Lecture 16: $H_\infty$ and Summary of Linear Analysis
So far we know:

- The Fourier Transform, $\phi$ maps $L_2(-\infty, \infty)$ to $\hat{L}_2$.
- The Laplace Transform, $\Lambda$ maps $L_2[0, \infty)$ to $H_2$.
- A Transfer Function is any element $\hat{G} \in \hat{L}_\infty$.
- A Transfer function defines a multiplication operator $M_{\hat{G}}$ which maps $\hat{L}_2$ to $\hat{L}_2$.
- Any Linear, Time-Invariant System $G : L_2 \rightarrow L_2$ can be represented by a transfer function as $\phi^{-1}M_{\hat{G}}\phi$ for some $\hat{G} \in \hat{L}_\infty$.

**Question:** How do we represent Causal Systems, which map $H_2 \rightarrow H_2$?
The Space $H_\infty$

**Definition 1.**

A function $\hat{G} : \mathbb{C}^+ \rightarrow \mathbb{C}^{n \times m}$ is in $H_\infty$ if

1. $\hat{G}(s)$ is analytic on the CRHP, $\mathbb{C}^+$.
2. \( \lim_{\sigma \to 0^+} \hat{G}(\sigma + \omega) = \hat{G}(\omega) \)
3. \( \sup_{s \in \mathbb{C}^+} \bar{\sigma}(\hat{G}(s)) < \infty \)

- Similar to $\hat{L}_\infty$, but analytic.
- Elements of $\hat{L}_\infty$ with an analytic continuation to the right half-plane.
- A Banach Space with norm

\[
\|\hat{G}\|_{H_\infty} = \text{ess sup}_{\omega \in \mathbb{R}} \bar{\sigma}(\hat{G}(\omega))
\]
The Space $H_{\infty}$

For any analytic functions, $\hat{u}$ and $\hat{G}$, the function

$$\hat{y}(s) = \hat{G}(s)\hat{u}(s)$$

is analytic. Thus if $\hat{G} \in H_{\infty}$,

- $\hat{G}$ is analytic on CRHP
- $M_{\hat{G}} : H_2 \rightarrow H_2$.
- $G = \Lambda^{-1} M_{\hat{G}} \Lambda$ maps $L_2[0, \infty) \rightarrow L_2[0, \infty)$.
- $G = \Lambda^{-1} M_{\hat{G}} \Lambda$ is causal, LTI.
Indeed, this is necessary and sufficient.

**Theorem 2.**

*G is a Causal, Linear, Time-Invariant Operator on \( L_2 \) if and only if there exists some \( \hat{G} \in H_\infty \) such that \( G = \Lambda^{-1} M \hat{G} \Lambda \).

\[
(\Lambda G u)(\omega) = \hat{G}(\omega) \hat{u}(\omega)
\]

**Conclusion:** \( H_\infty \) provides a complete parameterization of the Banach space of causal bounded linear time-invariant operators with

\[
\|G\|_{\mathcal{L}(L_2[0,\infty))} = \|\Lambda^{-1} M \hat{G} \Lambda\|_{\mathcal{L}(L_2[0,\infty))} = \|\hat{G}\|_{H_\infty}
\]

Optimal Control is an attempt to minimize the \( H_\infty \) norm of the closed-loop transfer function.
Example of $H_\infty$

Example:

\[
\hat{G}(i\omega) = \frac{e^{-i\omega\tau} - 1}{i\omega}
\]

which has

\[
\|\hat{G}\|_{H_\infty} = \tau
\]

which defines the system

\[
y(t) = \int_0^t (u(s - \tau) - u(s)) \, ds
\]

Question: How to parameterize $H_\infty$?
Rational Transfer Functions

The space of bounded analytic functions, $H_\infty$, is infinite-dimensional.

• this makes it hard to design optimal controllers.

We often restrict ourselves to state-space systems and state-space controllers.

**Definition 3.**

The space of rational functions is defined as

\[ R := \left\{ \frac{p(s)}{q(s)} : p, q \text{ are polynomials} \right\} \]

We define the following rational subspaces.

\[ RH_2 = R \cap H_2 \]
\[ R\hat{L}_2 = R \cap \hat{L}_2 \]
\[ RH_\infty = R \cap H_\infty \]

Note that $RH_2$, $R\hat{L}_2$ and $RH_\infty$ are not complete spaces.
Rational Transfer Functions

For rational transfer functions, the set of bounded LTI systems are precisely those with no unstable poles.

**Definition 4.**

- A rational function \( r(s) = \frac{p(s)}{q(s)} \) is **Proper** if the degree of \( p \) is less than or equal to the degree of \( q \).
- A rational function \( r(s) = \frac{p(s)}{q(s)} \) is **Strictly Proper** if the degree of \( p \) is less than the degree of \( q \).

**Proposition 1.**

1. \( \hat{G} \in R\hat{L}_\infty \) if and only if \( \hat{G} \) is proper with no poles (roots of \( q(s) \)) on the imaginary axis.
2. \( \hat{G} \in RH_\infty \) if and only if \( \hat{G} \) is proper with no poles on the closed right half-plane.
State-Space Systems

Recall a State-space

\[
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\]

**Theorem 5.**

- For any stable state-space system, \( G \), there exists some \( \hat{G} \in RH_\infty \) such that
  \[
  G = \Lambda^{-1} M \hat{G} \Lambda
  \]

- For any \( \hat{G} \in RH_\infty \), the operator \( G = \Lambda^{-1} M \hat{G} \Lambda \) can be represented in state-space for some \( A, B, C \) and \( D \) where \( A \) is Hurwitz.

For state-space system, \( (A, B, C, D) \),

\[
\hat{G}(s) = C(sI - A)^{-1}B + D
\]

**State-Space is NOT Unique**

- For a given Causal LTI system \( G \) with transfer function, \( \hat{G} \in RH_\infty \), there may be many state-space representations.
Definition 6.

Two state-space representations, \((A, B, C, D)\) and \((\hat{A}, \hat{B}, \hat{C}, \hat{D})\) are Equivalent if
\[
C(sI - A)^{-1} B + D = \hat{C}(sI - \hat{A})^{-1} \hat{B} + \hat{D}
\]

Definition 7.

A representation, \((A, B, C, D)\) is Minimal if it is controllable and observable.

Lemma 8.

Any transfer function \(\hat{G} \in RH_\infty\) has a minimal state-space representation.

We are skipping the section on minimality.
  - We will, however, return to the question of Grammians.