Modern Control Systems

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Lecture 09: Observability
Observability

For Static Full-State Feedback, we assume knowledge of the **Full-State**.

- In reality, we only have measurements

\[ y_m(t) = C_m x(t) \]

- How to implement our controllers?

Consider a system with no input:

\[
\begin{align*}
\dot{x}(t) &= Ax(t), & x(0) &= x_0 \\
y(t) &= Cx(t)
\end{align*}
\]

**Definition 1.**

The pair \((A, C)\) is **Observable** on \([0, T]\) if, given \(y(t)\) for \(t \in [0, T]\), we can find \(x_0\).
Let $\mathcal{F}(\mathbb{R}^{p_1}, \mathbb{R}^{p_2})$ be the space of functions which map $f : \mathbb{R}^{p_1} \to \mathbb{R}^{p_2}$.

**Definition 2.**

Given $(C, A)$, the flow map, $\Psi_T : \mathbb{R}^p \rightarrow \mathcal{F}(\mathbb{R}, \mathbb{R}^p)$ is

$$
\Psi_T : x_0 \mapsto C e^{At} x_0 \quad t \in [0, T]
$$

So $y = \Psi_T x_0$ implies $y(t) = C e^{At} x_0$.

**Proposition 1.**

*The pair $(C, A)$ is observable if and only if $\Psi_T$ is invertible*

$$
\ker \Psi_T = 0
$$
Theorem 3.

\[ \ker \Psi_T = \ker C \cap \ker CA \cap \ker CA^2 \cap \cdots \cap \ker CA^{n-1} = \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \]

Proof.

Similar to the Controllability proof: \( R_t = \text{image } C(A, B) \)

Definition 4.

The matrix \( O(C, A) \) is called the **Observability Matrix**

\[ O(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \]
Definition 5.
The Unobservable Subspace is $N_{CA} = \ker \Psi_T = \ker O(C, A)$.

Theorem 6.
$N_{AB}$ is $A$-invariant.
Duality

The Controllability and Observability matrices are related

\[
O(C, A) = C(A^T, C^T)^T \\
C(A, B) = O(B^T, A^T)^T
\]

For this reason, the study of controllability and observability are related.

\[
\ker O(C, A) = [\text{image } C(A^T, C^T)]^\perp \\
\text{image } C(A, B) = [\ker O(B^T, A^T)]^\perp
\]

We can investigate observability of \((C, A)\) by studying controllability of \((A^T, C^T)\)

- \((C, A)\) is observable if \(\text{image } C(A^T, C^T) = \mathbb{R}^n\)
**Definition 7.**

For pair \((C, A)\), the **Observability Grammian** is defined as

\[
Y = \int_0^\infty e^{ATs} CT Ce^{As} ds
\]

The following seminal result is not surprising:

**Theorem 8.**

For a given pair \((C, A)\), the following are equivalent.

- \(\ker Y = 0\)
- \(\ker \Psi_T = 0\)
- \(\ker O(C, A) = 0\)

If the state is observable, then it is observable arbitrarily fast.
There are several other results which fall out directly.

**Theorem 9 (PBH Test).**

\((C, A)\) is observable if and only if

\[
\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n
\]

for all \(\lambda \in \mathbb{C}\).

- Again, we can consider only eigenvalues \(\lambda\).
- No equivalent to Stabilizability?
Observability Form

**Theorem 10.**

For any pair \((C, A)\), there exists an invertible \(T\) such that

\[
TAT^{-1} = \begin{bmatrix}
\tilde{A}_{11} & 0 \\
\tilde{A}_{21} & \tilde{A}_{22}
\end{bmatrix} \quad CT^{-1} = \begin{bmatrix}
\tilde{C}_1 & 0
\end{bmatrix}
\]

where the pair \((\tilde{C}_1, \tilde{A}_{11})\) is observable.

Invariant Subspace Form

- What is the invariant subspace?

Dissecting the equations (and dropping the tilde), we have

\[
\begin{align*}
\dot{x}_1(t) &= A_{11}x_1(t) \\
\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) \\
y(t) &= Cx_1(t)
\end{align*}
\]

Then we can solve for the output:

\[
y(t) = Ce^{A_{11}t}x_1(0)
\]

The initial condition \(x_2(0)\) does not affect the output in any way!

- \(x_2(0) \in \ker \Psi_T\).
- No way to back out \(x_2(0)\).
Detectability

The equivalent to stabilizability

**Definition 11.**

The pair \((C, A)\) is detectable if, when in observability form, \(\tilde{A}_{22}\) is Hurwitz.

All unstable states are observable

**Theorem 12 (PBH for detectability).**

Suppose \((C, A)\) has observability form

\[
TAT^{-1} = \begin{bmatrix}
\tilde{A}_{11} & 0 \\
\tilde{A}_{21} & \tilde{A}_{22}
\end{bmatrix} \quad CT^{-1} = \begin{bmatrix}
\tilde{C}_1 & 0
\end{bmatrix}
\]

Then \(A_{22}\) is Hurwitz if and only if

\[
\text{rank} \begin{bmatrix}
A - \lambda I \\
C
\end{bmatrix} = n
\]

for all \(\lambda \in \mathbb{C}^+\).
Observers

Suppose we have designed a controller

\[ u(t) = Fx(t) \]

but we can only measure \( y(t) = Cx(t) \)!

