LMI Methods in Optimal and Robust Control

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Lecture 5: LMIs for Controllability and Feedback Stabilization

Recall: Stability and Solutions of the Lyapunov Equation

Lemma 1.

A is Hurwitz if and only if for any $Q > 0$, there exists a $P > 0$ such that

$$A^T P + PA = -Q < 0$$

One such solution is:

$$P = \int_0^\infty e^{A^T s} Q e^{As} ds$$

Lemma 2.

A is Schur if and only if for any $Q > 0$, there exists a $P > 0$ such that

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$$P = \sum_{k=0}^{\infty} (A^T)^k Q A^k$$

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Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

In the previous lecture, we focused on properties of the map

$$x_0 \mapsto x(t) = e^{At} x_0$$

in terms of the eigenvalues of A.

- We proposed LMIs to constrain the eigenvalues of A.

In this lecture we look at the effect of the *input*, $u(\cdot)$, on the *state*, $x(t)$ and on the *output*, $y(t)$.

$$u(\cdot) \mapsto y(\cdot)$$

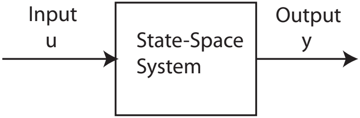
and

$$u(\cdot) \mapsto x(\cdot)$$

In this case, the map is more complicated. However, we will attempt to characterize its properties in terms of properties of the matrices (A, B, C)

State-Space System:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$



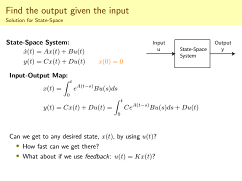
Input-Output Map:

$$x(t) = \int_0^t e^{A(t-s)} Bu(s)ds$$

$$y(t) = Cx(t) + Du(t) = \int_0^t Ce^{A(t-s)} Bu(s)ds + Du(t)$$

Can we get to any desired state, $x(t)$, by using $u(t)$?

- How fast can we get there?
- What about if we use *feedback*: $u(t) = Kx(t)$?



For a discrete-time system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \qquad x_0 = 0 \end{aligned}$$

The solution is

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} B u_i$$

$$y_k = CA^k x_0 + \sum_{i=0}^{k-1} CA^{k-i-1} B u_i + Du_k$$

Where, again, we usually take $x_0 = 0$.

Controllability

Definition 3.

For a given continuous-time system (A, B), the **state x_f is Reachable** if for any fixed T_f, there exists a $u(t)$ such that

$$x_f = \int_0^{T_f} e^{A(T_f - s)} B u(s) ds$$

Definition 4.

The **continuous-time system** (A, B) **is reachable** if any point $x_f \in \mathbb{R}^n$ is reachable.

Definition 5.

The **continuous-time system** (A, B) **is controllable** if for any T_f and any *initial* point $x_0 \in \mathbb{R}^n$, there exists a $u(t)$ such that $x(T_f) = 0$.

The reachable set and controllable set are Subspaces.

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The reachable set and controllable set are Subspaces.

- The difference between controllability and reachability and stabilizability is subtle. The difference from stabilizability is the finite-time question. The difference between reachability and controllability is away from origin vs. to origin. For a linear system, we can move the origin (introducing a bias in the input), which implies there is no difference at all for these systems.

The reachable set is defined by the map from input to solution at time T

$$u(\cdot) \mapsto x(T)$$

The reachable set is the image space of this map, $\Gamma_T : u \mapsto x$, where

$$x(T) = \underbrace{\int_0^T e^{A(T-s)} Bu(s) ds}_{\Gamma_T}$$

Review: Subspaces, Image, and Kernel

Definition 6.

A set, C, is a **Vector Space** if it is closed under addition and scalar multiplication.

1. $\alpha(u+v) = \alpha u + \alpha v \in C$ for all $\alpha \in \mathbb{R}$ and $u, v \in C$. (vector distributivity)
2. $(\alpha+\beta)u = \alpha u + \beta u \in C$ for all $\alpha, \beta \in \mathbb{R}$ and $u \in C$. (scalar distributivity)

Definition 7.

A **subspace** is a subset of a vector space which is also a vector space using the same definitions of addition and multiplication.

For right now, the most important subspaces are the image and kernel of a matrix ($M \in \mathbb{R}^{n \times m}$)/function/operator.

