Lecture 12: Interplanetary Mission Planning
Introduction

In this Lecture, you will learn:

Sphere of Influence
  • Definition

Escape and Re-insertion
  • The light and dark of the Oberth Effect

Patched Conics
  • Heliocentric Hohmann

Planetary Flyby
  • The Gravity Assist
Consider a Simple Earth-Moon Trajectory.

1. Launch
2. Establish Parking Orbit
3. Escape Trajectory
4. Arrive at Destination
5. Circularize or Depart Destination

The big difference is that now there are 3 bodies.

- We only know how to solve the 2-body problem.
- Solving the 3-body problem is beyond us.
Patched Conics

For interplanetary travel, the problem is even more complicated.

Consider the Figure

- The motion is elliptic about the sun.
- The motion is affected by the planets
  - Interference only occurs in the green bands.
  - Motion about planets is hyperbolic.

The solution is to break the mission into segments.

- During each segment we use *two-body motion*.
- The third body is a *disturbance*. 
**Question:** Who is in charge??

- The Sphere of Influence of A stops when A is no longer the dominant force.
- What do we mean by dominant?

**Wrong Definition:**
The Sphere of Influence of A is the region A exerts the largest gravitational force.

This would imply the moon is not in earth’s Sphere of Influence!!!
**Sun Perspective:** Let's group the forces as central and disturbing. Consider motion of a spacecraft relative to the sun:

\[
\ddot{\vec{r}}_{sv} + \frac{G(m_s + m_v)}{\|\vec{r}_{sv}\|^3} \vec{r}_{sv} = -Gm_p \left[ \frac{\vec{r}_{pv}}{\|\vec{r}_{pv}\|^3} + \frac{\vec{r}_{sp}}{\|\vec{r}_{sp}\|^3} \right]
\]

where \( p \) denotes planet, \( v \) denotes vehicles and \( s \) denotes sun.

**The Central Force is**

\[
\vec{F}_{central,s} = \frac{G(m_s + m_v)}{\|\vec{r}_{sv}\|^3} \vec{r}_{sv}
\]

**The Disturbing Force is**

\[
\vec{F}_{dist,s} = -Gm_p \left[ \frac{\vec{r}_{pv}}{\|\vec{r}_{pv}\|^3} + \frac{\vec{r}_{sp}}{\|\vec{r}_{sp}\|^3} \right]
\]
Planet Perspective: The motion of the spacecraft relative to the planet is

\[
\ddot{r}_{pv} + \frac{G(m_p + m_v)}{\|r_{pv}\|^3} r_{pv} = -Gm_s \left[ \frac{r_{sv}}{\|r_{sv}\|^3} + \frac{r_{sp}}{\|r_{sp}\|^3} \right]
\]

The Central Force for the planet is

\[
\vec{F}_{central,p} = \frac{G(m_p + m_v)}{\|r_{pv}\|^3} \vec{r}_{pv}
\]

The Disturbing Force for the planet is

\[
\vec{F}_{dist,p} = -Gm_s \left[ \frac{r_{sv}}{\|r_{sv}\|^3} + \frac{r_{sp}}{\|r_{sp}\|^3} \right]
\]
**Definition 1.**

An object is in the **Sphere of Influence** (SOI) of body 1 if

\[
\frac{\| \vec{F}_{\text{dist},1} \|}{\| \vec{F}_{\text{central},1} \|} < \frac{\| \vec{F}_{\text{dist},2} \|}{\| \vec{F}_{\text{central},2} \|}
\]

for any other body 2.

That is, the ratio of disturbing force to central force determines which planet is in control.

For planets, an approximation for determining the SOI of a planet of mass $m_p$ at distance $d_p$ from the sun is

\[
R_{SOI} \approx \left( \frac{m_p}{m_s} \right)^{2/5} d_p
\]
### Sphere of Influence Radii

