Modern Control Systems

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Lecture 10: Linear Systems Theory
We will now temporarily skip realization theory in favor of linear analysis.

\[ y_e = G u_e \]

We now view systems only in terms of inputs and outputs.

- We also have control inputs and outputs

\[ \begin{bmatrix} y_e \\ y_c \end{bmatrix} = G \begin{bmatrix} u_e \\ u_c \end{bmatrix} \]

- More on this later
Normed Spaces

Recall about normed Spaces

**Definition 1.**
A **Norm** on a vector space, $V$, is a function $\|\cdot\| : V \to \mathbb{R}^+$ such that

1. $\|x\| = 0$ if and only if $x = 0$
2. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in V$ and $\alpha \in \mathbb{R}$
3. $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$

Norms only satisfy Pythagorean Theorem

**Definition 2.**
A vector space with an associated norm is called a **Normed Space**.
Recall examples of normed spaces

On $\mathbb{R}^n$:

- $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ (Taxicab norm)
- $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$ (Euclidean norm)
- $\|x\|_p = \left(\sum_{i=1}^{n} x_i^p\right)^{1/p}$
- $\|x\|_\infty = \max |x_i|$

On infinite sequences $g : \mathbb{N} \to \mathbb{R}$:

- $\|f\|_\ell_1 = \sum_{i=1}^{\infty} |g_i|$
- $\|f\|_\ell_2 = \sqrt{\sum_{i=1}^{\infty} g_i^2}$
- $\|f\|_\ell_p = \left(\sum_{i=1}^{\infty} g_i^p\right)^{1/p}$
- $\|f\|_\ell_\infty = \max_{i=1,\ldots,\infty} |g_i|$

On functions $f : [0, 1] \to \mathbb{R}$:

- $\|f\|_{L_1} = \int_0^1 |f(s)| \, ds$
- $\|f\|_{L_2} = \sqrt{\int_0^1 f(s)^2 \, ds}$
- $\|f\|_{L_p} = \left(\int_0^1 f(s)^p \, ds\right)^{1/p}$
- $\|f\|_{L_\infty} = \sup_{s \in [0,1]} |f(s)|
Normed Spaces

Convergence of a Sequences

Norms define what is meant by convergence of a sequence.

**Definition 3.**

We say that

\[ \lim_{i \to \infty} x_i = y \]

if for every \( \epsilon > 0 \), there exists a \( N \) such that

\[ \| y - x_i \| \leq \epsilon \quad \text{for all } i > N. \]

Or the limit of a function \( f : X \to V \).

**Definition 4.**

For normed spaces \( X \) and \( Y \), we say that

\[ \lim_{x \to y} f(x) = z \]

if for every \( \epsilon > 0 \), there exists a \( \beta \) such that

\[ \| x - y \|_X \leq \beta \]

implies

\[ \| f(x) - z \|_Y \leq \epsilon. \]
Complete Spaces

Cauchy Sequences

For function $f : X \to V$, suppose that

$$\lim_{x \to y} f(x) = z$$

**Question:** does this imply that $z \in V$?

**Question:** Does every function have a limit?

**Answer:** It depends on the norm of $V$

**Definition 5.**

A sequence $x_i$ is a **Cauchy Sequence** if for any $\epsilon > 0$, there exists an $N$ such that

$$\|x_i - x_j\| \leq \epsilon$$

for all $i, j > N$.

This is a definition of a convergent sequence without the inconvenience of requiring the existence of a limit

- Otherwise, we need to find the limit to prove convergence.
- Now we just show the elements get closer together.
Question: Are all convergent sequences Cauchy?

Lemma 6.
Yes! Any convergent sequence is Cauchy.

Question: Are all Cauchy sequences convergent?
Whether all Cauchy sequences converge depends on the norm.

Definition 7.
A normed space, $V$, is Complete if every Cauchy sequence converges to a point in $V$.
- A complete normed space is called a Banach Space

In a Banach Space, if a sequence converges, it converges to a point in the space
Banach Space

Example

For any $p$, the space of functions $L_p(-\infty, \infty)$ is a Banach Space.

On infinite sequences $g : \mathbb{N} \to \mathbb{R}$

- $\|f\|_{\ell_1} = \sum_{i=1}^{\infty} |g_i|
- \|f\|_{\ell_2} = \sqrt{\sum_{i=1}^{\infty} g_i^2}
- \|f\|_{\ell_p} = \sqrt[p]{\sum_{i=1}^{\infty} g_i^p}
- \|f\|_{\ell_\infty} = \max_{i=1,\ldots,\infty} |g_i|

Also the $L_p[0, 1]$ spaces are complete

Lemma 8.

