Grammians

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Lecture 17: Grammians
Proposition 1.

Suppose $A$ is Hurwitz and $Q$ is a square matrix. Then

$$X = \int_0^\infty e^{AT}sQe^{As} \, ds$$

is the unique solution to the Lyapunov Equation

$$ATX + XA + Q = 0$$

Proposition 2.

Suppose $Q > 0$. Then $A$ is Hurwitz if and only if there exists a solution $X > 0$ to the Lyapunov equation

$$ATX + XA + Q = 0$$
Proposition 3.

Suppose $A$ is Hurwitz and $X_1 \geq 0$ satisfies

\[ A^T X_1 + X_1 A = -Q \]

Suppose $X_2$ satisfies

\[ A^T X_2 + X_2 A < -Q. \]

Then $X_2 > X_1$.

Proof.

\[
A^T (X_2 - X_1) + (X_2 - X_1) A = (A^T X_2 + X_2 A) - (A^T X_1 + X_1 A) \\
= A^T X_2 + X_2 A + Q < 0
\]

Since $A$ is Hurwitz and $Q > 0$, by the previous Proposition $X_2 - X_1 > 0$
Recall From State-Space Systems:

• Controllable means we can do eigenvalue assignment.
• Observable means we can design an observer.
• Controllable and Observable means we can design an observer-based controller.

Questions:

• How difficult is the control problem?
• What is the effect of an input on an output?
To give quantitative answers to these questions, we use Grammians.

**Definition 1.**

For pair \((C, A)\), the **Observability Grammian** is defined as

\[
Y = \int_0^\infty e^{A^T s} C^T C e^{As} \, ds
\]

**Definition 2.**

The **Controllability Grammian** of pair \((A, B)\) is

\[
W := \int_0^\infty e^{As} B B^T e^{A^T s} \, ds
\]
Grammians

Grammians are linked to Observability and Controllability

**Theorem 3.**

*For a given pair \((C, A)\), the following are equivalent.*

- \( \ker Y = 0 \)
- \( \ker \Psi_o = 0 \)
- \( \ker O(C, A) = 0 \)

**Theorem 4.**

*For any \( t \geq 0 \),*

\[
R_t = C_{AB} = \text{Image} \ (W_t)
\]
Recall the state-space system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) =Cx(t) \]

Assume that \( A \) is Hurwitz.

Recall the Observability Operator \( \Psi_o : \mathbb{R}^n \rightarrow L_2[0, \infty) \).

\[ (\Psi_o x_0)(t) = \begin{cases} 
    Ce^{At}x_0 & t \geq 0 \\
    0 & t \leq 0 
\end{cases} \]

• \( \Psi_o x_0 \in L_2 \) because \( A \) is Hurwitz.
• When \( u = 0 \), this is also the solution.
• We would like to look at the “size” of the output produced by an initial condition.
  ▶ Now we know how to measure the “size” of the output signal.

\[ \|y\|_{L_2}^2 = \langle \Psi_o x_0, \Psi_o x_0 \rangle_{L_2} = \langle x_0, \Psi_o^* \Psi_o x_0 \rangle_{\mathbb{R}^n} \]

• How to calculate the adjoint \( \Psi_o^* : L_2 \rightarrow \mathbb{R}^n \)?
It can be easily confirmed that the adjoint of the observability operator is

$$\Psi^*_o z = \int_0^\infty e^{A^T_s} C^T z(s) ds$$

Then

$$\Psi^*_o \Psi_o x_0 = \left[ \int_0^\infty e^{A^T_s} C^T C e^{A_s} ds \right] x_0$$

Which is simply the observability grammian

$$Y_o = \Psi^*_o \Psi_o = \int_0^\infty e^{A^T_s} C^T C e^{A_s} ds$$

Recall from the HW: $Y_o$ is the solution to

$$A^* Y_o + Y_o A + C^T C = 0$$

and $Y_o > 0$ if and only if $(C, A)$ is observable.
Observability Grammian

Proposition 4.

