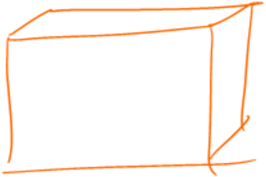
Spacecraft Dynamics and Control

Case 1 :



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$I_x \neq I_y$

Lecture 17: Stability of Torque-Free Motion

Case 2:

Damping

$\dot{T} < 0$

Attitude Dynamics

In this Lecture we will cover:

Non-Axisymmetric rotation

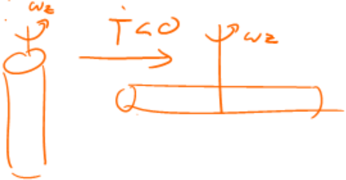
- Linearized Equations of Motion
- Stability

Case 1 :

Energy Dissipation

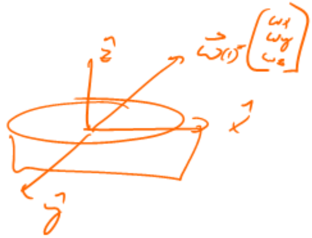
- The effect on stability of rotation

Case 2 :



Euler's Eqns

$$\dot{\omega}_x(t) = -\frac{I_z - I_y}{I_x}\omega_y(t)\omega_z(t)$$
$$\dot{\omega}_y(t) = -\frac{I_x - I_z}{I_y}\omega_x(t)\omega_z(t)$$
$$\dot{\omega}_z(t) = -\frac{I_y - I_x}{I_z}\omega_x(t)\omega_y(t)$$



Axisymmetric Case: $I_x = I_y$

- $\dot{\omega}_z = 0$ - ω_z is fixed
- Equations naturally become linear.
- Allows us to solve these linear equations explicitly

Non-axisymmetric Case $\boxed{I_x \neq I_y.}$

- We will have to rely on linear approximation

Linearization of the Euler Equations

Linearization allows us to consider small deviations about an equilibrium.

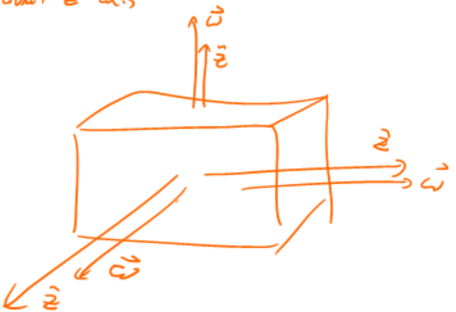
- We need to define the equilibrium

operating point: spin about z-axis

CASE: Stability of Spin about a principle axis.

- Nominal motion is

$$\boxed{\omega_0(t) = \begin{bmatrix} \omega_{x,0}(t) \\ \omega_{y,0}(t) \\ \omega_{z,0}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}}$$



- This is an equilibrium because

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\omega}_{x,0}(t) = -\frac{I_z - I_y}{I_x}\cancel{\omega_{y,0}}^{0}(t)\omega_{z,0}(t) = 0 \\ \dot{\omega}_{y,0}(t) = -\frac{I_x - I_z}{I_y}\cancel{\omega_{x,0}}^{0}(t)\omega_{z,0}(t) = 0 \\ \dot{\omega}_{z,0}(t) = -\frac{I_y - I_x}{I_z}\cancel{\omega_{x,0}}^{0}(t)\omega_{y,0}(t) = 0 \end{bmatrix}$$

Linearization of the Euler Equations

Now consider small disturbances to this equilibrium

$$\boxed{\vec\omega(t) = \vec\omega_0 + \Delta\vec\omega(t)}$$

$\vec\omega_0 = \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}$

Then $\Delta\omega(t) = \omega(t) - \omega_0$ and

$\Delta\omega(t) = \omega(t) - \omega_0$

$\Delta\dot\omega(t) = \dot\omega(t) - 0$

$$\boxed{\Delta\dot\omega(t) = \dot\omega(t)} - 0 = \begin{bmatrix} -\frac{I_z - I_y}{I_x}\omega_y(t)\omega_z(t) \\ -\frac{I_x - I_z}{I_y}\omega_x(t)\omega_z(t) \\ -\frac{I_y - I_x}{I_z}\omega_x(t)\omega_y(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{I_z - I_y}{I_x}(\omega_{y,0} + \Delta\omega_y(t))(\omega_{z,0} + \Delta\omega_z(t)) \\ -\frac{I_x - I_z}{I_y}(\omega_{x,0} + \Delta\omega_x(t))(\omega_{z,0} + \Delta\omega_z(t)) \\ -\frac{I_y - I_x}{I_z}(\omega_{x,0} + \Delta\omega_x(t))(\omega_{y,0} + \Delta\omega_y(t)) \end{bmatrix}$$

