Modern Control Systems

Matthew M. Peet
Illinois Institute of Technology

Lecture 11: Stabilizability and Eigenvalue Assignment
Stabilizability is weaker than controllability

**Definition 1.**

The pair \((A, B)\) is stabilizable if for any \(x(0) = x_0\), there exists a \(u(t)\) such that \(x(t) = \Gamma_t u\) satisfies

\[
\lim_{t \to \infty} x(t) = 0
\]

- Again, no restriction on \(u(t)\).
- Weaker than controllability
  - **Controllability:** Can we drive the system to \(x(T_f) = 0\)?
  - **Stabilizability:** Only need to *Approach* \(x = 0\).
- Stabilizable if uncontrollable subspace is naturally stable.
Consider the system in Controllability Form.

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
B_1 \\
0
\end{bmatrix} u(t)
\]

\[x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}\]

Note that

\[
\dot{x}_2(t) = A_{22}x_2(t)
\]

and so, we can solve explicitly

\[
x_2(t) = e^{A_{22}t}x_2(0)
\]

Clearly \(A_{22}\) must be Hurwitz if \((A, B)\) is stabilizable.

- Necessary and Sufficient
Lemma 2.

The pair \((A, B)\) is stabilizable if and only if \(A_{22}\) is Hurwitz.

This is an test for stabilizability, but requires conversion to controllability form.

- A more direct test is the PBH test

Theorem 3.

The pair \((A, B)\) is

- **Stabilizable** if and only if \(\text{rank} \begin{bmatrix} \lambda I - A & B \end{bmatrix} = n \) for all \(\lambda \in \mathbb{C}^+\)
- **Controllable** if and only if \(\text{rank} \begin{bmatrix} \lambda I - A & B \end{bmatrix} = n \) for all \(\lambda \in \mathbb{C}\)

**Note:** We need only check the eigenvalues \(\lambda\)
**Proof:** Controllable if and only if \( \text{rank} \left[ \lambda I - A \ B \right] = n \) for all \( \lambda \in \mathbb{C} \)

Proof.

We will use proof by contradiction. \((-2 \Rightarrow -1)\). Suppose \( \text{rank} \left[ \lambda I - A \ B \right] < n \).

- Thus \( \text{dim} \left( \text{Im} \left[ \lambda I - A \ B \right] \right) < n \)
- There exists an \( x \) such that \( x^T \left[ \lambda I - A \ B \right] = 0 \).
- Thus \( \lambda x^T = x^T A \) and \( x^T B = 0 \)
- Thus \( x^T A^2 = \lambda x^T A = \lambda^2 x^T \).
- Likewise \( x^T A^k = \lambda^k x^T \).
- Thus

\[
x^T C(A, B) = x^T \left[ B \ AB \ \cdots \ A^{n-1} B \right] = x^T \left[ B \ \lambda B \ \cdots \ \lambda^{n-1} B \right] = [0 \ \cdots \ 0]
\]

- Thus \( \text{dim}[\text{Im}C(A, B)] < n \), which means *Not Controllable*. \((-2 \Rightarrow -1)\).
- We conclude that controllable implies \( \text{rank} \left[ \lambda I - A \ B \right] = n \).
Proof.

For the second part, we will also use proof by contradiction. \((\neg 1 \Rightarrow \neg 2)\).

Suppose \((A, B)\) is not controllable. Then there exists an invertible \(T\) such that

\[
TAT^{-1} = \begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} \\
0 & \hat{A}_{22}
\end{bmatrix}, \quad TB = \begin{bmatrix}
\hat{B}_1 \\
0
\end{bmatrix}
\]

Now let \(\lambda\) be an eigenvalue of \(\hat{A}_{22}^T\) with eigenvector \(\hat{x}\). \(\hat{A}_{22}^T\hat{x} = \lambda \hat{x}\). Thus \(\hat{x}^T\hat{A}_{22} = \lambda \hat{x}^T\).