A subspace of a Banach Space is complete if and only if it is closed.

Example: The subspace

$$L_p[0, \infty) := \{ f \in L_p(-\infty, \infty) : f(t) = 0 \text{ for } t < 0 \}$$

Question: is $L_p(0, \infty)$ closed?
Banach Space
Example

Let $C[0,1]$ be the set of **continuous** functions with norm

$$\|f\| = \|f\|_{L_1} = \int_0^1 |f(s)| ds$$

To show that this is **NOT** a Banach space, define the sequence of functions $x_i \in C[0,1]$

$$x_i(t) = \begin{cases} 
0 & t \leq \frac{1}{2} - \frac{1}{n} \\
1 - \frac{n}{2} + nt & t \in \left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}\right] \\
1 & t \geq \frac{1}{2}
\end{cases}$$

The sequence is Cauchy since

$$\|x_i - x_j\| = \frac{1}{2} |1/i - 1/j| \to 0$$

However, there is obviously no **continuous** limit.
Now we get to a really important concept

**Definition 9.**

An **Inner Product** on a vector space $V$ is a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$, such that

1. $\langle x, x \rangle \geq 0$ for all $x \in V$.
2. $\langle x, x \rangle = 0$ if and only if $x = 0$.
3. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$. [Linearity]
4. $\langle x, y \rangle = \langle y, x \rangle$.

**Definition 10.**

A vector space with an inner product is called a **Inner Product Space**

- Any inner product space is a normed space using

$$\|x\|_V^2 = \langle x, x \rangle_V$$
An inner product space has the concept of an angle between vectors.

**Theorem 11 (Cauchy Schwartz).**

If $\|x\|^2 = \langle x, x \rangle$, then

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

**IMPORTANT:**

- Only norms derived from inner products satisfy Cauchy-Schwartz.
Inner Product Spaces allow for “right angles”.

**Definition 12.**

$x$ and $y$ are orthogonal in inner product space $V$, denoted $x \perp y$, if

$$\langle x, y \rangle_V = 0$$

**Pythagorean Theorem**

**Theorem 13.**

For $x$ and $y$ in inner product space $V$,

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

if and only if $x \perp y$. 

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Lecture 10:
Some inner product spaces:

**Euclidean Space** on $\mathbb{R}^n$

$$\langle x, y \rangle_2 = x^T y = \sum_{i=1}^{n} x_i y_i$$

**The Frobenius Norm on Matrices** $\mathbb{R}^{n \times m}$

$$\langle A, B \rangle = \operatorname{trace}(A^T B) = \sum_{i=1}^{n} \sum_{i=1}^{m} A_{ij} B_{ij}$$

which induces the Frobenius norm

$$\|X\|^2 = \langle X, X \rangle = \sum_{i=1}^{n} \sum_{i=1}^{m} X_{ij}^2$$
Hilbert Spaces

Definition 14.
An inner product space which is complete in the norm \( \|x\|^2 = \langle x, x \rangle \) is called a **Hilbert Space**.

Hilbert spaces are actually quite unusual.

**Example:** Define the following inner product on \( L_2[0, \infty) \):

\[
\langle x, y \rangle_{L_2} := \int_0^\infty x^T(s)y(s)ds
\]

Then

\[
\|x\|_{L_2}^2 = \int_0^\infty \|x(s)\|^2 ds
\]

And since \( L_2 \) is complete in this norm, \( L_2[0, \infty) \) is a Hilbert Space.
Hilbert Spaces

Example

\( \ell_p \)-Spaces

- \( \| f \|_{\ell_p} = \sqrt[p]{\sum_{i=1}^{\infty} g_i^p} \)
- \( \| f \|_{\ell_\infty} = \max_{i=1,\ldots,\infty} |g_i| \)

\( L_p \)-Spaces

- \( \| f \|_{L_p} = \sqrt[p]{\int_{-\infty}^{\infty} f(s)^p \, ds} \)
- \( \| f \|_{L_\infty} = \sup_{s \in [-\infty, \infty]} |f(s)| \)

Neither \( \ell_p \) nor \( L_p \) are Hilbert spaces for \( p \neq 2 \).
Definition 15.

$C[0, \infty)$ is the space of continuous functions with norm

$$\|f\|_\infty = \sup_t \|f(t)\|$$

- $C[0, \infty)$ is a Banach Space.
- $C[0, \infty)$ is not a Hilbert space.