$\omega_{y,0} = 0$, $\omega_{z,0} = n$, $\omega_{x,0} = 0$, $\omega_{z,0} = n$, $\omega_{x,0} = 0$, $\omega_{y,0} = 0$

$$\Delta\vec\omega(t) = \begin{bmatrix} \Delta\omega_x(t) \\ \Delta\omega_y(t) \\ \Delta\omega_z(t) \end{bmatrix}$$

$$\begin{bmatrix} \omega_x(t) \\ \omega_y(t) \\ \omega_z(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix} + \begin{bmatrix} \Delta\omega_x(t) \\ \Delta\omega_y(t) \\ \Delta\omega_z(t) \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \Delta\dot\omega_x \\ \Delta\dot\omega_y \\ \Delta\dot\omega_z \end{bmatrix} = \begin{bmatrix} -\frac{I_z - I_y}{I_x}\Delta\omega_y(t)(n + \Delta\omega_z(t)) \\ -\frac{I_x - I_z}{I_y}\Delta\omega_x(t)(n + \Delta\omega_z(t)) \\ -\frac{I_y - I_x}{I_z}\Delta\omega_x(t)\Delta\omega_y(t) \end{bmatrix}}$$

Linearization of the Euler Equations

Now because we have assumed that $\Delta\omega$ is small, products of the form $\Delta\omega_x \Delta\omega_y = 0$ are very small indeed. Using this observation, we make the following

$10^{-10}\, 10^{-10} = 10^{-20}$

Approximations:

$$\Delta\omega_x \Delta\omega_y = 0, \quad \Delta\omega_x \Delta\omega_z = 0, \quad \Delta\omega_z \Delta\omega_y = 0$$

This yields the following set of linearized equations:

$$\Delta\dot{\omega}(t) = \begin{bmatrix} -\frac{I_z - I_y}{I_x} \Delta\omega_y(t)(n + \Delta\omega_z(t)) \\ -\frac{I_x - I_z}{I_y} \Delta\omega_x(t)(n + \Delta\omega_z(t)) \\ -\frac{I_y - I_x}{I_z} \Delta\omega_x(t)\Delta\omega_y(t) \end{bmatrix}$$

$$\begin{bmatrix} \Delta\dot{\omega}_x \\ \Delta\dot{\omega}_y \\ \Delta\dot{\omega}_z \end{bmatrix} = \begin{bmatrix} -\frac{I_z - I_y}{I_x} n\Delta\omega_y(t) \\ -\frac{I_x - I_z}{I_y} n\Delta\omega_x(t) \\ 0 \end{bmatrix}$$

$\Delta\dot{\omega}_z = 0 \rightarrow \Delta\omega_z = c$

Linearization of the Euler Equations

$\omega_z(0)=n$

Thus the evolution of small disturbances is governed by a set of linear equations.

$$\boxed{\begin{bmatrix} \Delta\dot{\omega}_x(t) \\ \Delta\dot{\omega}_y(t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{I_z-I_y}{I_x}n \\ -\frac{I_x-I_z}{I_y}n & 0 \end{bmatrix} \begin{bmatrix} \Delta\omega_x(t) \\ \Delta\omega_y(t) \end{bmatrix}}$$

$$\boxed{\Delta\dot{\omega}_z(t) = 0}$$

$\omega_z(0)=n$

- The third equation $\Delta\dot{\omega}_z = 0$ implies $\Delta\omega_z(t) = constant$.
- The first two equations combine to yield

$$\boxed{\begin{aligned} \Delta\ddot{\omega}_x(t) &= -\frac{I_z - I_y}{I_x} n \Delta\dot{\omega}_y(t) \\ &= \frac{I_z - I_y}{I_x}\frac{I_x - I_z}{I_y} n^2 \Delta\omega_x(t) \end{aligned}}$$

