LMI Methods in Optimal and Robust Control

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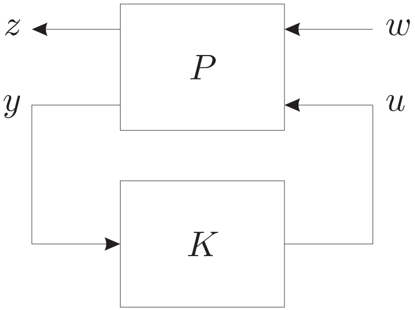
Lecture 09: An LMI for H_∞-Optimal Full-State Feedback Control

Recall: Linear Fractional Transformation



Plant:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad \text{where} \qquad P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Controller:

$$u = Ky \qquad \text{where} \qquad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

Optimal Control

Choose K to minimize

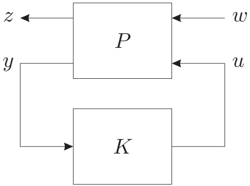
$$\|P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}\|$$

Equivalently choose $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ to minimize

$$\left\| \left[\begin{array}{c|c} \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} & \begin{matrix} B_1 + B_2 D_K Q D_{21} \\ B_K Q D_{21} \end{matrix} \\ \hline \begin{bmatrix} C_1 & 0 \end{bmatrix} + \begin{bmatrix} D_{12} & 0 \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} & D_{11} + D_{12} D_K Q D_{21} \end{array}\right] \right\|_{H_\infty}$$

where $Q = (I - D_{22}D_K)^{-1}$.

Optimal Full-State Feedback Control



For the full-state feedback case, we consider a controller of the form

$$u(t) = Fx(t)$$

Controller:

$$u = Ky \qquad \text{where} \qquad K = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array}\right]$$

Plant:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad \text{where} \qquad P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array}\right]$$

Optimal Full-State Feedback Control

Thus the closed-loop state-space representation is

$$\underline{S}(\hat{P},\hat{K}) = \left[\begin{array}{c|c} A + B_2F & B_1 \\ \hline C_1 + D_{12}F & D_{11} \end{array}\right]$$

By the KYP lemma, $\|\underline{S}(\hat{P},\hat{K})\|_{H_\infty} < \gamma$ if and only if there exists some $X > 0$ such that

$$\begin{bmatrix} (A+B_2F)^T X + X(A+B_2F) & XB_1 \\ B_1^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} (C_1+D_{12}F)^T \\ D_{11}^T \end{bmatrix}\begin{bmatrix} (C_1+D_{12}F) & D_{11} \end{bmatrix} < 0$$

This is a matrix inequality, but is nonlinear

- Quadratic (Not Bilinear)
- May NOT apply variable substitution trick.

Optimal Full-State Feedback Control

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$$\begin{bmatrix} (A+B_2F)^TX + X(A+B_2F) & XB_1 \\ B_1^TX & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} (C_1+D_{12}F)^T \\ D_{11}^T \end{bmatrix}\begin{bmatrix} (C_1+D_{12}F) & D_{11} \end{bmatrix} < 0$$

This is a matrix inequality, but is nonlinear

- Quadratic (Not Bilinear)
- May NOT apply variable substitution trick.

Recall the KYP Lemma

Lemma 1.

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- $\|\hat{G}\|_{H_\infty} \le \gamma$.
- *There exists a* $X>0$ *such that*

$$\begin{bmatrix} A^TX + XA & XB \\ B^TX & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C^T \\ D^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} < 0$$

Schur Complement

The KYP condition is

$$\begin{bmatrix} A^TX + XA & XB \\ B^TX & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

Recall the Schur Complement

Theorem 2 (Schur Complement).

For any $S \in \mathbb{S}^n$, $Q \in \mathbb{S}^m$ and $R \in \mathbb{R}^{n\times m}$, the following are equivalent.