**Question:** How to find \( x(t) \)?

- If \((C, A)\) observable, then we can observe \( y(t) \) on \( t \in [t, t + T] \).
  - But by then its too late!
  - we need \( x(t) \) in *real time!*
Definition 13.

An Observer, is an Artificial Dynamical System whose output tracks $x(t)$.

Suppose we want to observe the following system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Lets assume the system is state-space

- What are our inputs and output?
- What is the dimension of the system?
Observers

**Inputs:** $u(t)$ and $y(t)$.

**Outputs:** Estimate of the state: $\hat{x}(t)$.

Assume the observer has the same dimension as the system

\[
\dot{z}(t) = Mz(t) + Ny(t) + Pu(t) \\
\hat{x}(t) = Qz(t) + Ry(t) + Su(t)
\]

We want $\lim_{t \to 0} e(t) = \lim_{t \to 0} x(t) - \hat{x}(t) = 0$

- for any $u$, $z(0)$, and $x(0)$.
- We would also like internal stability, etc.
Observers

System:
\[
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\]

Observer:
\[
\dot{z}(t) = Mz(t) + Ny(t) + Pu(t) \\
\hat{x}(t) = Qz(t) + Ry(t) + Su(t)
\]

What are the dynamics of \( x - \hat{x} \)?
\[
\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) \\
= Ax(t) + Bu(t) - Q\dot{z}(t) + R\dot{y}(t) + S\dot{u}(t) \\
= Ax(t) + Bu(t) - Q(Mz(t) + Ny(t) + Pu(t)) + R(C\dot{x}(t) + D\dot{u}(t)) + S\dot{u}(t) \\
= Ax(t) + Bu(t) - QMz(t) - QN(Cx(t) + Du(t)) - QPu(t) \\
\hspace{2cm} + RC(Ax(t) + Bu(t)) + (S + RD)\dot{u}(t) \\
= (A + RCA - QNC)e(t) + (AQ + RCAAQ - QNCQ - QM)z(t) + (A + RCA - QNC)Ry(t) + (B + RCB - QP - QND)u(t) + (S + RD)\dot{u}(t)
\]

Designing an observer requires that these dynamics are Hurwitz.
Luenberger Observers

Initially, we consider a special class of observers, parameterized by the matrix $L$

\[
\dot{z}(t) = (A + LC)z(t) - Ly(t) + (B + LD)u(t) \tag{1}
\]

\[
\hat{x}(t) = z(t) \tag{2}
\]

In the general formulation, this corresponds to

\[
M = A + LC; \quad N = -L; \quad P = B + LD;
\]

\[Q = I; \quad R = 0; \quad S = 0;\]

So in this case $z(t) = \hat{x}(t)$ and $(A + RCA - QNC) = QM = A + LC$.

Furthermore $(A + RCA - QNC)R = 0$ and

\[AQ + RCAQ - QNCQ - QM = 0.\]

Thus the criterion for convergence is $A + LC$ Hurwitz.

\textbf{Question} Can we choose $L$ such that $A + LC$ is Hurwitz?

Similar to choosing $A + BF$. 

If turns out that controllability and detectability are useful

**Theorem 14.**

_The eigenvalues of \( A + LC \) are freely assignable through \( L \) if and only if \((C, A)\) is observable._

If we only need \( A + LC \) Hurwitz, then the test is easier.

- We only need detectability

**Theorem 15.**

_An observer exists if and only if \((C, A)\) is detectable_

**Note:** Theorem applies to ANY observer, not just Luenberger observers.
**Theorem 16.**

An observer exists if and only if \((C, A)\) is detectable

**Proof.**

We begin with \(1) \Rightarrow 2\). We use proof by contradiction. We show \(\neg 2) \Rightarrow \neg 1)\).

- Suppose \((C, A)\) is not detectable. We will show that for some initial conditions \(x(0)\) and \(z(0)\), The observer will not converge

\[
\dot{z}(t) = Mz(t) + Ny(t) + Pu(t)
\]

\[
\hat{x}(t) = Qz(t) + Ry(t) + Su(t)
\]

- Convert the system to obervability form where \(A_{22}\) is not Hurwitz.

\[
\dot{x}_1(t) = A_{11}x_1(t)
\]

\[
\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t)
\]

\[
y(t) = Cx_1(t)
\]
Proof.

- Choose \( x_1(0) = 0 \) and \( x_2(0) \) to be an eigenvector of \( A_{22} \) with associated eigenvalue \( \lambda \) having positive real part.

- Then \( x_1(t) = e^{A_{11}t}x_1(0) = 0 \) for all \( t > 0 \).