Then \((C, A)\) is observable if only if there exists a solution \(X > 0\) to the Lyapunov equation

\[
A^T X +XA + C^T C = 0
\]
The physical interpretation is clear: how much does an initial condition affect the output in the $L_2$-norm

$$\|y\|_{L_2} = x_0^T Y_o x_0$$

Since this is just a matrix, we can take this further by looking at which directions are most observable.

- Will correspond to $\bar{\sigma}(Y_o)$.

**Definition 5.**

The **Observability Ellipse** is

$$E_o := \left\{ x : x = Y_o^{1/2} x_0, \|x_0\| = 1 \right\}$$
Definition 6.

The **Observability Ellipse** is

\[ E_o := \{ x : x = Y_o^{1/2}x_0, \|x_0\| = 1 \} \]

**Notes:**

1. \( E_o \) is an ellipse.

   \[ E_o = \{ x : x^T Y_o^{-1} x = 1 \} \]

   For a proof,
   - let \( x \in E_o \). Then there exists some \( |x_0| = 1 \) such that \( x_0 = Y_o^{-1/2}x \).
   - Then \( x^T Y_o^{-1} x = x^T Y_o^{-1/2} Y_o^{-1/2} x = |x_0|^2 = 1 \).
   - Thus \( E_o \subset \{ x : x^T Y_o^{-1} x = 1 \} \). The other direction is similar

2. The Principal Axes of \( E_o \) are the eigenvectors of \( Y_o^{1/2} \), \( u_i \).

3. The lengths of the Principal Axes of \( E_o \) are \( \sigma_i(Y_o) \).

4. If \( \sigma_i(Y_o) = 0 \), the \( u_i \) is in the unobservable subspace.
Recall the Controllability Operator $\Psi_c : L_2(-\infty, 0] \rightarrow \mathbb{C}^n$

$$\Psi_c u = \int_{-\infty}^{0} e^{-As} Bu(s) ds$$

Which maps an input to a final state $x(0)$.

- Adjoint $\Psi_c^* : \mathbb{R}^n \rightarrow L_2(-\infty, 0]$

  $$(\Psi_c^* x)(t) = B^* e^{-A^* t} x$$

Recall: The system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is controllable if for any $x(0) \in \mathbb{R}^n$, there exists some $u \in L_2(-\infty, 0]$ such that

$$x(0) = \Psi_c u$$
Definition 7.

The Controllability Grammian is

\[ X_c := \Psi_c \Psi_c^* = \int_{-\infty}^{0} e^{-As} BB^T e^{-A^Ts} ds \]

\[ = \int_{0}^{\infty} e^{As} BB^T e^{A^Ts} ds \]

Recall

- \( X_c \) is the solution to

\[ AX_c + X_c A^T + BB^T = 0 \]

- \( X_c > 0 \) if and only if \((A, B)\) is controllable.
Proposition 5.

Then $(A, B)$ is controllable if only if there exists a solution $X > 0$ to the Lyapunov equation

$$AX + XA^T + BB^T = 0$$
Proposition 6.

Suppose \((A, B)\) is controllable. Then

1. \(X_c\) is invertible
2. Given \(x_0\), the solution to

\[
\min_{u \in L_2(-\infty, 0]} \|u\|_{L_2}:
\]

\[
x_0 = \Psi_c u
\]

is given by

\[
u_{opt} = \Psi^*_c X_c^{-1} x_0
\]
Proof.
The first is clear from $X_c > 0$. For the second part, we first show that $u_{opt}$ is feasible. We then show that it is optimal.

- For feasibility, we note that

$$
\Psi_c u_{opt} = \Psi_c \Psi_c^* X_c^{-1} x_0 \\
= X_c X_c^{-1} x_0 \\
= x_0
$$

which implies feasibility
Proof.

Now that we know that $u_{opt}$ is feasible, we show that for any other $\bar{u}$, if $\bar{u}$ is feasible, then $\|\bar{u}\|_{L^2} \geq \|u_{opt}\|_{L^2}$.

