Lecture 6: Controllability and Observability
Controllability

First add an input \( u(t) \)

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0
\]

The solution is

\[
x(t) = \int_0^t e^{A(t-s)} Bu(s) ds
\]

Use Leibnitz rule for differentiation of integrals

\[
\dot{x}(t) = e^{A(t-t)} Bu(t) + \int_0^t A e^{A(t-s)} Bu(s) ds
\]

\[
= Bu(t) + Ax(t)
\]

Controllability asks whether we can “control” the system states through appropriate choice of \( u(t) \).

- Note that we do not care how \( u(t) \) is chosen.

We start with a weaker definition
Controllability

**Definition 1.**
For a given \((A, B)\), the **state** \(x_f\) is **Reachable** if for any fixed \(T_f\), there exists a \(u(t)\) such that

\[
x_f = \int_0^{T_f} e^{A(T_f-s)} Bu(s) ds
\]

**Definition 2.**
The **system** \((A, B)\) is **reachable** if any point \(x_f \in \mathbb{R}^n\) is reachable.

For a fixed \(t\), the set of reachable states is defined as

\[
R_t := \{ x : x = \int_0^t e^{A(t-s)} Bu(s) ds \text{ for some function } u. \}
\]
The mapping $\Gamma_t : u \mapsto x_f$ is linear. Let $u = \alpha u_1 + \beta u_2$

$$\Gamma_t u = \int_0^{T_f} e^{A(T_f - s)} B (\alpha u_1(s) + \beta u_2(s)) \, ds$$

$$= \alpha \int_0^{T_f} e^{A(T_f - s)} Bu_1(s) \, ds + \beta \int_0^{T_f} e^{A(T_f - s)} Bu_2(s) \, ds$$

$$= \alpha \Gamma_t u_1 + \beta \Gamma_t u_2$$

Thus $R_t = \text{Image}(\Gamma_t)$.

- $R_t$ is a subspace.

**Definition 3.**

For a given system $(A, B)$, the **Controllability Matrix** is

$$C(A, B) := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. 
In Williams-Lawrence, the controllability matrix is denoted $P$.

**Definition 4.**
For a given $(A, B)$, the **Controllable Subspace** is

$$C_{AB} = \text{Image} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

**Definition 5.**
The system $(A, B)$ is **controllable** if

$$C_{AB} = \text{Im} C(A, B) = \mathbb{R}^n$$

**Question:** How does $R_t$ relate to $C_{AB}$?
**Definition 6.**

The finite-time **Controllability Grammian** of pair \((A, B)\) is

\[
W_t := \int_0^t e^{As} B B^T e^{A^T s} ds
\]

\(W_t\) is a positive semidefinite matrix.

The following relates these three concepts of controllability

**Theorem 7.**

For any \(t \geq 0\),

\[
R_t = C_{AB} = \text{Image} \ (W_t)
\]

or

\[
\text{Image} \ \Gamma_t = \text{Image} \ C(A, B) = \text{Image} \ (W_t)
\]
The most important consequence is

- $R_t$ does not depend on time!

If you can get there, you can get there arbitrarily fast. This says nothing about how you get $u(t)$

- This $u(t)$ comes from the proof (and $W_t$)

We can test reachability of a point $x$ by testing

$$x \in \text{Im} \left[ B \ AB \ A^2B \ \cdots \ A^{n-1}B \right]$$

The system is controllable if $W_t > 0$. Summary

1. $R_t$ is the set of reachable points
2. $C(A, B)$ is a fixed matrix, easily computable.
3. We need to find $u(t)$
Controllability

The following is a seminal result in state-space theory.

**Theorem 8 (Cayley-Hamilton Theorem).**

If

\[
\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_0
\]

then

\[
A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots a_0I = 0
\]

**Proof Sketch.**

The same principle as deriving the solution. Denote

\[
\text{char}_A(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0 = \det(sI - A)
\]

Then if \( A = T\Lambda T^{-1} \)

\[
\text{char}_A(A) = T\text{char}_A(\Lambda)T^{-1} = T \begin{bmatrix} \text{char}_A(\lambda_1) & \cdots & \text{char}_A(\lambda_n) \end{bmatrix} T^{-1}
\]
Controllability

Sketch.

But the $\lambda_i$ are eigenvalues of $A$, so

$$\text{char}_A(\lambda) = \det(\lambda I - A) = 0$$

hence

$$\text{char}_A(A) = T \begin{bmatrix} \text{char}_A(\lambda_1) & & \\ & \ddots & \\ & & \text{char}_A(\lambda_n) \end{bmatrix} T^{-1} = T \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} T = 0$$

The same approach works for Jordan Blocks.

Cayley-Hamilton says

$$A^n = -a_{n-1}A^{n-1} + \cdots + -a_0 I$$

thus $A^n \in \text{span}(A^{n-1}, \cdots, I)$. 
Proof.

We need to show that $\text{Im}(W_t) = \text{Im}(C(A, B)) = R_t$. To do this, we will prove that

- $\text{Im}(W_t) \subseteq R_t$
- $R_t \subseteq \text{Im}(C(A, B))$
- $\text{Im}(C(A, B)) \subseteq \text{Im}(W_t)$

We begin by showing that $R_t \subseteq \text{image}C_{AB}$ for any $t \geq 0$. Expand

$$e^{At} = \left[ I + At + \cdots + \frac{A^mt^m}{m!} + \cdots \right]$$

By Cayley-Hamilton

$$A^n = -a_{n-1}A^{n-1} + \cdots + -a_0I$$

Grouping by $A^i$, we get

$$e^{At} = \left[ I\phi_0(t) + A^1\phi_1(t) + \cdots + A^{n-1}\phi_{n-1}(t) \right]$$

for some scalar functions $\phi_i(t)$. 