Let

\[
x = T^T \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}, \quad \text{then} \quad x^T = \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T T
\]

Then

\[
x^T \begin{bmatrix} \lambda I - A & B \\
0 & TB
\end{bmatrix} = x^T T^{-1} \begin{bmatrix} \lambda T - TAT^{-1}T & TB \\
0 & T
\end{bmatrix}
\]

\[
= \hat{x}^T TT^{-1} \begin{bmatrix} \lambda T - \begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} \\
0 & \hat{A}_{22}
\end{bmatrix} T & \hat{B}_1 \\
0 & 0
\end{bmatrix}
\]

\[
= \lambda \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T T - \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T \begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} \\
0 & \hat{A}_{22}
\end{bmatrix} T \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T \begin{bmatrix}
\hat{B}_1 \\
0
\end{bmatrix}
\]
Proof.

\[
x^T [\lambda I - A \quad B] = \begin{bmatrix} \lambda \hat{x}^T \\ \hat{x} \end{bmatrix}^T \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} T \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & \lambda \hat{x}^T \\ \hat{x}^T & \hat{x}^T A_{22} \end{bmatrix} T - \begin{bmatrix} 0 & \hat{x}^T A_{22} \\ 0 & 0 \end{bmatrix} T = 0
\]

\[
= \begin{bmatrix} 0 & \lambda I - \hat{A}_{22} \\ \hat{A}_{22}^T \hat{x} & 0 \end{bmatrix} T = 0
\]

- Thus \( x^T [\lambda I - A \quad B] = 0 \).
- Thus \( \text{rank} [\lambda I - A \quad B] < n \).
- Finally \( (\neg 1 \Rightarrow \neg 2) \).
- We conclude that \( \text{rank} [\lambda I - A \quad B] = n \) implies controllability.
Definition 4.

A **Companion Matrix** is any matrix of the form:

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
& \ddots & \ddots \\
& & 0 & 1 \\
-a_0 & \cdots & -a_{n-1}
\end{bmatrix}
\]

A companion matrix has the convenient property that

\[
\det(sI - A) = \sum_{i=0}^{n-1} a_i s^i = a_0 + a_1 s + \cdots + a_{n-1} s^{n-1} + s^n
\]
Theorem 5. Suppose \((A, B)\) is controllable. \(B \in \mathbb{R}^{n \times 1}\). Then there exists an invertible \(T\) such that

\[
TAT^{-1} = \begin{bmatrix}
0 & 1 & 0 \\
& \ddots & \\
-a_0 & & -a_{n-1}
\end{bmatrix}, \quad TB = \begin{bmatrix}
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

This is **Controllable Canonical Form**

- Different from controllability form
- This is useful for reading off transfer functions

\[
G(s) = C(sI - A)^{-1}B + D
\]

which has a denominator

\[
det(sI - A) = a_0 + \cdots + a_{n-1}s^{n-1}
\]
The problem of designing a controller

- We have touched on this problem in reachability
  - \( u(t) = B^T e^{A(T_f - t)T^{-1}}z_f \)
  - This controller is open-loop
- It assumes perfect knowledge of system and state.

Problems
- Prone to Errors, Disturbances, Errors in the Model

Solution
- Use continuous measurements of state to generate control

Static Full-State Feedback Assumes:
- We can directly and continuously measure the state \( x(t) \)
- Controller is a static linear function of the measurement
  \[ u(t) = Fx(t), \quad F \in \mathbb{R}^{m \times n} \]
Eigenvalue Assignment
Static Full-State Feedback

**State Equations:** \( u(t) = Fx(t) \)

\[
\dot{x}(t) = Ax(t) + Bu(t) \\
= Ax(t) + BFx(t) \\
= (A + BF)x(t)
\]

**Stabilization:** Find a matrix \( F \in \mathbb{R}^{m \times n} \) such that \( A + BF \) is Hurwitz.

**Eigenvalue Assignment:** Given \( \{\lambda_1, \ldots, \lambda_n\} \), find \( F \in \mathbb{R}^{m \times n} \) such that

\[
\lambda_i \in \text{eig}(A + BF) \quad \text{for } i = 1, \ldots, n
\]

is Hurwitz.