- Define $P := \Psi_c^* X_c^{-1} \Psi_c$.

\[
P^2 = \Psi_c^* X_c^{-1} \Psi_c \Psi_c^* X_c^{-1} \Psi_c = \Psi_c^* X_c^{-1} \Psi_c = P
\]

- Furthermore $P^* = P$.
- Thus $P$ is a projection operator, which means

\[
\langle Pu, (I - P)u \rangle = 0
\]

- Thus for any $\bar{u}$

\[
\|\bar{u}\|^2 = \|P\bar{u} + (I - P)\bar{u}\|^2 = \|P\bar{u}\|^2 + \|(I - P)\bar{u}\|^2.
\]
Proof.

- If $\bar{u}$ is feasible, then

\[
\| P \bar{u} \|^2 = \| \Psi_c^* X_c^{-1} \Psi_c \bar{u} \|
\]
\[
= \| \Psi_c^* X_c^{-1} x_0 \| \quad \text{since } \bar{u} \text{ is feasible}
\]
\[
= \| u_{opt} \|^2
\]

- We conclude that

\[
\| \bar{u} \|^2 = \| u_{opt} \|^2 + \| (I - P) \bar{u} \|^2 \geq \| u_{opt} \|^2
\]

- Thus $u_{opt}$ is optimal

This shows that $u_{opt}$ is the minimum-energy input to achieve the final-state $x_0$. 

**Drawbacks:**

- Don’t have infinite time.
- Open-loop
Controllability Grammian

Physical Interpretation

The controllability Grammian tells us the minimum amount of energy required to reach a state.

\[ \| u_{opt} \|_{L_2}^2 = x_0^T X_c^{-1} x_0 \]

**Definition 8.**

The **Controllability Ellipse** is the set of states which are reachable with 1 unit of energy.

\[ \{ \Psi_c u : \| u \|_{L_2} \leq 1 \} \]

**Proposition 7.**

The following are equivalent

1. \( \{ \Psi_c u : \| u \|_{L_2} \leq 1 \} \)
2. \( \left\{ X_c^{1/2} x : \| x \| \leq 1 \right\} \)
3. \( \{ x : x^T X_c^{-1} x \leq 1 \} \)
Because we don’t always have infinite time:

- What is the optimal way to get to $x$ in time $T$

**Finite-Time Controllability Operator:** $\Psi_T : L_2[0, T] \to \mathbb{R}^n$.

$$
\Psi_T u := \int_0^T e^{A(T-s)} Bu(s) ds
$$

**Finite-Time Controllability Grammian**

$$
X_T := \Psi_T \Psi_T^* = \int_0^T e^{As} BB^T e^{A^Ts} ds
$$

**Note:** $X_T \geq X_s$ for $t \geq s$. 
Controllability Grammian

Finite-Time Grammian

Cannot be found by solving the Lyapunov equation. Must be found by numerical integration of the matrix-differential equation:

\[ \dot{X}_T(t) = AX_T(t) + X_T(t)A^T + BB^T \]

from \( t = 0 \) to \( t = T \) with \( X_T(0) = 0 \).

- \( X_c \) is the steady-state solution.
Proposition 8.

Suppose \((A, B)\) is controllable. Then

1. \(X_T\) is invertible
2. The solution to

\[
\min_{u \in L_2[0, T]} \|u\|_{L_2} : x_f = \Psi_T u
\]

is given by

\[
u_{opt} = \Psi^*_T X^{-1}_T x_f
\]
Finite-Time Grammian

Example

Consider the Spring-mass system \( (k_i = m_i = 1, b_i = 0.8) \)

\[
\dot{x}(t) = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-2 & 1 & 0 & -1.6 & .8 & 0 \\
1 & -2 & 1 & .8 & -1.6 & .8 \\
0 & 1 & -1 & 0 & .9 & -0.8
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
0
\end{bmatrix} u(t)
\]

with desired final state

\[
x_f = [1 \ 2 \ 3 \ 0 \ 0 \ 0]^T
\]
Finite-Time Grammian

Example

\[ \dot{x}_f = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \end{bmatrix}^T \]