Modern Control Systems

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Lecture 8: Eigenvalue Assignment
The problem of designing a controller

- We have touched on this problem in reachability
  \[ u(t) = B^T e^{A(T_f - t)T}^{-1} z_f \]
- This controller is open-loop

- It assumes perfect knowledge of system and state.

**Problems**

- Prone to Errors, Disturbances, Errors in the Model

**Solution**

- Use continuous measurements of state to generate control

**Static Full-State Feedback Assumes:**

- We can directly and continuously measure the state \( x(t) \)
- Controller is a static linear function of the measurement

\[ u(t) = Fx(t), \quad F \in \mathbb{R}^{m \times n} \]
**Eigenvalue Assignment**

Static Full-State Feedback

**State Equations:** $u(t) = Fx(t)$

\[
\dot{x}(t) = Ax(t) + Bu(t) = Ax(t) + BFx(t) = (A + BF)x(t)
\]

**Stabilization:** Find a matrix $F \in \mathbb{R}^{m \times n}$ such that

\[
A + BF
\]

is Hurwitz.

**Eigenvalue Assignment:** Given $\{\lambda_1, \cdots, \lambda_n\}$, find $F \in \mathbb{R}^{m \times n}$ such that

\[
\lambda_i \in \text{eig}(A + BF) \quad \text{for} \ i = 1, \cdots, n.
\]

**Note:** A solution to the eigenvalue assignment problem can also solve the stabilization problem.

**Question:** Is eigenvalue assignment actually harder?
Theorem 1.

Suppose $B \in \mathbb{R}^{n \times 1}$. Eigenvalues of $A + BF$ are freely assignable if and only if $(A, B)$ is controllable.

Proof.

1. (Controllable Canonical Form) There exists a $T$ such that

$$\hat{A} = TAT^{-1} = \begin{bmatrix} 0 & I \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \quad \hat{B} = TB = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

2. Define $\hat{F} = [\hat{f}_0 \ \cdots \ \hat{f}_{n-1}] \in \mathbb{R}^{1 \times n}$. Then

$$\hat{B}\hat{F} = \begin{bmatrix} 0 \\ \hat{f}_0 & \hat{f}_1 & \cdots & \hat{f}_{n-1} \end{bmatrix}$$
Proof.

\[ \hat{B}\hat{F} = \begin{bmatrix} 0 & \hat{f}_0 \\ \hat{f}_1 & \cdots & \hat{f}_{n-1} \end{bmatrix} \]

- Then

\[ \hat{A} + \hat{B}\hat{F} = \begin{bmatrix} O \\ -a_0 + \hat{f}_0 \\ -a_1 + \hat{f}_1 & \cdots & -a_{n-1} + \hat{f}_{n-1} \end{bmatrix} \]

- This has the characteristic equation

\[ \det \left( sI - (\hat{A} + \hat{B}\hat{F}) \right) = s^n + (a_{n-1} - \hat{f}_{n-1})s^{n-1} + \cdots + (a_0 - \hat{f}_0) \]

- Suppose we want eigenvalues \( \{\lambda_1, \cdots, \lambda_n\} \). Then define \( b_i \) as

\[ p(s) = (s - \lambda_1) \cdots (s - \lambda_n) = s^n + b_{n-1}s^{n-1} + \cdots + b_0 \]

- Choose \( \hat{f}_i = a_i - b_i \).

- Now let \( F = \hat{F}T \). Then \( A + BF = T^{-1}(\hat{A} + \hat{B}\hat{F})T \).
Proof.

• Then
\[
\det (sI - (A + BF)) = \det \left( T \left( sI - (\hat{A} + \hat{B}\hat{F}) \right) T^{-1} \right) \\
= \det \left( sI - (\hat{A} + \hat{B}\hat{F}) \right) \\
= (s - \lambda_1) \cdots (s - \lambda_n)
\]

• Hence \( A + BF \) has eigenvalues \( \{\lambda_1, \cdots, \lambda_n\} \).

Suppose we want the eigenvalues \( \{\lambda_1, \cdots, \lambda_n\} \).

1. Find the \( b_i \)
2. Choose \( \hat{f}_i = a_i - b_i \).
3. Then use \( F = [\hat{f}_0 \cdots \hat{f}_{n-1}] T \).

**Conclusion:** For Single-Input, controllability implies eigenvalue assignability.

• Requires conversion to controllable canonical form
• Matlab command `acker`
The multi-input case is harder

Lemma 2.

If \((A, B)\) is controllable, then for any \(x_0 \neq 0\), there exists a sequence \(\{u_0, u_1, \cdots, u_{n-2}\}\) such that \(\text{span}\{x_0, x_1, \cdots, x_{n-1}\} = \mathbb{R}^n\), where

\[
x_{k+1} = Ax_k + Bu_k \quad \text{for} \; k = 0, \cdots, n - 1
\]

Proof.