**Note:** A solution to the eigenvalue assignment problem will also solve the stabilization problem.

**Question:** Is eigenvalue assignment actually harder?
Eigenvalue Assignment
Single-Input Case

Theorem 6.
Suppose $B \in \mathbb{R}^{n \times 1}$. Eigenvalues of $A + BF$ are freely assignable if and only if $(A, B)$ is controllable.

Proof.

1. There exists a $T$ such that

$$\hat{A} = TAT^{-1} = \begin{bmatrix} 0 & I \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \quad \hat{B} = TB = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

2. Define $\hat{F} = [\hat{f}_0 \cdots \hat{f}_{n-1}] \in \mathbb{R}^{1 \times n}$. Then

$$\hat{B}\hat{F} = \begin{bmatrix} 0 \\ \hat{f}_0 \\ [\hat{f}_1 \cdots \hat{f}_{n-1}] \end{bmatrix}$$
Proof.

\[ \hat{B}\hat{F} = \begin{bmatrix} 0 & \hat{f}_0 & \cdots & \hat{f}_{n-1} \\ \hat{f}_0 & \hat{f}_1 & \cdots & \hat{f}_{n-1} \end{bmatrix} \]

Then

\[ \hat{A} + \hat{B}\hat{F} = \begin{bmatrix} O & I \\ -a_0 + \hat{f}_0 & -a_1 + \hat{f}_1 & \cdots & -a_{n-1} + \hat{f}_{n-1} \end{bmatrix} \]

This has the characteristic equation

\[ \det \left( sI - (\hat{A} + \hat{B}\hat{F}) \right) = s^n + (\hat{f}_{n-1} - a_{n-1})s^{n-1} + \cdots + (\hat{f}_0 - a_0) \]

Suppose we want eigenvalues \( \{\lambda_1, \cdots, \lambda_n\} \). Then define \( b_i \) as

\[ p(s) = (s - \lambda_1) \cdots (s - \lambda_n) = s^n + b_{n-1}s^{n-1} + \cdots + b_0 \]

Choose \( \hat{f}_i = a_i - b_i \).

Now let \( F = \hat{F}T \). Then

\[ A + BF = T^{-1}(\hat{A} + \hat{B}\hat{F})T \]
Proof.

- Then
  \[
  \det (sI - (A + BF)) = \det \left( T \left( sI - (\hat{A} + \hat{B}\hat{F}) \right) T^{-1} \right)
  = \det \left( sI - (\hat{A} + \hat{B}\hat{F}) \right)
  = (s - \lambda_1) \cdots (s - \lambda_n)
  \]

- Hence \( A + BF \) has eigenvalues \( \{\lambda_1, \cdots, \lambda_n\} \).

Suppose we want the eigenvalues \( \{\lambda_1, \cdots, \lambda_n\} \).

1. Find the \( b_i \)
2. Choose \( \hat{f}_i = a_i - b_i \).
3. Then use \( F = \left[ \hat{f}_0 \cdots \hat{f}_{n-1} \right] T \).

**Conclusion:** For Single-Input, controllability implies eigenvalue assignability.

- Requires conversion to controllable canonical form
- Matlab command `acker`
The multi-input case is harder

**Lemma 7.**

If \((A, B)\) is controllable, then for any \(x_0 \neq 0\), there exists a sequence 
\(\{u_0, u_1, \cdots, u_{n-2}\}\) such that span\(\{x_0, x_1, \cdots, x_{n-1}\}\) = \(\mathbb{R}^n\), where

\[
x_{k+1} = Ax_k + Bu_k \quad \text{for } k = 0, \cdots, n - 1
\]

**Proof.**

For 1 \(\Rightarrow\) 2, we again use proof by contradiction. We show \((-2 \Rightarrow -1)\).