$\geq 0\ ?$

If we take the Laplace transform of this equation, we get

$$s^2\Delta\hat{\omega}_x(s) = \frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2 \Delta\hat{\omega}_x(s)$$

Stability Analysis

From

$$\boxed{s^2\Delta\hat{\omega}_x(s) = \frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2 \Delta\hat{\omega}_x(s)}$$

we get the transfer function

$$\hat{G}(s) = \frac{1}{\boxed{s^2 - \frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2}} = \frac{1}{\lambda(s)}$$

$G(s)\,\Delta\hat{\omega}_x(s)$

or if you prefer, the characteristic equation

$$\lambda(s) = s^2 - \frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2 = 0$$

The roots of this characteristic equation are

Quadratic Formula

a+bi

≥0 or ≤0

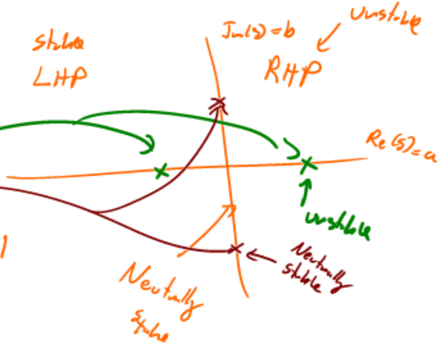
imaginary or real

$$\boxed{s = \pm\sqrt{\frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2}}$$

≥0

≥0

+ or − ?



Review Stability Analysis

Consider a differential equation with characteristic equation, $\lambda(s)$:

Recall that the roots of the characteristic equation tell us about the behaviour of the variable $\Delta\omega_x$.

- The roots may be real, imaginary, or a mixture: $s = a + b\imath$

There are **Three Cases**:

RHP

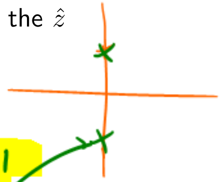
1. **[Instability:]** If $\boxed{\text{Real}(s) = a > 0}$ for any root of $\lambda(s)$, then small disturbances will grow over time.
2. **[Stability:]** If $\boxed{\text{Real}(s) = a < 0}$ LHP for all roots of $\lambda(s)$, then small disturbances will vanish over time.
3. **[Neutral Stability:]** If $\boxed{\text{Real}(s) = a = 0}$ for any root of $\lambda(s)$, then small disturbances will persist, but will not grow. and none in RHP

$$\lambda(s) = s^2 - \frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2$$

Now recall the roots of $\lambda(s)$ for the torque-free spaceraft spinning about the $\hat{z}$ axis with angular velocity n.

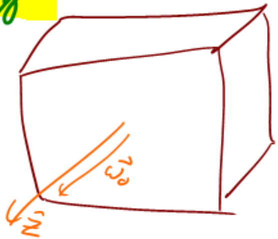
- The roots of $\lambda(s)$ are

$$s = \pm\sqrt{\frac{\overbrace{(I_z - I_y)}^{\geq 0}\overbrace{(I_x - I_z)}^{\leq 0}}{\underbrace{I_x I_y}_{\geq 0}} n^2}\ \ {}^{\geq 0}\quad < 0$$



Neutral Stability

We can break down our stability analysis into three cases:



CASE 1: Spin about the major axis ($I_z > I_x$ and $I_z > I_y$).

1. In this case $(I_z - I_y) > 0$ and $(I_x - I_z) < 0$.
2. Then the roots are purely imaginary
3. $s = \pm\imath b$ where $b = \sqrt{\frac{(I_z - I_y)(I_z - I_x)}{I_x I_y} n^2}$ is real.

Stable

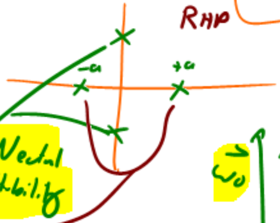
Stability of Torque-free Spin

$$s = \pm\sqrt{\frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2}$$

≥0 ≥0 ≤0 ≥0 ≥0

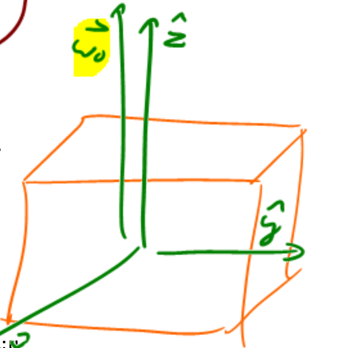
CASE 2: Spin about the minor axis ($I_z < I_x$ and $I_z < I_y$).

