LMI Methods in Optimal and Robust Control

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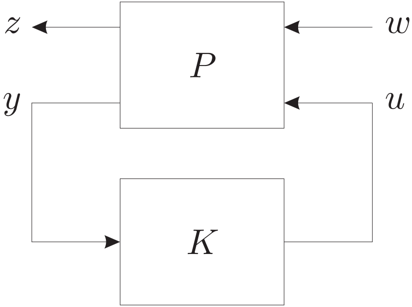
Lecture 10: An LMI for H_∞-Optimal Output Feedback Control

Optimal Output Feedback

Recall: Linear Fractional Transformation



Plant:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad \text{where} \qquad P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Controller:

$$u = Ky \qquad \text{where} \qquad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

Defining the System Variables

Choose K to minimize

$$\|P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}\|$$

Equivalently choose $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ to minimize

$$\left\| \left[\begin{array}{c|c} \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} & \begin{matrix} B_1 + B_2 D_K Q D_{21} \\ B_K Q D_{21} \end{matrix} \\ \hline \begin{bmatrix} C_1 & 0 \end{bmatrix} + \begin{bmatrix} D_{12} & 0 \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} & D_{11} + D_{12} D_K Q D_{21} \end{array}\right] \right\|_{H_\infty}$$

where $Q = (I - D_{22}D_K)^{-1}$.

Representing the Closed-Loop System

Recall the Matrix Inversion Lemma:

Lemma 1.

$$
\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}
= \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}
= \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix}
$$

Closed Loop System is Nonlinear Function of A_K, B_K, C_K, D_K.

Recall that

$$\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} I + D_K Q D_{22} & D_K Q \\ Q D_{22} & Q \end{bmatrix}$$

where $Q = (I - D_{22} D_K)^{-1}$. Then

$$\begin{aligned} A_{cl} &:= \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I + D_K Q D_{22} & D_K Q \\ Q D_{22} & Q \end{bmatrix} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A + B_2 D_K Q C_2 & B_2 (I + D_K Q D_{22}) C_K \\ B_K Q C_2 & A_K + B_K Q D_{22} C_K \end{bmatrix} \end{aligned}$$

Likewise

$$\begin{aligned} C_{cl} &:= \begin{bmatrix} C_1 & 0 \end{bmatrix} + \begin{bmatrix} D_{12} & 0 \end{bmatrix} \begin{bmatrix} I + D_K Q D_{22} & D_K Q \\ Q D_{22} & Q \end{bmatrix} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} C_1 + D_{12} D_K Q C_2 & D_{12} (I + D_K Q D_{22}) C_K \end{bmatrix} \end{aligned}$$

A New Set of Decision Variables

Thus we have

$$\left[\begin{array}{cc|c} A + B_2 D_K Q C_2 & B_2(I + D_K Q D_{22})C_K & B_1 + B_2 D_K Q D_{21} \\ B_K Q C_2 & A_K + B_K Q D_{22} C_K & B_K Q D_{21} \\ \hline \left[C_1 + D_{12} D_K Q C_2 \right. & \left. D_{12}(I + D_K Q D_{22})C_K\right] & D_{11} + D_{12} D_K Q D_{21} \end{array}\right]$$

where $Q = (I - D_{22} D_K)^{-1}$.

- This is nonlinear in (A_K, B_K, C_K, D_K).
- Hence we make a change of variables (First of several).

$$\begin{aligned} A_{K2} &= A_K + B_K Q D_{22} C_K \\ B_{K2} &= B_K Q \\ C_{K2} &= (I + D_K Q D_{22})C_K \\ D_{K2} &= D_K Q \end{aligned}$$

A New parametrization of the closed-loop system.

This yields the system

$$\left[\begin{array}{c|c} \begin{bmatrix} A+B_2D_{K2}C_2 & B_2C_{K2} \\ B_{K2}C_2 & A_{K2} \end{bmatrix} & \begin{matrix} B_1+B_2D_{K2}D_{21} \\ B_{K2}D_{21} \end{matrix} \\ \hline \begin{bmatrix} C_1+D_{12}D_{K2}C_2 & D_{12}C_{K2} \end{bmatrix} & D_{11}+D_{12}D_{K2}D_{21} \end{array}\right]$$

Which is affine in $\left[\begin{array}{c|c} A_{K2} & B_{K2} \\ \hline C_{K2} & D_{K2} \end{array}\right]$.