Definition 8 (Image and Kernel of a Matrix, M).

$$\operatorname{Im} M := \{x \in \mathbb{R}^n \;:\; x = My \;\text{ for some }\; y \in \mathbb{R}^m\}$$
$$\ker M := \{x \in \mathbb{R}^m \;:\; Mx = 0\}$$

For a fixed t, the set of reachable states is defined as

$$R_t := \{x \ : \ x = \int_0^t e^{A(t-s)} Bu(s)ds \text{ for some function } u.\}$$

Note: The mapping $\Gamma_t : u \mapsto x(t)$ is *linear*.

- Hence $R_t = \operatorname{Im} \Gamma_t$ is a subspace of $\mathbb{R}^n$

Definition 9.

For a given system (A, B), the **Controllability Matrix** is

$$C(A, B) := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

Definition 10.

For a given (A, B), the **Controllable Subspace** is

$$C_{AB} = \text{Image} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

Fact: The system (A, B) is **Controllable** if $C_{AB} = \operatorname{Im} C(A, B) = \mathbb{R}^n$.

Controllability

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Fact: The system (A, B) is Controllable if $C_{AB} = \operatorname{Im} C(A,B) = \mathbb{R}^n$.

$C(A, B)$ is a matrix and C_{AB} is the image of this matrix.
Controllability is a property of the *system*

- Thus we characterize a property of the system map $u(\cdot) \mapsto x(\cdot)$ using properties of the matrices B and A.

There is a more physical interpretation:

- B is the set of states input u affects directly. AB is the set of states u affects by affecting one state and then that state affects another. etc. until we achieve $n - 1$ degrees of separation, which by Cayley-Hamilton, means we can stop looking.

Controllability

Definition 11.

The finite-time **Controllability Grammian** of pair (A, B) is

$$W_t := \int_0^t e^{As} B B^T e^{A^T s} ds$$

W_t is a positive semidefinite matrix.
The following relates these three concepts of controllability

Theorem 12.

For any $t \geq 0$,

$$R_t = C_{AB} = \textit{Image}\ (W_t)$$

or

$$\text{Image } \Gamma_t = \text{Image } C(A, B) = \text{Image } (W_t)$$

W_t is positive **Definite** if and only if (A, B) is controllable.
Note the reachable set does not depend on t!

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This result connects

1. Existence of positive matrix (not yet an LMI, however)
2. Properties of A, B
3. Properties of the map $u \mapsto x$

Controllability: $\text{Im}(W_t) \subset R_t$

Proposition 1.

$$Im(W_t) \subset R_t$$

Proof.

First, suppose that $x \in \text{Im}(W_t)$ for some $t > 0$. Then $x = W_t z$ for some z.

- Now let $u(s) = B^T e^{A^T(t-s)} z$. Then
$$\begin{aligned}\Gamma_t u &= \int_0^t e^{A(t-s)} B u(s) ds \\ &= \int_0^t e^{A(t-s)} B B^T e^{A^T(t-s)} z ds = W_t z = x\end{aligned}$$
- Thus $x \in \text{Im}(\Gamma_t) = R_t$. Hence $\text{Im}(W_t) \subset R_t$. □

This proof is useful, since for any $x_d \in R_{T_f}$, it gives us the input

- Let $u(t) = B^T e^{A^T(T_f - t)} W_{T_f}^{-1} x_d$.
- Then for $\dot{x}(t) = Ax(t) + B(t)$, $x(0) = 0$, we have $x(T_f) = x_d$.

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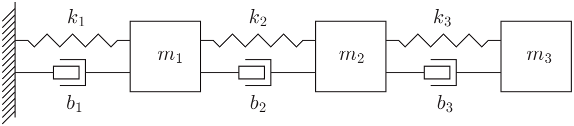
- Then for $\dot{x}(t) = Ax(t) + B(t)$, $x(0) = 0$, we have $x(T_f) = x_d$.

