Spacecraft Dynamics and Control

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Lecture 18: Feedback Control of Attitude Dynamics
In this Lecture we will cover:
An Review of Feedback control for Attitude Dynamics
  • Transfer Functions
  • PID Control
  • Root Locus

**Problem:** 3-axis Stabilization
  • Detumble (if $\vec{\omega} \not\equiv 0$)
  • Attitude Tracking (assuming $\vec{\omega} \cong 0$)
Recall:
\[
\begin{bmatrix}
L \\
M \\
N
\end{bmatrix} =
\begin{bmatrix}
I_{xx} \dot{\omega}_x + \omega_y \omega_z (I_{zz} - I_{yy}) \\
I_{yy} \dot{\omega}_y + \omega_x \omega_z (I_{xx} - I_{zz}) \\
I_{zz} \dot{\omega}_z + \omega_x \omega_y (I_{yy} - I_{xx})
\end{bmatrix}
\]

Non-axisymmetric Case \( I_x \neq I_y \neq I_z \).

We first assume the spacecraft has been detumbled, so we have

**Small Spin Assumption:** \( \omega_x = \omega_y = \omega_z \approx 0 \).

- Nominal motion is
\[
\omega_0(t) =
\begin{bmatrix}
\omega_{x,0}(t) \\
\omega_{y,0}(t) \\
\omega_{z,0}(t)
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

In this case, the Linearized dynamics become:
\[
\begin{bmatrix}
L \\
M \\
N
\end{bmatrix} =
\begin{bmatrix}
I_{xx} \dot{\omega}_x \\
I_{yy} \dot{\omega}_y \\
I_{zz} \dot{\omega}_z
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
\dot{\omega}_x \\
\dot{\omega}_y \\
\dot{\omega}_z
\end{bmatrix} =
\begin{bmatrix}
\frac{M}{I_x} \\
\frac{M}{I_y} \\
\frac{M}{I_z}
\end{bmatrix}
\]

The Dynamics are all uncoupled.
**Problem:** We don't measure rotation rates. We measure rotation angles.

- Now we need to choose an inertial coordinate system.

The Euler Angles define the transformation from the body-fixed to inertial coordinates

\[
\begin{bmatrix}
    p \\
    q \\
    r \\
    \omega_x \\
    \omega_y \\
    \omega_z
\end{bmatrix} = \vec{\omega}/B = R_1(\phi)R_2(\theta)R_3(\psi)\vec{\omega}/I = R(\phi)R(\theta)R(\psi) \begin{bmatrix}
    \dot{\phi} \\
    \dot{\theta} \\
    \dot{\psi}
\end{bmatrix}
\]

\[
\begin{bmatrix}
    \dot{\phi} \\
    \dot{\theta} \\
    \dot{\psi}
\end{bmatrix}
= \begin{bmatrix}
    \dot{\phi} - \dot{\psi} \sin \theta \\
    \dot{\theta} \cos \phi + \dot{\psi} \cos \theta \sin \phi \\
    \dot{\psi} \cos \theta \cos \psi - \dot{\theta} \sin \phi
\end{bmatrix}
\]

Of course, we often have \(\vec{\omega}/B\) and are trying to find \(\vec{\omega}/I\). In this case, the rotation matrices can be inverted to obtain

\[
\begin{bmatrix}
    \dot{\phi} \\
    \dot{\theta} \\
    \dot{\psi}
\end{bmatrix} = \begin{bmatrix}
    p + (q \sin \phi r \cos \phi) \tan \theta \\
    q \cos \phi - r \sin \phi \\
    (q \sin \phi + r \cos \phi) \sec \theta
\end{bmatrix}
\]

Notice the Singularity at \(\theta = \pm 90^\circ\) (can be avoided with quaternions).

Equations are also different for 2-1-3 and 1-2-3 rotation sequences.
Kinematics and Euler Angles

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} = 
\begin{bmatrix}
\dot{p} + (q \sin \phi r \cos \phi) \tan \theta \\
q \cos \phi - r \sin \phi \\
(q \sin \phi + r \cos \phi) \sec \theta
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
\dot{p} \\
\dot{q} \\
\dot{w}
\end{bmatrix} = 
\begin{bmatrix}
\dot{\omega}_x \\
\dot{\omega}_y \\
\dot{\omega}_z
\end{bmatrix} = 
\begin{bmatrix}
\frac{L}{I_x} \\
\frac{M}{I_y} \\
\frac{N}{I_z}
\end{bmatrix}
\]

Which is a set of 6 nonlinear coupled differential equations.