Inverting the Variable Substitution

Recovering D_K

Hence we can optimize over our new variables.

- System is linear in new variables and eliminates original variables.
- However, the change of variables must be **invertible**.

Now suppose we have D_{K2}. Then

$$D_{K2} = D_K Q = D_K (I - D_{22} D_K)^{-1}$$

implies that

$$D_K = D_{K2}(I - D_{22} D_K) = D_{K2} - D_{K2} D_{22} D_K$$

or

$$(I + D_{K2} D_{22}) D_K = D_{K2}$$

which can be inverted to get

$$D_K = (I + D_{K2} D_{22})^{-1} D_{K2}$$

Inverting the Variable Substitution

Recovering C_K

If we recall that

$$(I - QM)^{-1} = I + Q(I - MQ)^{-1}M$$

then we get

$$I + D_K Q D_{22} = I + D_K(I - D_{22}D_K)^{-1}D_{22} = (I - D_K D_{22})^{-1}$$

Examine the variable C_{K2}

$$\begin{aligned} C_{K2} &= (I + D_K(I - D_{22}D_K)^{-1}D_{22})C_K \\ &= (I - D_K D_{22})^{-1}C_K \end{aligned}$$

Hence, given D_K and C_{K2}, we can recover C_K as

$$C_K = (I - D_K D_{22})C_{K2}$$

Inverting the Variable Substitution

Once we have C_K and D_K, the other variables are easily recovered as

$$B_K = B_{K2}Q^{-1} = B_{K2}(I - D_{22}D_K)$$
$$A_K = A_{K2} - B_K(I - D_{22}D_K)^{-1}D_{22}C_K$$

To summarize, the original variables can be recovered as

$$D_K = (I + D_{K2}D_{22})^{-1}D_{K2}$$
$$B_K = B_{K2}(I - D_{22}D_K)$$
$$C_K = (I - D_K D_{22})C_{K2}$$
$$A_K = A_{K2} - B_K(I - D_{22}D_K)^{-1}D_{22}C_K$$

Closed Loop System Parameters

$$\left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right] := \left[\begin{array}{c|c} \begin{bmatrix} A + B_2 D_{K2} C_2 & B_2 C_{K2} \\ B_{K2} C_2 & A_{K2} \end{bmatrix} & \begin{matrix} B_1 + B_2 D_{K2} D_{21} \\ B_{K2} D_{21} \end{matrix} \\ \hline \begin{bmatrix} C_1 + D_{12} D_{K2} C_2 & D_{12} C_{K2} \end{bmatrix} & D_{11} + D_{12} D_{K2} D_{21} \end{array}\right]$$

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix}$$

Or

$$A_{cl} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}$$

$$B_{cl} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}$$

$$C_{cl} = \begin{bmatrix} C_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}$$

$$D_{cl} = \begin{bmatrix} D_{11} \end{bmatrix} + \begin{bmatrix} 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}$$

Optimal Output Feedback Control

However, if we apply the KYP Lemma, the result is bilinear in X and A_K, B_K, C_K, D_K

- Dual KYP Lemma is used for Controller Synthesis
- Primal KYP Lemma is used for observer Synthesis
- For Observer-Based Controller Synthesis, we need both Primal AND Dual forms....

Lemma 2 (Transformation Lemma).

Suppose that

$$\begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix} > 0$$

Then there exist X_2, X_3, Y_2, Y_3 such that

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1} = Y^{-1} > 0$$

where $Y_h = \begin{bmatrix} Y_1 & I \\ Y_2^T & 0 \end{bmatrix}$ has full rank.

Optimal Output Feedback Control

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where $Y_h = \begin{bmatrix} Y_1 & I \\ Y_2^T & 0 \end{bmatrix}$ has full rank.

The primal variable is

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}$$

The dual variable is

$$\begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}$$

Optimal Output Feedback Control

Proof.