1. $\begin{bmatrix} M & R \\ R^T & Q \end{bmatrix} < 0$
2. $Q < 0$ *and* $M - RQ^{-1}R^T < 0$

In this case, let $Q = -\frac{1}{\gamma} I < 0$,

$$M = \begin{bmatrix} A^TX + XA & XB \\ B^TX & -\gamma I \end{bmatrix} \qquad R = \begin{bmatrix} C & D \end{bmatrix}^T$$

Note we are making the LMI Larger.

Schur Complement

The Schur Complement says that

$$\begin{bmatrix} A^TX + XA & XB \\ B^TX & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C^T \\ D^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} < 0$$

if and only if

$$\begin{bmatrix} A^TX + XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

This leads to the

Full-State Feedback Condition

$$\begin{bmatrix} (A + B_2F)^TX + X(A + B_2F) & XB_1 & (C_1 + D_{12}F)^T \\ B_1^TX & -\gamma I & D_{11}^T \\ (C_1 + D_{12}F) & D_{11} & -\gamma I \end{bmatrix} < 0$$

which is now bilinear in X and F.

Schur Complement

The Schur Complement says that

$$\begin{bmatrix} A^TX+XA & XB \\ B^TX & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C^T \\ D^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} < 0$$

if and only if

$$\begin{bmatrix} A^TX+XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

This leads to the
Full-State Feedback Condition

$$\begin{bmatrix} (A+B_2F)^TX+X(A+B_2F) & XB_1 & (C_1+D_{12}F)^T \\ B_1^TX & -\gamma I & D_{11}^T \\ (C_1+D_{12}F) & D_{11} & -\gamma I \end{bmatrix} < 0$$

which is now bilinear in X and F.

Statement of the Dilated KYP Lemma

Lemma 3.

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- $\|\hat{G}\|_{H_\infty} \leq \gamma$.
- *There exists a* $X > 0$ *such that*

$$\begin{bmatrix} A^TX+XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

Dual KYP Lemma

To apply the variable substitution trick, we must also construct the dual form of this LMI.

Lemma 4 (KYP Dual).

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- $\|G\|_{H_\infty} \leq \gamma$.
- *There exists a* $Y > 0$ *such that*

$$\begin{bmatrix} YA^T + AY & B & YC^T \\ B^T & -\gamma I & D^T \\ CY & D & -\gamma I \end{bmatrix} < 0$$

Dual KYP Lemma

Proof.

Let $X = Y^{-1}$. Then

$$\begin{bmatrix} YA^T + AY & B & YC^T \\ B^T & -\gamma I & D^T \\ CY & D & -\gamma I \end{bmatrix} < 0 \qquad \text{and} \qquad Y > 0$$

if and only if $X > 0$ and

$$\begin{bmatrix} Y^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} YA^T + AY & B & YC^T \\ B^T & -\gamma I & D^T \\ CY & D & -\gamma I \end{bmatrix} \begin{bmatrix} Y^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} A^T X + XA & XB & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0.$$

By the Schur complement this is equivalent to

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

By the KYP lemma, this is equivalent to $\|G\|_{H_\infty} \leq \gamma$. □

Full-State Feedback Optimal Control

We can now apply this result to the state-feedback problem.

Theorem 5.

The following are equivalent:

- *There exists an F such that*

$$\left\| \underline{S}\left(\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array}\right] \right) \right\|_{H_\infty} \le \gamma.$$

- *There exist $Y > 0$ and Z such that*

$$\begin{bmatrix} YA^T + AY + Z^T B_2^T + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0$$

Then $F = ZY^{-1}$.

Proof.