This tells us a specific $u(\cdot)$ where

$$u(\cdot) \mapsto x(T_f) = x_d$$

Numerical Example



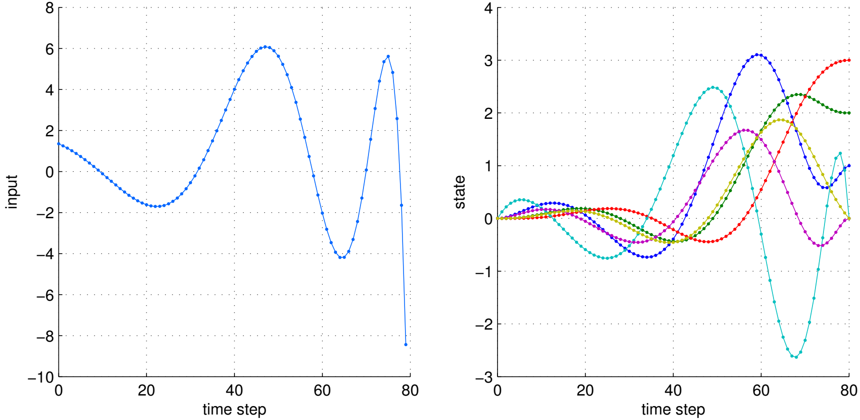
masses $m_i = 1$, spring constants $k = 1$, damping constants $b = 0.8$

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -1.6 & 0.8 & 0 \\ 1 & -2 & 1 & 0.8 & -1.6 & 0.8 \\ 0 & 1 & -1 & 0 & 0.8 & -0.8 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$u(t)$ is force applied to mass 1

$x_{\text{des}} = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \end{bmatrix}^T$ at time step $T = 80$.

Numerical Example



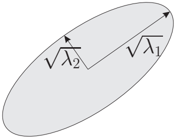
sampling period $h = 0.1$, optimal input achieves desired state

$$x(80) = x_{\text{des}} = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \end{bmatrix}^T$$

The Controllability Gramian and Ellipsoid

Minimum Energy Ellipsoid: The set of states reachable in time T with an input u of size $\|u\|_{L_2}^2 = \int_0^T \|u(t)\|^2 dt = 1$ is

$$\{x \in \mathbb{R}^n \; : \; x^T W_T^{-1} x \leq 1\}$$



- Ellipsoid with semiaxis lengths $\lambda_i(W_T)$
- Ellipsoid with semiaxis directions given by eigenvectors of W_T

Extend this to infinite time:

Definition 13.

The **Controllability Gramian** of pair (A, B) is

$$W := \int_0^\infty e^{As} B B^T e^{A^T s} ds$$

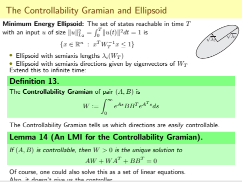
The Controllability Gramian tells us which directions are *easily* controllable.

Lemma 14 (An LMI for the Controllability Gramian).

If (A, B) is controllable, then $W > 0$ is the unique solution to

$$AW + WA^T + BB^T = 0$$

Of course, one could also solve this as a set of linear equations.



- To get from $x(0) = 0$ to $x(T_f) = x_d$ we may use $u(t) = B^T e^{A^T (T_f - t)} W_{T_f}^{-1} x_d$
- We can see that the input u which achieves x_d has magnitude

$$\begin{aligned}
\|u\|_{L_2}^2 = \int_0^t \|u(s)\|^2 ds &= \int_0^t x_d^T W_{T_f}^{-1} e^{A(T_f - s)} BB^T e^{A^T (T_f - s)} W_{T_f}^{-1} x_d ds \\
&= x_d^T W_{T_f}^{-1} \left(\int_0^t e^{A(T_f - s)} BB^T e^{A^T (T_f - s)} ds \right) W_{T_f}^{-1} x_d \\
&= x_d^T W_{T_f}^{-1} W_{T_f} W_{T_f}^{-1} x_d = x_d^T W_{T_f}^{-1} x_d
\end{aligned}$$

  Hence if $x^T W_t^{-1} x \le 1$, x is reachable at time t with input of size $\|u\| \le 1$.
- The u defined in this way is actually *optimal*.
- Eigenvalues and SVD are the same here: $W_t = U \Sigma U^T$ so $W_t^{-1} = U^T \Sigma^{-1} U$. Hence if x_d is a unit eigenvector/singular vector v_i with eigenvalue/singular value σ_i, $\|u\|_{L_2} = \sqrt{\sigma_i^{-1}}$.

Stabilizability

Stabilizability is weaker than controllability

Definition 15.