- Since
$$\begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix} > 0,$$
by the Schur complement $X_1 > 0$ and $X_1^{-1} - Y_1 < 0$. Since $I - X_1 Y_1 = X_1(X_1^{-1} - Y_1)$, we conclude that $I - X_1 Y_1$ is invertible.
- Choose any two square invertible matrices X_2 and Y_2 such that
$$X_2 Y_2^T = I - X_1 Y_1$$
- Because X_2 and Y_2 are invertible,
$$Y_h^T = \begin{bmatrix} Y_1 & Y_2 \\ I & 0 \end{bmatrix} \text{ and } X_h = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix}$$
are also non-singular.

□

Optimal Output Feedback Control

Proof.

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are also non-singular.

□

$$\begin{bmatrix} Y_h^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

will be the half-dual transformation.

- Y_h^T contains the top half of the dual Lyapunov variable.
- X_h contains the bottom half of the primal Lyapunov variable.

Proof.

- Now define X and Y as

$$X = Y_h^{-T} X_h \qquad \text{and} \qquad Y = X_h^{-1} Y_h^T.$$

Then

$$XY = Y_h^{-1} X_h X_h^{-1} Y_h = I$$

□

Are X_2, Y_2 actually the completion of their respective matrices and what are X_3, Y_3? If X_2, Y_2 are square, then

$$X_h^{-1} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -X_2^{-1} X_1 & X_2^{-1} \end{bmatrix} \qquad Y_h^{-T} = \begin{bmatrix} Y_1 & Y_2 \\ I & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I \\ Y_2^{-1} & -Y_2^{-1} Y_1 \end{bmatrix}.$$

So, since $X_2 Y_2^T = I - X_1 Y_1$, we have $X_3 = -Y_2^{-1} Y_1 X_2$ since

$$X = Y_h^{-T} X_h = \begin{bmatrix} X_1 & X_2 \\ Y_2^{-1} - Y_2^{-1} Y_1 X_1 & -Y_2^{-1} Y_1 X_2 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_2^T & -Y_2^{-1} Y_1 X_2 \end{bmatrix}$$

and $Y_3 = -X_2^{-1} X_1 Y_2$ since

$$Y = X_h^{-1} Y_h^T = \begin{bmatrix} Y_1 & Y_2 \\ X_2^{-1} - X_2^{-1} X_1 Y_1 & -X_2^{-1} X_1 Y_2 \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & -X_2^{-1} X_1 Y_2 \end{bmatrix}$$

Optimal Output Feedback Control

Proof.

- Now define X and Y as

$$X = Y_h^{-T} X_h \quad \text{and} \quad Y = X_h^{-1} Y_h^T.$$

Then

$$XY = Y_h^{-1} X_h X_h^{-1} Y_h = I$$

□

Are X_2, Y_2 actually the completion of their respective matrices and what are X_3, Y_3? If X_2, Y_2 are square, then

$$X_h^{-1} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -X_2^{-1}X_1 & X_2^{-1} \end{bmatrix} \qquad Y_h^{-T} = \begin{bmatrix} Y_1 & Y_2 \\ I & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I \\ Y_2^{-1} & -Y_2^{-1}Y_1 \end{bmatrix}.$$

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and $Y_3 = -X_2^{-1} X_1 Y_2$ since

$$Y = X_h^{-1} Y_h^T = \begin{bmatrix} Y_1 & Y_2 \\ X_2^{-1} - X_2^{-1}X_1Y_1 & -X_2^{-1}X_1Y_2 \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & -X_2^{-1}X_1Y_2 \end{bmatrix}$$

We will be applying the half-dual transformation

$$\begin{bmatrix} Y_h^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \overbrace{\begin{bmatrix} A^TX + XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix}}^{\text{primal KYP lemma}} \begin{bmatrix} Y_h & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} Y_h^TA^TXY_h + Y_h^TXAY_h & Y_h^TXB & Y_h^TC^T \\ B^TXY_h & -\gamma I & D^T \\ CY_h & D & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} Y_h^TA^TX_h^T + X_hAY_h & X_hB & Y_h^TC^T \\ B^TX_h^T & -\gamma I & D^T \\ CY_h & D & -\gamma I \end{bmatrix}$$

Since

$$Y_h^T = \begin{bmatrix} Y_1 & Y_2 \\ I & 0 \end{bmatrix} \text{ and } X_h = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix},$$

this will only work if Y_2 and X_2 are somehow eliminated from the expression.