- We have already linearized the second set.
- We should also linearize the first set.
  - Assume $\theta \approx 0$, $\phi \approx 0$, and $\psi \approx 0$. 

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} = 
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]

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Decoupling the Equations

Combining these two sets of equations, we get

\[
\begin{bmatrix}
\ddot{\phi} = \frac{L}{I_x} \\
\ddot{\theta} = \frac{M}{I_y} \\
\ddot{\psi} = \frac{N}{I_z}
\end{bmatrix}
\]

If \(L\), \(M\), and \(N\) are decoupled, we can decouple the equations

- Orthogonal reaction wheel for each body-fixed axis.

Lets design a controller for roll

\[
\ddot{\phi} = \frac{L}{I_x}
\]

Consider **Proportional Feedback**

\[
\frac{L(t)}{I_x} = -K(\phi(t) - \phi_0)
\]

Then the closed-loop poles are at \(s = \pm \imath \sqrt{K}\)

- Neutrally Stable
Let's look at the effect of PD control on a 2nd-order system:

\[ \hat{G}(s) = \frac{1}{s^2 + bs + c} \]

**Controller:** \( \hat{K}(s) = -K [1 + T_D s] \)

**Closed Loop Transfer Function:**

\[
\begin{align*}
\frac{\hat{K}(s)\hat{G}(s)}{1 + \hat{K}(s)\hat{G}(s)} &= \frac{K [1 + T_D s]}{s^2 + bs + c + K [1 + T_D s]} \\
&= \frac{K [1 + T_D s]}{s^2 + (b + KT_D) s + (c + K)}
\end{align*}
\]

The poles of the system are freely assignable for a 2nd order system.

- \( T_D \) and \( K \) allow us to construct any denominator we desire.
Suppose we want poles at $s = p_1, p_2$.

We want the closed loop of the form:

$$\frac{1}{(s - p_1)(s - p_2)} = \frac{1}{(s^2 - (p_1 + p_2)s + p_1p_2)}$$

Thus we want

- $c + K = p_1p_2$ which means $K = p_1p_2 - c$.
- $b + KT_D = -(p_1 + p_2)$ which means $T_D = -\frac{p_1 + p_2 + b}{K} = -\frac{p_1 + p_2 + b}{p_1p_2 - c}$

PD feedback gives Total Control over a 2nd-order system.
Factor in Constraints on the Step Response:

\[
\sigma < -\frac{4.6}{T_{s,\text{desired}}} \\
\omega_d < \frac{\pi}{\ln(M_{p,\text{desired}})} \sigma \\
\omega_n > \frac{1.8}{T_{r,\text{desired}}}
\]

Any pole locations not prohibited are allowed.

- \(\sigma\) is real part of \(s\)
- \(\omega_d\) is imaginary part of \(s\)
- \(\omega_n\) is magnitude of \(s\)
- \(T_s\) is settling time
- \(M_p\) is percent overshoot
- \(T_r\) is rise time
**Steady-State Error**

**Definition 1.**

Steady-State Error \( (e_{ss}) \) for a stable system \( G \) is the final difference between input and output.

\[
e_{ss} = \lim_{t \to \infty} u(t) - y(t)
\]

**Theorem 2 (Final Value Theorem).**

\[
\lim_{t \to \infty} y(t) = \lim_{s \to 0} s\hat{y}(s)
\]

For Step response: \( \hat{u} = \frac{1}{s} \). So if \( G(s) \) is the transfer function of \( G \),

\[
e_{ss} = \lim_{s \to 0} \frac{1}{1 + G(s)K(s)}
\]

Now, for roll control, \( G(s) = \frac{1}{s^2} \),

\[
e_{ss} = \lim_{s \to 0} \frac{1}{1 + G(s)K(s)} = \lim_{s \to 0} \frac{s^2}{s^2 + K T_D s + K} = \frac{0}{K} = 0
\]

So \( e_{ss} = 0! \)
Ramp Response - Important for tracking (Since spacecraft is moving). In this case the input is

\[ \hat{u}(s) = \frac{1}{s^2} \]

The stead-state error can be found as:

\[ e_{ss} = \lim_{s \to 0} \left( \frac{1}{s^2} - \frac{G(s)K(s)}{1 + G(s)K(s)} \frac{1}{s^2} \right) = \lim_{s \to 0} \frac{1}{s} \left( \frac{1}{1 + G(s)K(s)} \right) = 0 \]

