body model by entering viscous damping coefficients and wheel inertias. The user can then watch resulting motion. This motion can also be plotted for further analysis.

Figure 5: Reorientation of the Spin Axis

Angular Momentum Vector & New Spin Axis
Old Spin Axis

Conclusion

ASMOS is a simulation tool that incorporates animated 3-D computer graphics to visualize spacecraft attitude motion. The program runs on Macintosh personal computers and features pull down menus and dialog boxes making the program accessible and easy to use. The program is capable of simulating and animating a wide range of rigid body attitude motion. The rigid body model includes an energy sink for investigating stability and energy dissipation effects.

References


Note we say a body is axisymmetric if $I_x = I_y$.

- We don’t need rotational symmetry...
An important case is spin-stabilization of an axisymmetric spacecraft.

- Assume symmetry about z-axis \((I_x = I_y)\)

Then recall

\[
\dot{\omega}_z = -\frac{I_y - I_x}{I_z} \omega_x(t) \omega_y(t) = 0
\]

Thus \(\omega_z = \text{constant}\).

The equations for \(\omega_x\) and \(\omega_y\) are now

\[
\begin{bmatrix}
\dot{\omega}_x \\
\dot{\omega}_y
\end{bmatrix} = \begin{bmatrix}
0 & -\frac{I_z - I_y}{I_x} \omega_z \\
-\frac{I_x - I_z}{I_y} \omega_z & 0
\end{bmatrix} \begin{bmatrix}
\omega_x(t) \\
\omega_y(t)
\end{bmatrix}
\]

Which is a linear ODE.
Spin Stabilization
Axisymmetric Case

Fortunately, linear systems have closed-form solutions. Let \( \lambda = \frac{I_z - I_x}{I_x} \omega_z \). Then

\[
\dot{\omega}_x(t) = -\lambda \omega_y(t) \\
\dot{\omega}_y(t) = \lambda \omega_x(t)
\]

Combining, we get

\[
\ddot{\omega}_x(t) = -\lambda^2 \omega_x(t)
\]

which has solution

\[
\omega_x(t) = \omega_x(0) \cos(\lambda t) + \frac{\dot{\omega}_x(0)}{\lambda} \sin(\lambda t)
\]

Differentiating, we get

\[
\omega_y(t) = -\frac{\dot{\omega}_x(t)}{\lambda} = \omega_x(0) \sin(\lambda t) - \frac{\dot{\omega}_x(0)}{\lambda} \cos(\lambda t)
\]

\[
= \omega_x(0) \sin(\lambda t) + \omega_y(0) \cos(\lambda t)
\]

\[
\omega_x(t) = \omega_x(0) \cos(\lambda t) - \omega_y(0) \sin(\lambda t)
\]
Spin Stabilization

Axisymmetric Case

Define $\omega_{xy} = \sqrt{\omega_x^2 + \omega_y^2}$.

$$\omega_{xy}^2 = (\omega_x(0) \sin(\lambda t) + \omega_y(0) \cos(\lambda t))^2 + (\omega_x(0) \cos(\lambda t) - \omega_y(0) \sin(\lambda t))^2$$

$$= \omega_x(0)^2 \sin^2(\lambda t) + \omega_y(0)^2 \cos^2(\lambda t) + 2\omega_x(0)\omega_y(0) \cos(\lambda t) \sin(\lambda t)$$

$$+ \omega_x(0)^2 \cos^2(\lambda t) + \omega_y(0)^2 \sin^2(\lambda t) - 2\omega_x(0)\omega_y(0) \cos(\lambda t) \sin(\lambda t)$$

$$= \omega_x(0)^2 (\sin^2(\lambda t) + \cos^2(\lambda t)) + \omega_y(0)^2 (\cos^2(\lambda t) + \sin^2(\lambda t))$$

$$= \omega_x(0)^2 + \omega_y(0)^2$$

Thus

- $\omega_z$ is constant
  - rotation about axis of symmetry
- $\sqrt{\omega_x^2 + \omega_y^2}$ is constant
  - rotation perpendicular to axis of symmetry

This type of motion is often called **Precession**!
Circular Motion in the Body-Fixed Frame

Thus

\[
\omega(t) = \begin{bmatrix}
\omega_x(t) \\
\omega_y(t) \\
\omega_z(t)
\end{bmatrix} = \begin{bmatrix}
\cos(\lambda t) & -\sin(\lambda t) & 0 \\
\sin(\lambda t) & \cos(\lambda t) & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\omega_x(0) \\
\omega_y(0) \\
\omega_z(0)
\end{bmatrix} = R_3(\lambda t) \begin{bmatrix}
\omega_x(0) \\
\omega_y(0) \\
\omega_z(0)
\end{bmatrix}
\]
For $\lambda > 0$, this is a Positive (counterclockwise) rotation, about the z-axis, of the angular velocity vector $\omega$ as expressed in the body-fixed coordinates!
Prolate vs. Oblate

The speed of the precession is given by the natural frequency:

$$\lambda = \frac{I_z - I_x}{I_x} \omega_z$$

with period

$$T = \frac{2\pi}{\lambda} = \frac{2\pi I_x}{I_z - I_x} \omega_z^{-1}.$$  

**Direction of Precession:** There are two cases

**Definition 4 (Direct).**

An axisymmetric (about $z$-axis) rigid body is **Prolate** if $I_z < I_x = I_y$.