Lemma 3 (Converse Transformation Lemma).

Given $X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} > 0$ *where* X_2 *has full column rank. Let*

$$X^{-1} = Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}$$

then

$$\begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix} > 0$$

and $Y_h = \begin{bmatrix} Y_1 & I \\ Y_2^T & 0 \end{bmatrix}$ *has full column rank.*

This result shows the Transformation Lemma is not conservative.

Proof.

Since X_2 is full rank, $X_h = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix}$ also has full column rank. Note that $YX = I$ implies

$$Y_h^T X = \begin{bmatrix} Y_1 & Y_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix} = X_h.$$

Hence

$$Y_h^T = \begin{bmatrix} Y_1 & Y_2 \\ I & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix} Y = X_h Y$$

has full column rank. Now, since $XY = I$ implies $X_1 Y_1 + X_2 Y_2^T = I$, we have

$$X_h Y_h = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix} \begin{bmatrix} Y_1 & I \\ Y_2^T & 0 \end{bmatrix} = \begin{bmatrix} Y_1 & I \\ X_1 Y_1 + X_2 Y_2^T & X_1 \end{bmatrix} = \begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix}$$

Furthermore, because Y_h has full rank,

$$\begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix} = X_h Y_h = X_h Y X_h^T = Y_h^T X Y_h > 0$$

Optimal Output Feedback Control

Proof.

Since X_2 is full rank, $X_h = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix}$ also has full column rank. Note that $YX = I$ implies

$$Y_h^T X = \begin{bmatrix} Y_1 & Y_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix} = X_h.$$

Hence

$$Y_h^T = \begin{bmatrix} Y_1 & Y_2 \\ I & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix} Y = X_h Y$$

has full column rank. Now, since $XY = I$ implies $X_1 Y_1 + X_2 Y_2^T = I$, we have

$$X_h Y_h = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix} \begin{bmatrix} Y_1 & I \\ Y_2^T & 0 \end{bmatrix} = \begin{bmatrix} Y_1 & I \\ X_1 Y_1 + X_2 Y_2^T & X_1 \end{bmatrix} = \begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix}$$

Furthermore, because Y_h has full rank,

$$\begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix} = X_h Y_h = X_h Y X_h^T = Y_h^T X Y_h > 0$$

□

Note the relationship:

$$\begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix} = X_h Y_h$$

Note how both X_2 and Y_2 vanish?

H_∞-optimal Dynamic Output Feedback Control

Theorem 1.

The following are equivalent.

1. *There exists a* $K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *such that* $\|\underline{S}(K,P)\|_{H_\infty} < \gamma$.
2. $\exists X, Y \in \mathbb{S}^{n_s}$ *and* $K_3 \in \mathbb{R}^{n_c+n_s \times n_m+n_s}$ *such that* $\begin{bmatrix} Y & I \\ I & X \end{bmatrix} > 0$ *and*

$$\begin{bmatrix} AY+YA^T & A & B_1 & YC_1^T \\ A^T & XA+A^TX & XB_1 & C_1^T \\ B_1^T & B_1^TX & -\gamma I & D_{11}^T \\ C_1Y & C_1 & D_{11} & -\gamma I \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & I \\ 0 & 0 \\ D_{12} & 0 \end{bmatrix} K_3 \begin{bmatrix} 0 & C_2 & D_{21} & 0 \\ I & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ C_2^T & 0 \\ D_{21}^T & 0 \\ 0 & 0 \end{bmatrix} K_3^T \begin{bmatrix} B_2^T & 0 & 0 & D_{12}^T \\ 0 & I & 0 & 0 \end{bmatrix} < 0 \qquad (1)$$

Moreover, if X, Y, K_3 *satisfy 2), then* $D_K = (I + D_{K2}D_{22})^{-1}D_{K2}$,
$B_K = B_{K2}(I - D_{22}D_K)$, $C_K = (I - D_K D_{22})C_{K2}$,
$A_K = A_{K2} - B_K(I - D_{22}D_K)^{-1}D_{22}C_K$ *where*

$$\begin{bmatrix} D_{K2} & C_{K2} \\ B_{K2} & A_{K2} \end{bmatrix} = \begin{bmatrix} I & 0 \\ XB_2 & X_2 \end{bmatrix}^{-1} \left(K_3 - \begin{bmatrix} 0 & 0 \\ 0 & XAY \end{bmatrix} \right) \begin{bmatrix} I & C_2Y \\ 0 & Y_2^T \end{bmatrix}^{-1}$$

for any full-rank X_2 *and* Y_2 *such that* $X_2Y_2^T = I - XY$.