The pair (A, B) is stabilizable if for any $x(0) = x_0$, there exists a $u(t)$ such that $x(t) = \Gamma_t u$ satisfies

$$\lim_{t\to\infty} x(t) = 0$$

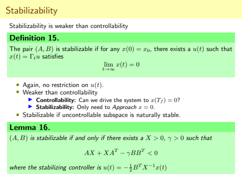
- Again, no restriction on $u(t)$.
- Weaker than controllability
  - **Controllability:** Can we drive the system to $x(T_f) = 0$?
  - **Stabilizability:** Only need to *Approach* $x = 0$.
- Stabilizable if uncontrollable subspace is naturally stable.

Lemma 16.

(A, B) is stabilizable if and only if there exists a $X > 0$, $\gamma > 0$ such that

$$AX + XA^T - \gamma BB^T < 0$$

where the stabilizing controller is $u(t) = -\frac{1}{2}B^T X^{-1} x(t)$



- Note that feasibility of the stabilizability LMI does NOT require A to be stable.

  $$AX + XA^T < \gamma BB^T$$

  means that $AX + XA^T < 0$ only for those x in the perp of the image space of B.
- The stabilizing controller is a feedback gain!

For this LMI, the solution used in the proof is

$$X = \hat{W}_t = \gamma \int_0^t e^{-As} BB^T e^{-A^T s} ds$$

The reverse-time controllability grammian...

Recall our result on Reachability:

- To reach $x(T_f) = z_f$
  - $u(t) = B^T e^{A(T_f - t)} W_T^{-1} z_f$
  - This controller is open-loop
- It assumes perfect knowledge of system and state.

Problems

- Prone to Errors, Disturbances, Errors in the Model

Solution

- Use continuous measurements of state to generate control

Static Full-State Feedback Assumes:

- We can directly and continuously measure the state $x(t)$
- Controller is a static linear function of the measurement

$$u(t) = Kx(t), \qquad K \in \mathbb{R}^{m \times n}$$

Eigenvalue Assignment

Static Full-State Feedback

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Previously the problem was *Analysis*:

- Determine properties of the maps $x_0 \mapsto x(\cdot)$ and $u(\cdot) \mapsto x(t)$

Now the problem becomes *Synthesis*:

- Alter the dynamics
- Change the matrix, A, so that it has desired properties

Synthesis is always a two-part, non-convex (typically bilinear) problem:

- Modify $A \mapsto A + BK$
- Ensure $A + BK$ has desired properties

These must be done *simultaneously*!

State Equations: $u(t) = Kx(t)$

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ &= Ax(t) + BKx(t)\\ &= (A + BK)x(t)\end{aligned}$$

Stabilization: Find a matrix $K \in \mathbb{R}^{m\times n}$ such that

$$A + BK$$

is Hurwitz.

Eigenvalue Assignment: Given $\{\lambda_1, \cdots, \lambda_n\}$, find $K \in \mathbb{R}^{m\times n}$ such that

$$\lambda_i \in \text{eig}(A + BK) \qquad \text{for } i = 1, \cdots, n.$$

Note: A solution to the eigenvalue assignment problem can also solve the stabilization problem.

Question: Is eigenvalue assignment actually harder?

Theorem 17.

Suppose $B \in \mathbb{R}^{n\times 1}$. Eigenvalues of $A + BK$ are freely assignable if and only if (A, B) is controllable.

Use `place` to assign eigenvalues.
But this is a course on LMIs, so we take a different approach.

The Static State-Feedback Problem

Lets start with the problem of stabilization.

Definition 18.

The **Static State-Feedback Problem** is to find a feedback matrix K such that

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$u(t) = Kx(t)$$

is stable

- Find K such that $A + BK$ is Hurwitz.

Can also be put in LMI format:

$$\textbf{Find } X > 0,\ K\ :$$
$$X(A + BK) + (A + BK)^T X < 0$$

Problem: Bilinear in K and X.

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State-feedback refers to the fact that $u(t) = Kx(t)$ is a function of all the states, which we assume are all individually measurable. Static refers to the fact that the linear function Kx does not vary in time.

- Resolving this bilinearity is a quintessential step in the controller synthesis process.
- Carries over throughout the course in various generalizations
- The resolution is quite simple and elegant.

An Equivalent LMI for Static State-Feedback

- The bilinear problem in K and X is a common paradigm.
- Bilinear optimization is not convex.
- To convexify the problem, we use a change of variables.