<table>
<thead>
<tr>
<th>Celestial Body</th>
<th>Equatorial Radius ($km$)</th>
<th>SOI Radius ($km$)</th>
<th>SOI Radius (body radii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>2487</td>
<td>$1.13 \times 10^5$</td>
<td>45</td>
</tr>
<tr>
<td>Venus</td>
<td>6187</td>
<td>$6.17 \times 10^5$</td>
<td>100</td>
</tr>
<tr>
<td>Earth</td>
<td>6378</td>
<td>$9.24 \times 10^5$</td>
<td>145</td>
</tr>
<tr>
<td>Mars</td>
<td>3380</td>
<td>$5.74 \times 10^5$</td>
<td>170</td>
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<tr>
<td>Jupiter</td>
<td>71370</td>
<td>$4.83 \times 10^7$</td>
<td>677</td>
</tr>
<tr>
<td>Neptune</td>
<td>22320</td>
<td>$8.67 \times 10^7$</td>
<td>3886</td>
</tr>
<tr>
<td>Moon</td>
<td>1738</td>
<td>$6.61 \times 10^4$</td>
<td>38</td>
</tr>
</tbody>
</table>
Example: Lunar Lander

**Problem:** Suppose we want to plan a lunar-lander mission. Determine the spheres of influence to consider for a patched-conic approach.

- The moon orbits at a distance of 384,000km.
- The Sphere of influence of the earth is of radius 924,000km.
- The sphere of influence of the moon is of radius 66,100km.

**Solution:** The spacecraft will transition to the lunar sphere at distance

\[ r = 384,000 - 66,100 = 317,900 km \]

Thus we will need a plane change. A reasonable mission design is

1. Depart earth on a Hohmann transfer to radius 317,900 km.
2. Perform inclination change near apogee.
3. Enter sphere of influence of the moon.
4. Establish parking orbit.
Example: Lunar Lander

Additionally, a **Plane Change** is needed.

- Note that the lunar orbit is inclined at about 5.8° to the ecliptic plane.
- The inclination of the lunar orbit is almost fixed with respect to the ecliptic.
- Not fixed but not the equatorial plane.
- Inclination to equator varies by $21.3° \pm 5.8°$ every 18 years.
5 Stage Lunar Intercept Mission

First Stage Lunar Tug Assist
Interplanetary Mission Planning

Every mission is different.
- It is impossible to cover every scenario
Instead, Let’s go through an example.
- Can serve as a template.

**Problem:** Design an Earth-Venus rendez-vous. Final Venus orbit should be posigrade of altitude 500km.

**Solution:** We begin in an initial parking orbit.
- Orbital plane aligned with ecliptic plane  
  - $i \approx 23^\circ$
- Circular orbit.
  - Radius $r \approx 6578\, km$
Interplanetary Hohmann Transfer

Design a Hohmann transfer from Earth to Venus.

Naturally, the perigee and apogee velocities of the transfer ellipse are

\[
v_p = \sqrt{2\mu_{\text{sun}} \frac{r_e}{r_v(r_e + r_v)}}
\]

\[
v_a = \sqrt{2\mu_{\text{sun}} \frac{r_v}{r_e(r_e + r_v)}}
\]

Note that because Venus is an inner planet, apogee velocity occurs at Earth

The Hohmann transfer is defined using the Sphere of Influence of the Sun

- Velocities are in the Heliocentric Frame!
Interplanetary Hohmann Transfer

We can use the Hohmann transfer because the voyage will take place exclusively in the sun’s frame of reference.

- The earth orbits at radius $1au = 1.5 \cdot 10^8 km = 23,518ER$.
- The SOI of the earth is only $145ER$, or $.5\%$. 
Interplanetary Hohmann Transfer

Injection ($v_a$)

**Problem:** How to achieve the initial $v_a$?

The initial velocities $v_a$ and $v_p$ are in the *Heliocentric* frame.

- To achieve $v_a$ requires an initial $\Delta v$
- Initial $\Delta v$ will be in the *Geocentric* frame.
  - Preferably in low orbit (*Oblenrth Effect*)

In the Geocentric Frame, we require

$$v_\infty + v_e = v_a$$

$v_e$ is velocity of the earth in heliocentric frame. Thus the expression for $v_\infty$ is

$$v_\infty = \sqrt{-\frac{\mu}{E}} = \sqrt{v_f^2 - \frac{2\mu}{r_{park}}}$$

where $v_f$ is the speed at injection and $r_{park}$ is the parking radius.
Interplanetary Hohmann Transfer Injection

\[ \mu_{\text{sun}} = 1.327 \cdot 10^{11}, a_{\text{earth}} = 1.49 \cdot 10^8 \]