The first step in the proof is to use the KYP lemma to show that $\|\underline{\mathsf{S}}(P,K)\|_{H_\infty} = \left\| \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right] \right\|_{H_\infty} < \gamma$ is equivalent to

$$\begin{bmatrix} Y_h^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}^T X + X A_{cl} & X B_{cl} & C_{cl}^T \\ B_{cl}^T X_h & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} \begin{bmatrix} Y_h & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} Y_h^T A_{cl}^T X_h^T + X_h A_{cl} Y_h & X_h B_{cl} & Y_h^T C_{cl}^T \\ B_{cl}^T X_h^T & -\gamma I & D_{cl}^T \\ C_{cl} Y_h & D_{cl} & -\gamma I \end{bmatrix} < 0$$

The next, and key step in the proof is

$$\begin{bmatrix} X_hA_{cl}Y_h & X_hB_{cl} & 0 \\ 0 & 0 & 0 \\ C_{cl}Y_h & D_{cl} & 0 \end{bmatrix} = \begin{bmatrix} X_h & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} Y_h & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \qquad (2)$$

$$= \begin{bmatrix} I & 0 & 0 \\ X & X_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \left(\begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & I \\ D_{12} & 0 \end{bmatrix} K_2 \begin{bmatrix} C_2 & 0 & D_{21} \\ 0 & I & 0 \end{bmatrix} \right) \begin{bmatrix} Y_1 & I & 0 & 0 \\ Y_2^T & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}$$

$$= \begin{bmatrix} AY & A & B_1 & 0 \\ XAY & XA & XB_1 & 0 \\ 0 & 0 & 0 & 0 \\ C_1Y & C_1 & D_{11} & 0 \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & I \\ 0 & 0 \\ D_{12} & 0 \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ XB_2 & X_2 \end{bmatrix} K_2 \begin{bmatrix} I & C_2Y \\ 0 & Y_2^T \end{bmatrix} \right) \begin{bmatrix} 0 & C_2 & D_{21} & 0 \\ I & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} AY & A & B_1 & 0 \\ 0 & XA & XB_1 & 0 \\ 0 & 0 & 0 & 0 \\ C_1Y & C_1 & D_{11} & 0 \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & I \\ 0 & 0 \\ D_{12} & 0 \end{bmatrix} \underbrace{\left(\begin{bmatrix} 0 & 0 \\ 0 & XAY \end{bmatrix} + \begin{bmatrix} I & 0 \\ XB_2 & X_2 \end{bmatrix} K_2 \begin{bmatrix} I & C_2Y \\ 0 & Y_2^T \end{bmatrix} \right)}_{K_3} \begin{bmatrix} 0 & \cdots \\ I & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} AY & A & B_1 & 0 \\ 0 & XA & XB_1 & 0 \\ 0 & 0 & 0 & 0 \\ C_1Y & C_1 & D_{11} & 0 \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & I \\ 0 & 0 \\ D_{12} & 0 \end{bmatrix} K_3 \begin{bmatrix} 0 & C_2 & D_{21} & 0 \\ I & 0 & 0 & 0 \end{bmatrix}.$$

Then we are done, since

$$\begin{bmatrix} Y_h^T A_{cl}^T X_h^T + X_h A_{cl} Y_h & X_h B_{cl} & Y_h^T C_{cl}^T \\ B_{cl}^T X_h^T & -\gamma I & D_{cl}^T \\ C_{cl} Y_h & D_{cl} & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} X_h A_{cl} Y_h & X_h B_{cl} & 0 \\ 0 & 0 & 0 \\ C_{cl} Y_h & D_{cl} & 0 \end{bmatrix} + \begin{bmatrix} X_h A_{cl} Y_h & X_h B_{cl} & 0 \\ 0 & 0 & 0 \\ C_{cl} Y_h & D_{cl} & 0 \end{bmatrix}^T + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\gamma I & 0 \\ 0 & 0 & -\gamma I \end{bmatrix}$$

Conclusion

The H_∞-optimal controller is a dynamic system.