**Definition 5 (Retrograde).**

An axisymmetric (about $z$-axis) rigid body is **Oblate** if $I_z > I_x = I_y$.

Thus we have two cases:

- $\lambda > 0$ if object is *Oblate* (CCW rotation)
- $\lambda < 0$ if object is *Prolate* (CW rotation)

Note that these are rotations of $\omega$, as expressed in the **Body-Fixed** Frame.
Pay Attention to the Body-Fixed Axes

The black arrow is $\vec{\omega}$.

- The body-fixed $x$ and $y$ axes are indicated with red and green dots.
- Notice the direction of rotation of $\omega$ with respect to these dots.
- The angular momentum vector is the inertial $z$ axis.
As these videos illustrate, we are typically interested in motion in the Inertial Frame.

- Use of Rotation Matrices is complicated.
  - Which coordinate system to use???
- Lets consider motion relative to $\vec{h}$.
  - Which is fixed in inertial space.

We know that in Body-Fixed coordinates,

$$\vec{h} = I\vec{\omega} = \begin{bmatrix} I_x\omega_x \\ I_y\omega_y \\ I_z\omega_z \end{bmatrix}$$

Now lets find the orientation of $\omega$ and $\hat{z}$ with respect to this fixed vector.
Motion in the Inertial Frame

Let \( \hat{x}, \hat{y} \) and \( \hat{z} \) define the body-fixed unit vectors.

We first note that since \( I_x = I_y \) and

\[
\vec{h} = I_x \omega_x \hat{x} + I_y \omega_y \hat{y} + I_z \omega_z \hat{z}
\]

\[
= I_x (\omega_x \hat{x} + \omega_y \hat{y} + \omega_z \hat{z}) + (I_z - I_x) \omega_z \hat{z}
\]

we have that

\[
\vec{\omega} = \frac{1}{I_x} \vec{h} + \frac{I_z - I_x}{I_x \omega_z} \hat{z}
\]

which implies that \( \vec{\omega} \) lies in the \( \hat{z} - \vec{h} \) plane.
We now focus on two constants of motion

- \( \theta \) - The angle \( \vec{h} \) makes with the body-fixed \( \hat{z} \) axis.
- \( \gamma \) - The angle \( \vec{\omega} \) makes with the body-fixed \( \hat{z} \) axis.

Since

\[
\vec{h} = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \begin{bmatrix} I_x \omega_x \\ I_y \omega_y \\ I_z \omega_z \end{bmatrix}
\]

The angle \( \theta \) is defined by

\[
\tan \theta = \frac{\sqrt{h_x^2 + h_y^2}}{h_z} = \frac{I_x \sqrt{\omega_x^2 + \omega_y^2}}{I_z \omega_z} = \frac{I_x \omega_{xy}}{I_z \omega_z}
\]

Since \( \omega_{xy} \) and \( \omega_z \) are fixed, \( \theta \) is a constant of motion.
We now focus on two constants of motion:

- \( \theta \) - The angle \( \vec{h} \) makes with the body-fixed \( \hat{z} \) axis.
- \( \gamma \) - The angle \( \vec{\omega} \) makes with the body-fixed \( \hat{z} \) axis.

Since

\[
\vec{h} = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \begin{bmatrix} I_x \omega_x \\ I_y \omega_y \\ I_z \omega_z \end{bmatrix}
\]

The angle \( \theta \) is defined by

\[
\tan \theta = \sqrt{h_x^2 + h_y^2} = \sqrt{I_x \omega_x^2 + I_y \omega_y^2} = \frac{I_x \omega_x}{I_z \omega_z}
\]

Since \( \omega_{xy} \) and \( \omega_z \) are fixed, \( \theta \) is a constant of motion. 

Again, \( \vec{h} \) here is in the body-fixed frame.

- This is why it changes over time.
The second angle to consider is

- \( \gamma \) - The angle \( \vec{\omega} \) makes with the body-fixed \( \hat{z} \) axis.

As before, the angle \( \gamma \) is defined by

\[
\tan \gamma = \frac{\sqrt{\omega_x^2 + \omega_y^2}}{\omega_z} = \frac{\omega_{xy}}{\omega_z}
\]

Since \( \omega_{xy} \) and \( \omega_z \) are fixed, \( \gamma \) is a constant of motion.