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Proof.

Because the $\phi_i$ are scalars,

$$\Gamma_t u = \int_0^t e^{A(t-s)} Bu(s) ds$$

$$= B \int_0^t \phi_0(t - s) u(s) ds + \cdots + A^{n-1} B \int_0^t \phi_{n-1}(t - s) u(s) ds$$

Let

$$y_i = \int_0^t \phi_i(t - s) u(s) ds,$$

then

$$\Gamma_t u = By_0 + \cdots + A^{n-1} By_{n-1}$$

$$= \begin{bmatrix} B & \cdots & A^{n-1} B \end{bmatrix} \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix} = C(A, B) \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Thus $\Gamma_t u \in \text{image } [B \cdots A^{n-1} B]$. Therefore, $R_t \subset C_{AB}$. \qed
We need to prove

- \( \text{Im}(W_t) \subseteq R_t \)
- \( R_t \subseteq \text{Im}(C(A, B)) \)
- \( \text{Im}(C(A, B)) \subseteq \text{Im}(W_t) \)

So far, we have shown that

\[ R_t \subseteq C_{AB} \]

Next, we will show that

\[ \text{Im}(W_t) \subseteq R_t \]
Controllability: $\text{Im}(W_t) \subset R_t$

Proposition 1.

$$\text{Im}(W_t) \subset R_t$$

Proof.

First, suppose that $x \in \text{Im}(W_t)$ for some $t > 0$. Then $x = W_tz$ for some $z$.

- Now let $u(s) = B^T e^{A^T(t-s)} z$. Then

  $$\Gamma_t u = \int_0^t e^{A(t-s)} Bu(s) ds$$

  $$= \int_0^t e^{A(t-s)} B B^T e^{A^T(t-s)} z ds$$

  $$= W_t z = x$$

- Thus $x \in \text{Im}(\Gamma_t) = R_t$.

We conclude that $\text{Im}(W_t) \subset R_t$
The last proof is a proof by contradiction. 

*Once you eliminate the impossible, whatever remains, no matter how improbable, must be the truth.*

There are two mutually exclusive possibilities

1. \( x \in \text{Im}(C(A, B)) \) implies \( x \in \text{Im}(W_t) \)
2. There exists some \( x \in \text{Im}(C(A, B)) \) such that \( x \not\in \text{Im}(W_t) \).

We eliminate the second possibility by showing that:

- If \( x \not\in \text{Im}(W_t) \) then \( x \not\in \text{Im}(C(A, B)) \).

In shorthand:

\[
(\neg 2 \Rightarrow \neg 1) \iff (1 \rightarrow 2)
\]

Also, recall:

**Theorem 9.**

*For any \( M \in \mathbb{R}^{n \times m} \), \( \text{Im}(M)^	op = \text{Ker}[M^T] \).*
Proof by Contradiction: \( \text{Im}(C(A, B)) \subseteq \text{Im}(W_t) \)

**Theorem 10.**

\[
\text{Im}(C(A, B)) \subseteq \text{Im}(W_t).
\]

**Proof.**

Suppose \( x \notin \text{Im}(W_t) \). Then \( x \in \text{Im}(W_t)^\perp \).

- As we have shown, this means \( x \in \ker(W_t) \), so \( W_t x = 0 \).
- Thus

\[
x^T W_t x = \int_0^t x^T e^{A(t-s)} B B^T e^{A^T(t-s)} x \, ds
\]

\[
= \int_0^t u(s)^T u(s) \, ds = 0
\]

where \( u(s) = B^T e^{A^T(t-s)} x \).

- This implies \( u(s) = B^T e^{A^T(t-s)} x = 0 \) for all \( s \in [0, t] \).
Proof by Contradiction: $\text{Im}(C(A, B)) \subset \text{Im}(W_t)$

**Proof.**

- This means that for all $s \in [0, t],$

  $$\frac{d^k}{ds^k} B^T e^{AT} s x = B^T (A^T)^k e^{AT} s x = 0$$

- At $s = 0,$ this implies $B^T (A^T)^k x = 0$ for all $k.$
- We conclude that

  $$x^T \begin{bmatrix} B & AB & \cdots & A^{n-1} B \end{bmatrix} = 0$$

- Thus $C(A, B)^T x = 0,$ so $x \in \ker C(A, B)^T.$ As before, this means $x \in \text{Im}(C(A, B))^\perp.$
- We conclude that $x \not\in \text{Im}(C(A, B)).$ This proves by contradiction that $\text{Im}(C(A, B)) \subset \text{Im}(W_t).$
Summary: $R_t = \text{Im}(W_t) = \text{Im}(C(A, B))$

We have shown that

$$R_t = \text{Im}(W_t) = \text{Im}(C(A, B))$$

Moreover, we have shown that for any $x_d \in R_{T_f}$, we can find a controller

- Choose any $z$ such that $x_d = W_{T_f} z$. ($z = W_{T_f}^{-1} x$ if $W_{T_f}$ is invertible)
- Let $u(t) = B^T e^{A^T(T_f-t)} z$.
- Then the system $\dot{x}(t) = Ax(t) + B(t)$ with $x(0) = 0$ has solution with $x(T_f) = x_d$.
- $x_d = \Gamma_{T_f} u$. 