1. In this case $(I_z - I_y) < 0$ and $(I_x - I_z) > 0$.
2. The roots are also purely imaginary
3. $\boxed{s = \pm\imath b}$ where $\boxed{b = \sqrt{\frac{(I_z - I_y)(I_z - I_x)}{I_x I_y} n^2}}$ is real.



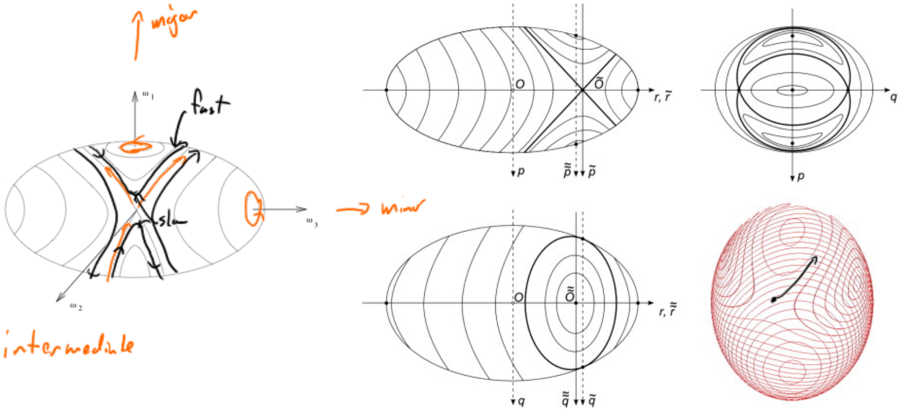
CASE 3: Spin about the intermediate axis ($I_y < I_z < I_x$ or $I_x < I_z < I_y$).

1. In this case $(I_z - I_y)(I_x - I_z) > 0$.
2. The roots are real ≥0 ≥0
3. $s = \pm a$ where $a = \sqrt{\frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2} > 0$ is real.
4. One of the roots has positive real part - **UNSTABLE**



Thus spin about an intermediate axis is always unstable (small deviations will eventually get big!)

Polhodes



This effect can be visualized using Polhodes.

- Positions of the axis of rotation, $\vec{\omega}$
- For fixed energy, lines are of constant angular momentum $\vec{h}$.

Instability of the intermediate axis

Figure: A Deck of Cards on the ISS (Garriott)

Figure: Simulated Ellipsoid

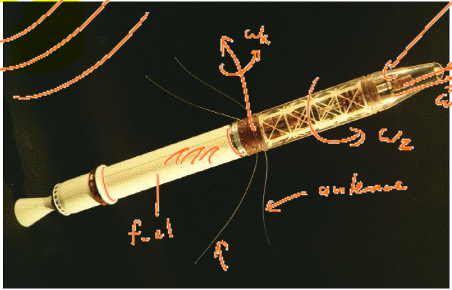
Figure: A Textbook on the ISS (Petit)

Summary:

- Spin about intermediate axis - **Unstable**
- Spin about major or minor axis - **Neutral Stability**

What about Disturbances?

- Fuel Sloshing
- Flexible Structures
- Heat dissipation



$\vec{h}$

Problem:

- Newton's Second Law predicts Conservation of **Momentum**
- It says nothing about **Kinetic Energy**!!!

Question: What is the effect of losses in Kinetic Energy?

Destabilization caused by Energy Dissipation

Summary:

- Spin about intermediate axis - **Unstable**
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What about Disturbances?

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Question: What is the effect of losses in Kinetic Energy?

Spacecraft depicted is Explorer 1

- First American satellite
- Based on missile technology (no separation from rocket motor)
- Launched January 31, 1958
- Initially spin about minor axis.
- Quickly started precessing and decayed to spin about major axis
- Energy Dissipation from long flexible antennae
- Prompted development of Euler equations.

Destabilization caused by Energy Dissipation

Question: How to relate energy drain $\dot{T} < 0$ to changes in $\vec{\omega}$?