- We have the relationship

\[
\tan \theta = \frac{I_x \omega_{xy}}{I_z \omega_z} = \frac{I_x}{I_z} \tan \gamma
\]

Thus we have two cases:

1. \( I_x > I_z \) - Then \( \theta > \gamma \)
2. \( I_x < I_z \) - Then \( \theta < \gamma \) (As Illustrated)
Motion in the Inertial Frame

Figure: The case of $I_x > I_z$ ($\theta > \gamma$)

Figure: The case of $I_z > I_x$ ($\gamma > \theta$)
We illustrate the motion using the Space Cone and Body Cone

- The space cone is fixed in inertial space (doesn't move)
- The space cone has width $|\omega - \theta|$
- The body cone is centered around the z-axis of the body.
- In body-fixed coordinates, the space cone rolls around the body cone (which is fixed)
- In inertial coordinates, the body cone rolls around the space cone (which is fixed)
Motion in the Inertial Frame

The orientation of the body in the inertial frame is defined by the sequence of Euler rotations

- $\psi$ - $R_3$ rotation about $\vec{h}$.
  - Aligns $h_x$ perpendicular to $\hat{z}$.
- $\theta$ - $R_1$ rotation by angle $\theta$ about $h_x$.
  - Rotate $h_z$-axis to body-fixed $\hat{z}$ vector
  - We have shown that this angle is fixed!
  - $\dot{\theta} = 0$.
- $\phi$ - $R_3$ rotation about body-fixed $\hat{z}$ vector.
  - Aligns $h_x$ to $\hat{x}$.

The Euler angles are related to the angular velocity vector as

$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{\psi} \sin \theta \sin \phi \\ \dot{\psi} \sin \theta \cos \phi \\ \dot{\phi} + \dot{\psi} \cos \theta = constant \end{bmatrix}$
This comes from

\[ \mathbf{\ddot{\omega}} = R_3(\phi)R_1(\theta)R_3(\psi) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + R_3(\phi)R_1(\theta) \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + R_3(\phi) \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} \]

\[ = \begin{bmatrix} \dot{\psi} \sin \theta \sin \phi \\ \dot{\psi} \sin \theta \cos \phi \\ \dot{\psi} \cos \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} \]
To find the motion of $\omega$, we differentiate

$$
\begin{bmatrix}
\dot{\omega}_x \\
\dot{\omega}_y \\
\dot{\omega}_z
\end{bmatrix} =
\begin{bmatrix}
\dot{\psi} \phi \sin \theta \cos \phi \\
\dot{\psi} \phi \sin \theta \sin \phi \\
0
\end{bmatrix}
$$

Now, substituting into the Euler equations yields

$$
\dot{\psi} = \frac{I_z}{(I_x - I_z) \cos \theta} \dot{\phi}
$$

There are two cases here:

- $I_x > I_z$ - **Direct** precession
  - $\dot{\psi}$ and $\dot{\phi}$ aligned.

- $I_y > I_x$ - **Retrograde** precession
  - $\dot{\psi}$ and $\dot{\phi}$ are opposite.
Motion in the Inertial Frame

Recall $\dot{\omega}_x$ and $\dot{\omega}_y$ can be expressed in terms of $\omega_x$ and $\omega_y$.
Motion in the Inertial Frame

**Figure:** Retrograde Precession \((I_z > I_x)\)

**Figure:** Direct Precession \((I_z < I_x)\)
Mathematica Demonstrations

Mathematica Precession Demonstration
Prolate and Oblate Spinning Objects

**Figure:** Prolate Object: \( I_x = I_y = 4 \) and \( I_z = 1 \)

**Figure:** Oblate Object: Vesta
Note Bene: Precession of a spacecraft is often called nutation ($\theta$ is called the nutation angle).

- By most common definitions, for torque-free motions, $N = 0$
  - Free rotation has NO nutation.
  - This is confusing
Precession

Example: Chandler Wobble

**Problem:** The earth is 42.72 km wider than it is tall. How quickly will the rotational axis of the earth precess due to this effect?

**Solution:** for an axisymmetric ellipsoid with height $a$ and width $b$, we have $I_x = I_y = \frac{1}{5}m(a^2 + b^2)$ and $I_z = \frac{2}{5}mb^2$.

Thus $b = 6378km$, $a = 6352km$ and we have $(m_e = 5.974 \cdot 10^{24}kg)$

$I_z = 9.68 \cdot 10^{37}kg\cdot m^2$, $I_x = I_y = 9.72 \cdot 10^{37}kg\cdot m^2$

If we take $\omega_z = \frac{2\pi}{T} \equiv 2\pi day^{-1}$, then we have

$\lambda = \frac{I_z - I_x}{I_x} \omega_z = .0041 day^{-1}$

That gives a period of $T = \frac{2\pi}{\lambda} = 243.5 days$. This motion of the earth is known as the **Chandler Wobble**.

**Note:** This is only the Torque-free precession.
• Actual period is 434 days
  ▶ Actual $I_x = I_y = 8.008 \cdot 10^{37} \text{kg} \cdot \text{m}^2$.
  ▶ Actual $I_z = 8.034 \cdot 10^{37} \text{kg} \cdot \text{m}^2$.
  ▶ Which would predict $T = 306$ days
The precession of the earth was first noticed by Euler, D’Alembert and Lagrange as slight variations in latitude.

Error partially due to fact Earth is not a rigid body (Chandler + Newcomb).

Magnitude of around 9m

Previous plot scale is milli-arc-seconds (mas)
Next Lecture

In the next lecture we will cover

Non-Axisymmetric rotation
- Linearized Equations of Motion
- Stability

Energy Dissipation
- The effect on stability of rotation