Suppose there exists an F such that $\|\underline{S}(P, K(0,0,0,F))\|_{H_\infty} \le \gamma$. By the Dual KYP lemma, this implies there exists a $Y > 0$ such that

$$\begin{bmatrix} Y(A+B_2F)^T + (A+B_2F)Y & B_1 & Y(C_1+D_{12}F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1+D_{12}F)Y & D_{11} & -\gamma I \end{bmatrix} < 0.$$

Let $Z = FY$. Then

$$\begin{bmatrix} YA^T + Z^TB_2^T + AY + B_2Z & B_1 & YC_1^T + Z^TD_{12}^T)^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1Y + D_{12}Z & D_{11} & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} YA^T + YF^TB_2^T + AY + B_2FY & B_1 & YC_1^T + YF^TD_{12}^T)^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1Y + D_{12}FY & D_{11} & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} Y(A+B_2F)^T + (A+B_2F)Y & B_1 & Y(C_1+D_{12}F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1+D_{12}F)Y & D_{11} & -\gamma I \end{bmatrix} < 0.$$

Full-State Feedback Optimal Control

Proof.

Suppose there exists an F such that $\|\underline{\mathsf{S}}(P, K(0,0,0,F))\|_{H_\infty} \le \gamma$. By the Dual KYP lemma, this implies there exists a $Y > 0$ such that

$$\begin{bmatrix} Y(A+B_2F)^T + (A+B_2F)Y & B_1 & Y(C_1+D_{12}F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1+D_{12}F)Y & D_{11} & -\gamma I \end{bmatrix} < 0.$$

Let $Z = FY$. Then

$$\begin{bmatrix} YA^T + Z^TB_2^T + AY + B_2Z & B_1 & YC_1^T + Z^TD_{12}^T)^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1Y + D_{12}Z & D_{11} & -\gamma I \end{bmatrix}$$
$$= \begin{bmatrix} YA^T + YF^TB_2^T + AY + B_2FY & B_1 & YC_1^T + YF^TD_{12}^T)^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1Y + D_{12}FY & D_{11} & -\gamma I \end{bmatrix}$$
$$= \begin{bmatrix} Y(A+B_2F)^T + (A+B_2F)Y & B_1 & Y(C_1+D_{12}F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1+D_{12}F)Y & D_{11} & -\gamma I \end{bmatrix} < 0.$$

□

For convenience, we use

$$\underline{\mathsf{S}}(P, K(0,0,0,F)) = \underline{\mathsf{S}}\left(\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array}\right] \right)$$

Proof.

Now suppose there exists a $Y > 0$ and Z such that

$$\begin{bmatrix} YA^T + Z^T B_2^T + AY + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0.$$

Let $F = ZY^{-1}$. Then

$$\begin{bmatrix} Y(A + B_2 F)^T + (A + B_2 F)Y & B_1 & Y(C_1 + D_{12} F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1 + D_{12} F)Y & D_{11} & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} YA^T + YF^T B_2^T + AY + B_2 FY & B_1 & YC_1^T + YF^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} FY & D_{11} & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} YA^T + Z^T B_2^T + AY + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0.$$

Full-State Feedback Optimal Control

Therefore the following optimization problems are equivalent

Form A

$$\min_F \left\| \underline{S} \left(\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array} \right], \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array} \right] \right) \right\|_{H_\infty}$$

Form B

$$\min_{\gamma, Y, Z} \gamma :$$

$$\begin{bmatrix} -Y & 0 & 0 & 0 \\ 0 & YA^T + AY + Z^T B_2^T + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ 0 & B_1^T & -\gamma I & D_{11}^T \\ 0 & C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0$$

The optimal controller is given by $F = ZY^{-1}$.

Optimal Estimation

Objective: Design

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \hat{z}(t) \end{bmatrix} = \left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right] \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Initially, we want $\hat{z}(t) \to x(t)$ where $x(t)$ is the state of the plant

- The exogenous input is the disturbances, w
- The controlled input is the estimate of the state, $\hat{z}(t)$.
- The observed output, obviously, is the output from the plant, $y(t)$
- The regulated output is the error, $e(t) = \hat{x}(t) - x(t)$.