- Then
  \[
  \dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) = A_{22}x_2(t).
  \]

  Hence \( x_2(t) = e^{A_{22}t}x_2(0) = x_2(0)e^{\lambda t} \). Thus \( \lim_{t \to \infty} x_2(t) = \infty \).

- However, \( y(t) = Cx_1(t) = 0 \) for all \( t > 0 \).

- For any observer, choose \( z(0) = 0 \) and \( u(t) = 0 \). Then
  \[
  \dot{z}(t) = Mz(t) + Ny(t) + Pu(t) = Mz(t)
  \]

  Hence \( z(t) = e^{Mt}z(0) = 0 \) for all \( t > 0 \) and \( \hat{x}(t) = 0 \) for all \( t > 0 \).

- We conclude that \( \lim_{t \to \infty} e(t) = \lim_{t \to \infty} x(t) - \hat{x}(t) = \infty \).
Theorem 17.  
An observer exists if and only if $(C, A)$ is detectable

Proof.

Next we prove that $2) \Rightarrow 1)$. We do this directly by constructing the observer.

- If $(C, A)$ is detectable, then there exists a $L$ such that $A + LC$ is Hurwitz.
- Choose the Luenberger observer
  \[ \dot{z}(t) = (A + LC)z(t) - Ly(t) + (B + LD)u(t) \]
  \[ \hat{x}(t) = z(t) \]

- Referencing previous slide, $A + RCA - QNC = QM = A + LC$ and $B + RCB - QP - QND = 0$ and $S + RD = 0$

- Then the error dynamics become
  \[ \dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = (A + LC)e(t) \]

- Which has solution \( \lim_{t \to \infty} e^{(A+LC)t}e(0) = 0 \).
- Thus the observer converges.
\[ \dot{z}(t) = (A + LC)z(t) - Ly(t) + (B + LD)u(t) \]  \hspace{1cm} (3)

\[ \hat{x}(t) = z(t) \]  \hspace{1cm} (4)

**Theorem 18.**

The eigenvalues of \( A + LC \) are freely assignable through \( L \) if and only if \( (C, A) \) is observable.

**Theorem 19.**

An observer exists if and only if \( (C, A) \) is detectable.
**Question:** How to compute $L$?

- The eigenvalues of $A + LC$ and $(A + LC)^T = A^T + C^T L^T$ are the same.
- This is the same problem as controller design!

**Answer:** Choose a vector of eigenvalues $E$.

- $L = \text{place}(A^T, C^T, E)^T$

So now we know how to design an Luenberger observer.

- Also called an estimator

The error dynamics will be dictated by the eigenvalues of $A + LC$.

- For fast convergence, chose very negative eigenvalues.
- generally a good idea for the observer to converge faster than the plant.
Observer-Based Controllers

Summary: What do we know?
• How to design a controller which uses the full state.
• How to design an observer which converges to the full state.

Question: Is the combined system stable?
• We know the error dynamics converge.
• Let's look at the coupled dynamics.

Proposition 2.

The system defined by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t) \\
u(t) &= F\hat{x}(t) \\
\dot{\hat{x}}(t) &= (A + LC + BF + LDF) \hat{x}(t) - Ly(t)
\end{align*}
\]

has eigenvalues equal to that of \( A + LC \) and \( A + BF \).

Note we have reduced the dependence on \( u(t) \).
Observer-Based Controllers

The proof is relatively easy

**Proof.**

The state dynamics are

\[ \dot{x}(t) = Ax(t) + BF\hat{x}(t) \]

Rewrite the estimation dynamics as

\[ \dot{x}(t) = (A + LC + BF + LDF') \hat{x}(t) - Ly(t) \]

\[ \quad = (A + LC') \hat{x}(t) + (B + LD) F\hat{x}(t) - LCx(t) - LDu(t) \]

\[ \quad = (A + LC') \hat{x}(t) + (B + LD) u(t) - LCx(t) - LDu(t) \]

\[ \quad = (A + LC') \hat{x}(t) + Bu(t) - LCx(t) \]

\[ \quad = (A + LC + BF) \hat{x}(t) - LCx(t) \]

In state-space form, we get

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\hat{x}}(t)
\end{bmatrix}
= \begin{bmatrix}
A & BF \\
-LC & A + LC + BF
\end{bmatrix}
\begin{bmatrix}
x(t) \\
\hat{x}(t)
\end{bmatrix}
\]
Observer-Based Controllers

Proof.

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\hat{x}}(t)
\end{bmatrix} = \begin{bmatrix} A & BF \\
-LC & A + LC + BF
\end{bmatrix} \begin{bmatrix} x(t) \\
\hat{x}(t)
\end{bmatrix}
\]

Use the similarity transform \( T = T^{-1} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \).

\[
T\bar{A}T^{-1} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A & BF \\
-LC & A + LC + BF \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = \begin{bmatrix} A + BF & -BF \\ A + BF & -(A + LC + BF) \end{bmatrix}
\]

which has eigenvalues \( A + LC \) and \( A + BF \).