- Suppose that for any \(x_0\), and any \(\{u_0, u_1, \cdots, u_{n-2}\}\), span\(\{x_0, \cdots, x_{n-1}\}\) \(\neq \mathbb{R}^n\). Then there exists some \(y\) such that \(y^T x_k = 0\) for any \(k = 0, \cdots, n - 1\). We can solve explicitly for \(x_k\):

\[
x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} Bu_j
\]
Proof.

\[ x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} Bu_j \]

- Let \( k = n - 1 \), and \( x_0 = Bu_{n-1} \) for some \( u_{n-1} \). Then for any \( u \)

\[ y^T x_{n-1} = y^T \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u_{n-1} \\ u_0 \\ \vdots \\ u_{n-2} \end{bmatrix} = y^T C(A, B) u = 0 \]

- Therefore, \( \text{image}(C(A, B)) \neq \mathbb{R}^n \). Hence \((A, B)\) is not controllable. This proves the lemma.
Lemma 8.

Suppose $(A, B)$ is controllable. Then for any nonzero column, $B_1 \in \mathbb{R}^n$, of $B$, there exists a $F_1 \in \mathbb{R}^{m \times n}$ such that $(A + BF_1, B_1)$ is controllable.

Proof.

Suppose $(A, B)$ is controllable. Let $x_0 = B_1$ and apply the previous Lemma to find some input $u_0, \ldots, u_{n-2}$ such that $\text{span}\{x_0, \ldots x_{n-1}\} = \mathbb{R}^n$ where

$$x_{k+1} = Ax_k + Bu_k$$

Let $T = \begin{bmatrix} x_0 & \cdots & x_{n-1} \end{bmatrix}$. Then $T$ is invertible. Let

$$F_1 = \begin{bmatrix} u_0 & \cdots & u_{n-2} \end{bmatrix} T^{-1} = UT^{-1}$$

- This implies $F_1 T = U$ and hence $F_1 x_i = u_i$ for $i = 0, \ldots, n - 1$.
- Now expand

$$x_{k+1} = Ax_k + Bu_k = Ax_k + BF_1 x_k = [A + BF_1] x_k$$
Proof.

\[ x_{k+1} = Ax_k + Bu_k = Ax_k + BF_1 x_k = [A + BF_1]x_k \]

Which means that \( x_k = [A + BF_1]^k x_0 \). However, since \( x_0 = B_1 \), we have

\[
T = \begin{bmatrix}
  x_0 & \cdots & x_{n-1}
\end{bmatrix}
= \begin{bmatrix}
  B_1 & \cdots & (A + BF_1)^{n-1} B_1
\end{bmatrix}
= C(A + BF_1, B_1)
\]

- Since \( T \) is invertible, \( C(A + BF_1, B_1) \) is full rank and hence \( (A + BF_1, B_1) \) is controllable.
**Theorem 9.**

The eigenvalues of $A + BF$ are freely assignable if and only if $(A, B)$ is controllable.

**Proof.**

The “only if” direction is clear. Suppose $(A, B)$ is controllable and we want eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$. Let $B_1$ be the first column of $B$.

- By Lemma, there exists a $F_1$ such that $(A + BF_1, B_1)$ is controllable.
- By other Lemma, since the $(A + BF_1, B_1)$ is controllable, the eigenvalues of $(A + BF_1, B_1)$ are assignable. This we can find a $F_2$ such that $A + BF_1 + B_1F_2$ has eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$.

- Choose $F = F_1 + \begin{bmatrix} F_2 \\ 0 \end{bmatrix}$. Then

$$A + BF = A + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = A + BF_1 + B_1F_2$$

has the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$. 

M. Peet
**Theorem 10.**

The eigenvalues of \( A + BF \) are freely assignable if and only if \( (A, B) \) is controllable.

Note that the proof was not very constructive: Need to find \( F_1 \) and \( F_2 \)...

**Matlab Commands**

- \( K = \text{acker}(A, B, p) \) for 1-D
- \( K = \text{place}(A, B, p) \) for n-D. \( p \) is the vector of pole locations.

**Theorem 11.**

If \( (A, B) \) is stabilizable, then there exists a \( F \) such that \( A + BF \) is Hurwitz.

**Proof.**

Apply the previous result to the controllability form.

**Conclusion:** If \( (A, B) \) is stabilizable, then it can be stabilized using only static state feedback. \( u(t) = Kx(t) \).