We thus obtain the 9-matrix representation of the optimal control problem given by

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \left[\begin{array}{c||c|c} A & B & 0 \\ \hline\hline -I & 0 & I \\ \hline C & D & 0 \end{array}\right] \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix}.$$

Optimal Filtering

Objective: Design

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \hat{z}(t) \end{bmatrix} = \left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right] \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

This time, however, we want $\hat{z}(t) \to z(t)$

- The exogenous input is the disturbances, w
- The controlled input is the estimate of $z(t)$ – i.e. $\hat{z}(t)$.
- The observed output is still the output from the plant, $y(t)$
- The regulated output is the error, $e_z(t) = \hat{z}(t) - z(t)$.

The main difference here is that the estimator is trying to reject the effect of the disturbance, w, on the regulated output. However, in this case, w is not sensor noise since it affects the regulated output.

The 9-matrix representation of the optimal control problem given by

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \left[\begin{array}{c||c|c} A & B & 0 \\ \hline\hline -C_1 & D_{11} & I \\ \hline C & D & 0 \end{array}\right] \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix}.$$

The filtering problem is harder than the estimation problem in that we cannot assume Luenberger structure in all cases.

Luenberger Observer Structure

For the estimation problem, we may assume

$$\dot{\hat{x}}(t) = A\hat{x}(t) + L(C\hat{x}(t) - y(t))$$

Which yields the optimization problem

$$\min_{L \in \mathbb{R}^{n_x \times n_y}} \left\| \underline{\mathsf{S}} \left(\left[\begin{array}{c||c|c} A & B & 0 \\ \hline\hline -C_1 & -D_{11} & I \\ \hline C & D & 0 \end{array} \right], \left[\begin{array}{c|c} A + LC & -L \\ \hline I & 0 \end{array} \right] \right) \right\|_{H_\infty}.$$

Now, applying the LFT in Subsec. **??**, we have the closed-loop dynamics are given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \\ z(t) \end{bmatrix} = \left[\begin{array}{cc|c} A & 0 & B \\ -LC & A + LC & -LD \\ \hline -C_1 & I & D_{11} \end{array} \right] \begin{bmatrix} x(t) \\ \hat{x}(t) \\ w(t) \end{bmatrix},$$

Duality and Optimal Observers

Actually, the optimal observer problem can be reduced to the optimal state-feedback problem by noticing that

$$\underline{\mathsf{S}}\left(\left[\begin{array}{c||c|c} A & B & 0 \\ \hline\hline -C_1 & -D_{11} & C_1 \\ \hline C & D & 0 \end{array}\right], \left[\begin{array}{c|c} A+LC & -L \\ \hline I & 0 \end{array}\right]\right) = \left[\begin{array}{c|c} A+LC & -(B+LD) \\ \hline C_1 & -D_{11} \end{array}\right]$$

$$= \left[\begin{array}{c|c} A^T + C^T L^T & C_1^T \\ \hline -(B^T + D^T L^T) & -D_{11}^T \end{array}\right]^T = \underline{\mathsf{S}}\left(\left[\begin{array}{c|cc} A^T & C_1^T & C^T \\ \hline -B^T & -D_{11}^T & -D^T \\ I & 0 & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & L^T \end{array}\right]\right)^T$$

and hence

$$\left\|\underline{\mathsf{S}}\left(\left[\begin{array}{c||c|c} A & B & 0 \\ \hline\hline -C_1 & D_{11} & I \\ \hline C & D & 0 \end{array}\right], \left[\begin{array}{c|c} A+LC & -L \\ \hline I & 0 \end{array}\right]\right)\right\|_{H_\infty} = \left\|\underline{\mathsf{S}}\left(\left[\begin{array}{c|cc} A^T & C_1^T & C^T \\ \hline -B^T & -D_{11}^T & -D^T \\ I & 0 & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & L^T \end{array}\right]\right)\right\|_{H_\infty}$$

So, just solve the optimal state-feedback problem and use $L = K^T$.