Problem 1:

$$\textbf{Find } X > 0, K \ :$$
$$X(A+BK) + (A+BK)^T X < 0$$

Problem 2:

$$\textbf{Find } P > 0, Z \ :$$
$$AP + BZ + PA^T + Z^T B^T < 0$$

Definition 19.

Two optimization problems are equivalent if a solution to one will provide a solution to the other.

Theorem 20.

Problem 1 is equivalent to Problem 2.

The Dual Lyapunov LMI

Problem 1:

Find $X > 0, :$

$XA + A^T X < 0$

Problem 2:

Find $Y > 0, :$

$YA^T + AY < 0$

Lemma 21.

Problem 1 is equivalent to problem 2.

Proof.

First we show 1) solves 2). Suppose $X > 0$ is a solution to Problem 1. Let $Y = X^{-1} > 0$.

- If $XA + A^T X < 0$, then

$$X^{-1}(XA + A^T X)X^{-1} < 0$$

- Hence

$$X^{-1}(XA + A^T X)X^{-1} = AX^{-1} + X^{-1}A^T = AY + YA^T < 0$$

- Therefore, Problem 2 is feasible with solution $Y = X^{-1}$.

The Dual Lyapunov LMI

Problem 1:

Find $X > 0,$:

$XA + A^T X < 0$

Problem 2:

Find $Y > 0,$:

$YA^T + AY < 0$

Proof.

Now we show 2) solves 1) in a similar manner. Suppose $Y > 0$ is a solution to Problem 1. Let $X = Y^{-1} > 0$.

- Then

$$\begin{aligned} XA + A^T X &= X(AX^{-1} + X^{-1}A^T)X \\ &= X(AY + YA^T)X < 0 \end{aligned}$$

□

Conclusion: If $V(x) = x^T P x$ proves stability of $\dot{x} = Ax$,

- Then $V(x) = x^T P^{-1} x$ proves stability of $\dot{x} = A^T x$.

A Stabilization LMI using the Variable Substitution Trick

Thus we rephrase Problem 1

Problem 1:

Find $P > 0, K$:

$(A + BK)P + P(A + BK)^T < 0$

Problem 2:

Find $X > 0, Z$:

$AX + BZ + XA^T + Z^T B^T < 0$

Theorem 22.

Problem 1 is equivalent to Problem 2.

Proof.

We will show that 2) Solves 1). Suppose $X > 0$, Z solves 2). Let $P = X > 0$ and $K = ZP^{-1}$. Then

$$\begin{aligned}(A + BK)P + P(A + BK)^T &= AP + PA^T + BKP + PK^T B^T \\ &= AP + PA^T + BZ + Z^T B^T < 0\end{aligned}$$

Now suppose that $P > 0$ and K solve 1). Let $X = P > 0$ and $Z = KP$. Then

$$AP + PA^T + BZ + Z^T B^T = (A + BK)P + P(A + BK)^T < 0$$

The result can be summarized more succinctly

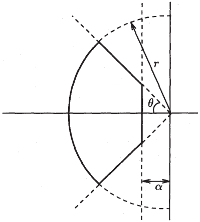
Theorem 23.

(A, B) *is static-state-feedback stabilizable if and only if there exists some* $P > 0$ *and* Z *such that*

$$AP + PA^T + BZ + Z^T B^T < 0$$

with $u(t) = ZP^{-1}x(t)$.

Controllers for D-stability



Recall the LMI for D-stability where we add in the controller $u = Kx$.

Theorem 24.

The pole locations, $z \in \mathbb{C}$ of $A + BK$ satisfy $|z| \le r$, $\operatorname{Re} x \le -\alpha$ and $z + z^* \le -c|z - z^*|$ if and only if there exists some $P > 0$ such that

$$\begin{bmatrix} -rP & (A+BK)P \\ ((A+BK)P)^T & -rP \end{bmatrix} < 0,$$

$$(A+BK)P + ((A+BK)P)^T + 2\alpha P < 0, \quad \textit{and}$$

$$\begin{bmatrix} (A+BK)P + ((A+BK)P)^T & c((A+BK)P - ((A+BK)P)^T) \\ c(((A+BK)P)^T - (A+BK)P) & (A+BK)P + ((A+BK)P)^T \end{bmatrix} < 0$$

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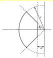

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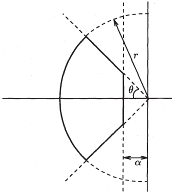
Note the D-stability LMI already appears in "Dual" Form

- LMIs are particularly useful in that they allow one to directly and sequentially impose constraints on the variables by combining different LMI constraints into a single LMI.
- So we can add closed-loop eigenvalue constraints.
- Or robustness constraints.
- However, this is limited by the variable substitution process $Z = KQ$ and $P = Q^{-1}$.
- Old variables K, P must not appear anywhere in the LMI.