Consider the expression for Kinetic Energy:

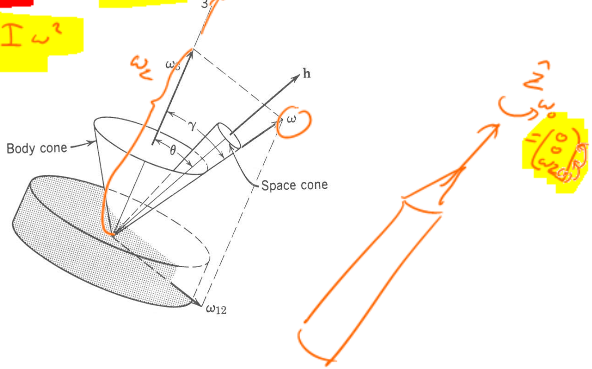
$$2T = \omega_x^2 I_x + \omega_y^2 I_y + \omega_z^2 I_z$$

$\vec{H} = I\vec{\omega}$

$T_R = \frac{1}{2} I \omega^2$

Meanwhile, the total angular momentum is

$$h^2 = I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2$$



Consider the Axisymmetric Case: $\boxed{I_x = I_y}$. Then

ω_{xy}

$$\boxed{\begin{aligned} 2T &= I_x(\omega_x^2 + \omega_y^2) + \omega_z^2 I_z \\ h^2 &= I_x^2(\omega_x^2 + \omega_y^2) + I_z^2\omega_z^2 \end{aligned}}$$

ω_{xy}

the second equation implies

ω_{xy}

$$\boxed{\omega_x^2 + \omega_y^2} = \frac{h^2 - I_z^2\omega_z^2}{I_x^2}$$

ω_{xy}

We substitute $\omega_x^2 + \omega_y^2 = \frac{h^2 - I_z^2\omega_z^2}{I_x^2}$ into the expression for T to get

$$2\boxed{T} = \frac{\boxed{h^2} - I_z^2\boxed{\omega_z^2}}{I_x} + \omega_z^2 I_z = \frac{h^2}{I_x} + \omega_z^2 I_z\left(1 - \frac{I_z}{I_x}\right)$$

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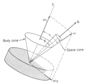
Consider the Axisymmetric Case: $I_x = I_y$. Then

$$2T = I_x(\omega_x^2 + \omega_y^2) + \omega_z^2 I_z$$
$$h^2 = I_x^2(\omega_x^2 + \omega_y^2) + I_z^2\omega_z^2$$

the second equation implies

$$\omega_x^2 + \omega_y^2 = \frac{h^2 - I_z^2\omega_z^2}{I_x^2}$$



We substitute $\omega_x^2 + \omega_y^2 = \frac{h^2 - I_z^2\omega_z^2}{I_x^2}$ into the expression for T to get

$$2T = \frac{h^2 - I_z^2\omega_z^2}{I_x} + \omega_z^2 I_z = \frac{h^2}{I_x} + \omega_z^2 I_z\left(1 - \frac{I_z}{I_x}\right)$$

$$2T = \frac{h^2}{I_x} + \omega_z^2 I_z\left(1 - \frac{I_z}{I_x}\right)$$

- T may decrease, but h is invariant
- $\boxed{\omega_{xy}}$ and ω_z may change as T decreases.
- The expression only include ω_z, however.

Destabilization caused by Energy Dissipation

Now consider the angle (θ) by which $\vec{h}$ differs from $\hat{z}$.

in BF frame

$$\cos\theta = \frac{h_z}{h} = \frac{\boxed{I_z\omega_z}}{h}$$

$I_z \omega_z = h \cos\theta$

We would like to express T in terms of θ.