Now, for roll control, \( G(s) = \frac{1}{s^2} \),

\[ e_{ss} = \lim_{s \to 0} \frac{1}{s(1 + G(s)K(s))} = \lim_{s \to 0} \frac{s}{s + KT_D s + K} = 0 \]

So still, \( e_{ss} = 0! \)

Conclusion: Integral Control is not necessary in space!!
Lyapunov Stability for Detumbling Spacecraft
Controller Design for Nonlinear Dynamics

A VERY Brief Introduction to Lyapunov Stability

Consider a Nonlinear ODE

\[ \dot{x}(t) = f(x(t)) \]

with \( x(0) \in \mathbb{R}^n \).

Theorem 3 (Lyapunov Stability).

Suppose there exists a continuous \( V \) and \( \alpha, \beta, \gamma > 0 \) where

\[ V(x) > 0 \quad \text{and} \]

\[ \dot{V}(x(t)) = \nabla V(x(t))^T f(x(t)) < 0 \]

for all \( x \in \mathbb{R}^n \). Then \( \dot{x} = f(x) \) is Globally Stable.
The literature on Lyapunov functions is vast

- **BY FAR** the most commonly used tool for control of nonlinear systems.
- Represents the potential energy of a particular state.
  - Potential Energy as measured by the size (square integral) of the resulting trajectory.
Lyapunov Stability for Detumbling Spacecraft

The Dynamics of a tumbling spacecraft with input torque $u(t)$:

- Uses the matrix form of cross-product ($\omega \times$) and inertia tensor ($I$)

$$I\dot{\omega}(t) = -\omega \times (t)I\omega(t) + u(t)$$

Now we Propose a Lyapunov Function:

$$V(\omega) = \frac{1}{2} \omega^T I \omega = \frac{1}{2} \|I^{\frac{1}{2}} \omega\|^2 > 0$$

Take the time-derivative of this Lyapunov Function:

- Uses the matrix form of cross-product ($\omega \times$)

$$\dot{V}(t) = \omega(t)^T I \dot{\omega}(t) = \omega(t)^T (-\omega \times (t)\omega(t) + u(t))$$

$$= -\omega(t)^T (\omega(t) \times \omega(t)) + \omega(t)^T u(t))$$

$$= \omega(t)^T u(t)$$

Choose Controller

$$u(t) = -P\omega(t) \quad \text{where} \quad P > 0$$

Since all the eigenvalues of $P$ are positive, $\omega^T P \omega > 0$ and hence

$$\dot{V}(t) = -\omega(t)P\omega(t) < 0$$

Which proves global stability!
Lyapunov Stability for Detumbling Spacecraft

- $I$ is a positive matrix, so we can take its square root.
- Note this does not control to any particular orientation.
- Assumes spacecraft capable of large torques.
- Typically we use nonlinear control for detumble and piecewise-linearized control for attitude tracking.

Lyapunov Stability for Detumbling Spacecraft

The dynamics of a tumbling spacecraft with input torque $u(t)$:

- Uses the matrix form of cross-product ($\omega \times$) and inertia tensor ($I$)

\[
I\dot{\omega}(t) = -\omega(t)\times I\omega(t) + u(t)
\]

Now we propose a Lyapunov function:

\[
V(\omega) = \frac{1}{2}\omega^T I \omega = \frac{1}{2}||I^{1/2}\omega||^2 > 0
\]

Take the time-derivative of this Lyapunov function:

- Uses the matrix form of cross-product ($\omega \times$)

\[
\dot{V}(t) = \omega(t)^T I \dot{\omega}(t) = \omega(t)^T (-\omega(t)\times I\omega(t) + u(t)) = -\omega(t)^T I \omega(t) + \omega(t)^T u(t)
\]

Choose controller

\[
u(t) = -P\omega(t)\quad \text{where} \quad P > 0
\]

Since all the eigenvalues of $P$ are positive, $\omega^TP\omega > 0$ and hence

\[
\dot{V}(t) = -\omega(t)^T P \omega(t) < 0
\]

Which proves global stability!
In this Lecture we have covered:
Kinematics Coupled with Dynamics
  • Linearized to uncoupled version
Feedback Control
  • Proportion-Differential Feedback
  • Steady-State Error.