For \(1 \Rightarrow 2\), we again use proof by contrapositive. We show \((\neg 2 \Rightarrow \neg 1)\).

- Suppose that for any \(x_0\), and any \(\{u_0, u_1, \cdots, u_{n-2}\}\), \(\text{span}\{x_0, \cdots, x_{n-1}\} \neq \mathbb{R}^n\). Then there exists some \(y\) such that \(y^T x_k = 0\) for any \(k = 0, \cdots, n - 1\). We can solve explicitly for \(x_k\):

\[
x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} Bu_j
\]
Proof.

\[ x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} B u_j \]

- Let \( k = n - 1 \), and \( x_0 = B u_{n-1} \) for some \( u_{n-1} \). Then for any \( u \)

\[
y^T x_{n-1} = y^T \begin{bmatrix} A^{n-1} B & A^{n-2} B & \cdots & B \end{bmatrix} \begin{bmatrix} u_{n-1} \\ u_0 \\ \vdots \\ u_{n-2} \end{bmatrix} = y^T C(A, B) u = 0
\]

- Therefore, \( \text{image}(C(A, B)) \neq \mathbb{R}^n \). Hence \((A, B)\) is not controllable. This proves the lemma.
Lemma 3.

Suppose \((A, B)\) is controllable. Then for any nonzero column, \(B_1 \in \mathbb{R}^n\), of \(B\), there exists a \(F_1 \in \mathbb{R}^{m \times n}\) such that \((A + BF_1, B_1)\) is controllable.

Proof.

Suppose \((A, B)\) is controllable. Let \(x_0 = B_1\) and apply the previous Lemma to find some input \(u_0, \cdots, u_{n-2}\) such that \(\text{span}\{x_0, \cdots x_{n-1}\} = \mathbb{R}^n\) where

\[ x_{k+1} = Ax_k + Bu_k \]

Let \(T = [x_0 \ \cdots \ x_{n-1}]\). Then \(T\) is invertible. Let

\[ F_1 = [u_0 \ \cdots \ u_{n-2}] T^{-1} = UT^{-1} \]

- This implies \(F_1T = U\) and hence \(F_1x_i = u_i\) for \(i = 0, \cdots, n-1\).
- Now expand

\[ x_{k+1} = Ax_k + Bu_k = Ax_k + BF_1x_k = [A + BF_1]x_k \]
Proof.

\[ x_{k+1} = Ax_k + Bu_k = Ax_k + BF_1 x_k = [A + BF_1]x_k \]

Which means that \( x_k = [A + BF_1]^k x_0 \). However, since \( x_0 = B_1 \), we have

\[
T = \begin{bmatrix} x_0 & \cdots & x_{n-1} \\ B_1 & \cdots & (A + BF_1)^{n-1} B_1 \end{bmatrix} = C(A + BF_1, B_1)
\]

- Since \( T \) is invertible, \( C(A + BF_1, B_1) \) is full rank and hence \( (A + BF_1, B_1) \) is controllable.
Theorem 4.

The eigenvalues of $A + BF$ are freely assignable if and only if $(A, B)$ is controllable.

Proof.

The “only if” direction is clear. Suppose $(A, B)$ is controllable and we want eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$. Let $B_1$ be the first column of $B$.

- By Lemma, there exists a $F_1$ such that $(A + BF_1, B_1)$ is controllable.
- By other Lemma, since the $(A + BF_1, B_1)$ is controllable, the eigenvalues of $(A + BF_1, B_1)$ are assignable. This we can find a $F_2$ such that $A + BF_1 + B_1F_2$ has eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$.

- Choose $F = F_1 + \begin{bmatrix} F_2 \\ 0 \end{bmatrix}$. Then

\[
A + BF = A + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} F_1 + \begin{bmatrix} F_2 \\ 0 \end{bmatrix} \end{bmatrix} = A + BF_1 + B_1F_2
\]

has the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$.
Theorem 5.

The eigenvalues of $A + BF$ are freely assignable if and only if $(A, B)$ is controllable.

Note that the proof was not very constructive: Need to find $F_1$ and $F_2 \ldots 2$

Matlab Commands

- K=acker(A,B,p) for 1-D
- K=place(A,B,p) for n-D. $p$ is the vector of pole locations.

Theorem 6.

If $(A, B)$ is stabilizable, then there exists a $F$ such that $A + BF$ is Hurwitz.

Proof.

Apply the previous result to the controllability form.

Conclusion: If $(A, B)$ is stabilizable, then it can be stabilized using only static state feedback. $u(t) = Kx(t)$. 