We solve this equation for ω_z to get $\omega_z = \frac{h}{I_z}\cos\theta$.
Combining with the equation for T, we get

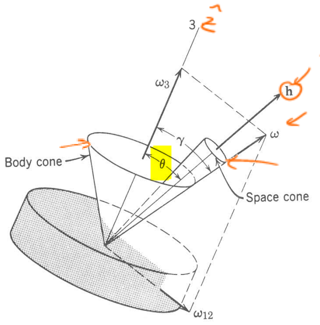
$$2T = \frac{h^2}{I_x} + \frac{h^2}{I_z}\cos^2\theta\left(1 - \frac{I_z}{I_x}\right)$$

$T \downarrow$ $\theta \uparrow$ or $\downarrow$

Taking the time-derivative, we find

$$\dot{T} = -\frac{h^2}{2I_z}2\cos\theta\sin\theta\left(1 - \frac{I_z}{I_x}\right)\dot{\theta}$$

$$\dot{T} < 0 \qquad = \frac{h^2}{2I_z}2\cos\theta\sin\theta\left(\frac{I_z}{I_x} - 1\right)\dot{\theta} \qquad \dot{\theta} < 0 \text{ or } \dot{\theta} > 0$$



Destabilization caused by Energy Dissipation

Now consider the angle (θ) by which $\vec{h}$ differs from $\hat{z}$.

$$\cos\theta = \frac{h_z}{h} = \frac{I_z\omega_z}{h}$$

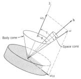
We would like to express T in terms of θ.

We solve this equation for ω_z to get $\omega_z = \frac{h}{I_z}\cos\theta$.
Combining with the equation for T, we get

$$2T = \frac{h^2}{I_x} + \frac{h^2}{I_z}\cos^2\theta\left(1 - \frac{I_z}{I_x}\right)$$

Taking the time-derivative, we find

$$\begin{aligned}\dot{T} &= -\frac{h^2}{2I_z}2\cos\theta\sin\theta\left(1 - \frac{I_z}{I_x}\right)\dot{\theta}\\ &= \frac{h^2}{2I_z}2\cos\theta\sin\theta\left(\frac{I_z}{I_x} - 1\right)\dot{\theta}\end{aligned}$$



Recall θ is the angle the angular momentum vector makes with the body-fixed axis.

- $\vec{h}$ and $\vec{\omega}$ are expressed in body-fixed coordinates.

Destabilization caused by Energy Dissipation

$$\dot{T} = \frac{h^2}{I_z}\cos\theta\sin\theta\left(\frac{I_z}{I_x} - 1\right)\dot{\theta}$$

or

$$\boxed{\dot{\theta} = \frac{\overbrace{I_z I_x}^{\geq 0}}{\underbrace{h^2}_{\geq 0}\underbrace{\cos\theta\sin\theta}_{\geq 0}\left(I_z - I_x\right)}\overbrace{\dot{T}}^{\leq 0}}$$

$\theta \approx 0$ but not exactly

≤ 0 for case 1

≥ 0 for case 2.

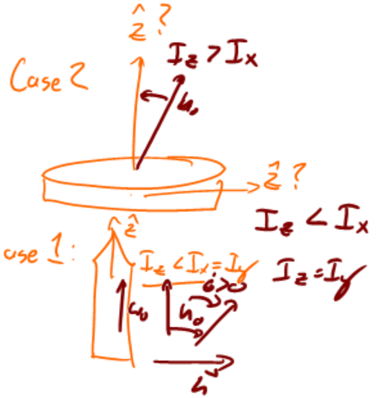
Recall that θ is the angle by which $\vec{h}$ differs from $\hat{z}$

- Initially, $\theta = 0$ - spin aligned with $\hat{z}$ axis.

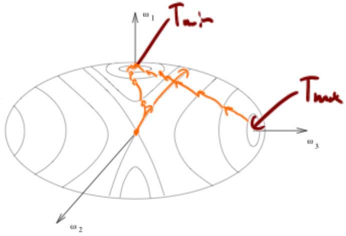
There are two cases

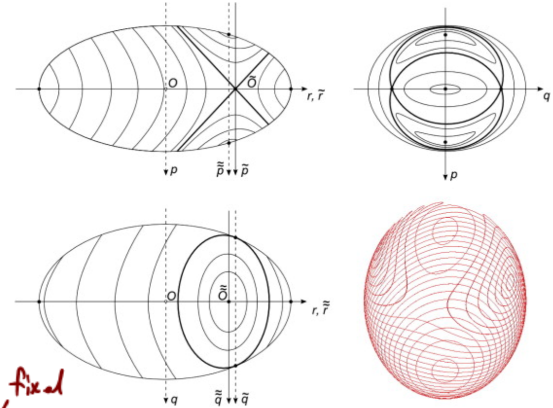
1. **CASE 1:** If $I_x > I_z$, then $\dot{T} < 0$ implies that $\dot{\theta} > 0$.
   - Spin Axis $\hat{z}$ is **UNSTABLE**
2. **CASE 2:** If $I_x < I_z$, then $\dot{T} < 0$ implies that $\dot{\theta} < 0$
   - Spin Axis $\hat{z}$ is **STABLE**

alignment of $\hat{z}$ and $\hat{h}$



Polhodes



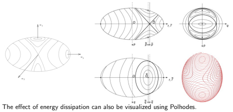


fixed

The effect of energy dissipation can also be visualized using Polhodes.