It is now easy to compute

\[ v_\infty = v_p - v_e = 27.34 - 29.84 = -2.48 \text{ km/s} \]

We can now solve for \( v_f \).

\[ v_f = \sqrt{(v_p - v_e)^2 + \frac{2\mu}{r_{park}}} \]

To calculate the initial \( \Delta v \), use \( v_i = \sqrt{\mu/r_{park}} \) for velocity of the parking orbit.

\[ \Delta v_1 = v_f - v_i = 11.28 \text{ km/s} - 7.78 \text{ km/s} = 3.5 \text{ km/s} \]
Arrival at Venus

\[ R_v = 6187\text{km}, \quad \mu_v = 324859, \quad a_{\text{Venus}} = 1.08 \cdot 10^8 \]

Our incoming velocity in the Venus-frame is

\[ v_{\infty,v} = v_p - v_v = 37.81\text{km/s} - 35.09\text{km/s} = 2.71\text{km/s} \]

Because the velocity is positive, we will enter from the back door.
Arrival at Venus

For orbital insertion, we want to perform a retrograde burn at periapse.

- We need our periapse to be \( r_{\text{des}} = 6687 \text{km} \).
- The \( a \) of the injected orbit is
  \[
  -\frac{\mu v}{2a} = E = \frac{1}{2}v_{\text{inf},v}^2
  \]
- \( a \) cannot be modified.

- We calculate \( a = -\frac{\mu v}{v_{\text{inf},v}^2} = -44,232 \).
- To achieve \( r_p = a(1 - e) \), we need
  \[
e = 1 - \frac{r_p}{a} = 1.15
  \]
Arrival at Venus

To achieve the desired \( e = 1.15 \), we control the conditions at the **Patch Point**.
- We do through the angular momentum, \( h \).

We can control the **Target Radius**, \( \Delta \) through small adjustments far from the planet. Angular momentum can be controlled exactly through target radius, \( \Delta \).

\[
h_v = \Delta v_\infty, v
\]
Arrival at Venus

Recall that $p$ is defined only by angular momentum

$$p = \frac{h^2}{\mu} = \frac{\Delta^2 v_{\infty,v}^2}{\mu v}$$

Since

$$p = a(1 - e^2)$$

and $a$ is fixed, we can solve for $\Delta$,

$$\Delta = \sqrt{\frac{p \mu v}{v_{\infty,v}^2}} = \sqrt{\frac{a(1 - e^2) \mu v}{v_{\infty,v}^2}} = 25, 120 km$$
Messenger Probe to Mercury
Gravity Assist Trajectories

The same approach can be used to design gravity assist trajectories. In 2-dimensions, this is

\[ \vec{v}_f = R_1(\delta) (\vec{v}_i - \vec{v}_{planet}) + \vec{v}_{planet} \]

Example: If \( \delta = 180^\circ \) and \( \vec{v}_i = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \) km/s and \( \vec{v}_p = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \) km/s, then

\[ v_f = R(180^\circ) \begin{bmatrix} -4 \\ 0 \end{bmatrix} \text{ km/s} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ km/s} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \text{ km/s} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ km/s} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \text{ km/s} \]

Thus the probe was able to triple its velocity!
Gravity Assist Trajectories

To achieve the desired turning angle, we must control the geometry. The turning angle $\delta$ is given by

$$2 \cos^{-1} \frac{1}{e}$$

Recall

$$a = -\mu_{planet} / \| \vec{v}_i - \vec{v}_{planet} \|^2$$

Then the eccentricity can be fixed by the target radius as

$$\Delta = \sqrt{\frac{a(1 - e^2)\mu_{planet}}{\| \vec{v}_i - \vec{v}_{planet} \|^2}} = 25,120 \text{ km}$$

In 3 dimensions, the calculations are more complex.
Trajectories for Voyager 1 and Voyager 2 Spacecraft
Trajectories for Voyager 1, Voyager 2, and Pioneer Spacecraft

Outer Solar System Probes
Pioneer-10: 3 March 1972
Pioneer 11: 6 April, 1973
Voyager 2: 20 August 1977
Voyager 1: 5 September 1977
This Lecture you have learned:

SPACECRAFT DYNAMICS

Next Lecture: Final Exam.