Then we have an LMI which gives us a controller for D-stabilization

Lemma 25 (An LMI for D-Stabilization).

Suppose there exists $X > 0$ *and* Z *such that*

$$\begin{bmatrix} -rP & AP + BZ \\ (AP + BZ)^T & -rP \end{bmatrix} < 0,$$

$$AP + BZ + (AP + BZ)^T + 2\alpha P < 0, \quad \textit{and}$$

$$\begin{bmatrix} AP + BZ + (AP + BZ)^T & c(AP + BZ - (AP + BZ)^T) \\ c((AP + BZ)^T - (AP + BZ)) & AP + BZ + (AP + BZ)^T \end{bmatrix} < 0$$

Then if $K = ZP^{-1}$, *the pole locations,* $z \in \mathbb{C}$ *of* $A + BK$ *satisfy* $|x| \leq r$, $\operatorname{Re} x \leq -\alpha$ *and* $z + z^* \leq -c|z - z^*|$.

The Discrete-Time Case

Now consider the discrete-time system:

$$x_{k+1} = Ax_k + Bu_k$$

For discrete-time, controllability and reachability are not equivalent!

- Consider $x_{k+1} = 0$. Controllable, but not reachable.
- Lets ignore these pathological cases

Definition 26.

The Discrete-Time **Controllability Gramian** of pair (A, B) is

$$W := \sum_0^\infty A^k B B^T (A^T)^k$$

Lemma 27 (An LMI for the Controllability Gramian).

If (A, B) is controllable, then $W > 0$ is the unique solution to

$$A^T W A - W = -BB^T$$

The Discrete-Time Stabilization Problem

Again, we seek a feedback controller $u_k = Kx_k$ for which the closed-loop is Schur.
State Equations: $u_k = Kx_k$

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ &= Ax_k + BKx_k \\ &= (A + BK)x_k \end{aligned}$$

Stabilization: Find a matrix $F \in \mathbb{R}^{m \times n}$ such that

$$A + BK$$

is Schur. Recall that $A + BK$ is Schur if and only if there exists a $P > 0$ such that $(A + BK)^T P(A + BK) - P < 0$. Hence the following non-LMI problem

Find $P > 0,\ K$:
$(A + BK)^T P(A + BK) - P < 0$

The Discrete-Time Stabilization Problem

Now consider the Schur Stability condition:

$$(A+BK)^T P(A+BK) - P < 0$$

Pre- and Post-multiplying by P^{-1} shows this matrix inequality is equivalent to

$$P^{-1} - P^{-1}(A+BK)^T P(A+BK)P^{-1} > 0$$

Applying the Schur Complement, this matrix inequality is equivalent to

$$\begin{bmatrix} P^{-1} & (A+BK)P^{-1} \\ P^{-1}(A+BK)^T & P^{-1} \end{bmatrix} > 0$$

The Discrete-Time Stabilization Problem

We now have the following two equivalent problems:

Problem 1:
Find $P > 0$, K such that

$$P - (A + BK)^T P(A + BK) > 0$$

Problem 2:
Find $X > 0$, K such that

$$\begin{bmatrix} X & (A+BK)X \\ X(A+BK)^T & X \end{bmatrix} > 0$$

Taking Problem 2 and using the change of variables $Z = KX$, we get an LMI:

Lemma 28.

Suppose there exists some $X > 0$ and Z such that

$$\begin{bmatrix} X & AX + BZ \\ (AX+BZ)^T & X \end{bmatrix} > 0$$

then if $K = ZX^{-1}$, the closed-loop system matrix $(A + BK)$ is Schur.

The Discrete-Time Stabilization Problem

Final Note: An Alternative LMI condition for stabilizability is as follows

Lemma 29.

The pair (A, B) is discrete-time stabilizable if and only if there there exists some $P > 0$ such that

$$APA^T - P < BB^T.$$