- Each line has constant $\boxed{\frac{h^2}{T}.}$ $\dot{T} < 0 \rightarrow$ go fr T_{max} to T_{min}
- Rotation proceeds from large T to small T.

Polhodes



The effect of energy dissipation can also be visualized using Polhodes.

- Each line has constant $\frac{h^2}{T}$.
- Rotation proceeds from large T to small T.

- Polhode represents intersection of energy and inertia ellipsoids.
- Poinsot's construction: Take the inertia ellipsoid, hold the center a fixed distance from an inkpad and where it rolls forms one of the lines.

Theorem 1 (Major Axis Rule).

1. *Spin about the major axis is stable* even with $\dot{T}<0$
2. *Spin about any other axis is unstable*

minor: if $\dot{T}<0$ intermediate: if $\dot{T}<0$ or $\dot{T}=0$

Conclusion:

- Spacecraft must be fat!

Problem:

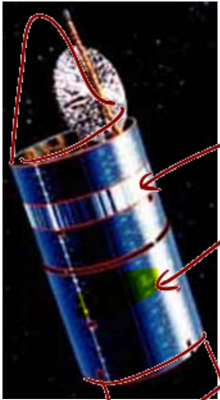
- Rockets are thin.

Satellites should not be prolate

Solution: Dual Spinners

- Only a fat slice of the spacecraft is spun up
- Allows nutation dampers to stabilize the spin axis.

dissipate Energy



overall sat is prolate

despin

spinning (oblate)

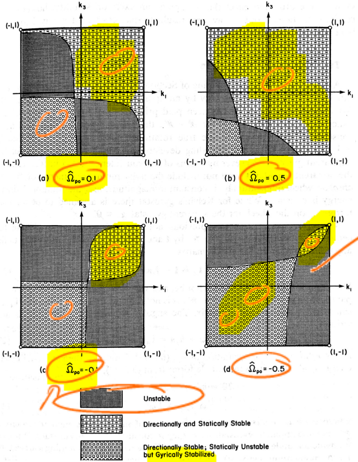
Booster (Discard)

A De-spun section can increase the stability about a minor axis.

$$k_1 = \frac{I_2 - I_3}{I_3}, \qquad k_3 = \frac{I_2 - I_1}{I_3},$$

$$\hat{\Omega}_{po} := \frac{\Omega_{po}}{\nu}, \qquad \Omega_{po} = \frac{h_s}{\sqrt{I_1 I_3}}$$

- $h_s = I_s \omega_s$ is angular momentum of spinning section
- ν is angular speed of body about the 2-axis



Unstable

Figure: Stability Regions for Dual-Spinner

Dual Spinners (General Case)

A De-spun section can increase the stability about a minor axis.

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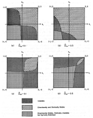

Figure: Stability Regions for Dual-Spinner

Alternately, we can redefine k_1, k_3

$$k_{1h} := k_1 + \hat{\Omega}_{po}\sqrt{\frac{1-k_1}{1-k_3}}$$

$$k_{2h} := k_2 + \hat{\Omega}_{po}\sqrt{\frac{1-k_3}{1-k_1}} - \Xi$$

- Stable iff $k_{1h}k_{3h} > 0$. With energy dissipation: if $k_{1h} > 0$ and $k_{3h} > 0$
- $\hat{\Omega}_{po} > 0$ if body and wheel spinning in same direction.
- Ξ is an energy damping term

Nutation Damper

Attitude Dynamics

In this Lecture we have covered:
Non-Axisymmetric rotation

- Linearized Equations of Motion
- Stability

Energy Dissipation

- The effect on stability of rotation