- Transfer Function $\hat{K}(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$

Minimizes the effect of external input (w) on external output (z).

$$\|z\|_{L_2} \le \|\underline{\mathsf{S}}(P,K)\|_{H_\infty} \|w\|_{L_2}$$

- Minimum Energy Gain

H_2-optimal dynamic output feedback control

Theorem 2.

The following are equivalent.

1. *There exists a* $\hat{K} = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *such that* $\|S(K,P)\|_{H_2} < \gamma$.
2. *There exist* X_1, Y_1, Z, K_3 *such that*

$$\begin{bmatrix} AY+YA^T & A & B_1 \\ A^T & XA+A^TX & XB_1 \\ B_1^T & B_1^TX & -\gamma I \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} K_3 \begin{bmatrix} 0 & C_2 & D_{21} \\ I & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ C_2^T & 0 \\ D_{21}^T & 0 \end{bmatrix} K_3^T \begin{bmatrix} B_2^T & 0 & 0 \\ 0 & I & 0 \end{bmatrix} < 0,$$

$$\begin{bmatrix} Y & I & YC_1^T \\ I & X & C_1^T \\ C_1Y & C_1 & W \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ D_{12} & 0 \end{bmatrix} K_3 \begin{bmatrix} 0 & C_2 & 0 \\ I & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ C_2^T & 0 \\ 0 & 0 \end{bmatrix} K_3^T \begin{bmatrix} 0 & 0 & D_{12}^T \\ 0 & 0 & 0 \end{bmatrix} > 0,$$

$$D_{11} + \begin{bmatrix} D_{12} & 0 \end{bmatrix} K_3 \begin{bmatrix} D_{21} \\ 0 \end{bmatrix} = 0, \qquad \text{trace}(Z) < \gamma$$

Moreover, $D_K = (I + D_{K2}D_{22})^{-1}D_{K2}$, $B_K = B_{K2}(I - D_{22}D_K)$, $C_K = (I - D_KD_{22})C_{K2}$, $A_K = A_{K2} - B_K(I - D_{22}D_K)^{-1}D_{22}C_K$ *where*

$$\begin{bmatrix} D_{K2} & C_{K2} \\ B_{K2} & A_{K2} \end{bmatrix} = \begin{bmatrix} I & 0 \\ XB_2 & X_2 \end{bmatrix}^{-1} \left(K_3 - \begin{bmatrix} 0 & 0 \\ 0 & XAY \end{bmatrix} \right) \begin{bmatrix} I & C_2Y \\ 0 & Y_2^T \end{bmatrix}^{-1}$$

for any full-rank X_2 *and* Y_2 *such that* $X_2Y_2^T = I - XY$.

Theorem 3.

suppose there exist X, Y, K_3 *such that*

$$\begin{bmatrix} AY+YA^T & A & B_1 & YC_1^T \\ A^T & XA+A^TX & XB_1 & C_1^T \\ B_1^T & B_1^TX & -\gamma_1 I & D_{11}^T \\ C_1Y & C_1 & D_{11} & -\gamma_1 I \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & I \\ 0 & 0 \\ D_{12} & 0 \end{bmatrix} K_3 \begin{bmatrix} 0 & C_2 & D_{21} & 0 \\ I & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ C_2^T & 0 \\ D_{21}^T & 0 \\ 0 & 0 \end{bmatrix} K_3^T \begin{bmatrix} B_2^T & 0 & 0 & D_{12}^T \\ 0 & I & 0 & 0 \end{bmatrix} < 0$$

$$\begin{bmatrix} AY+YA^T & A & B_1 \\ A^T & XA+A^TX & XB_1 \\ B_1^T & B_1^TX & -\gamma I \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} K_3 \begin{bmatrix} 0 & C_2 & D_{21} \\ I & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ C_2^T & 0 \\ D_{21}^T & 0 \end{bmatrix} K_3^T \begin{bmatrix} B_2^T & 0 & 0 \\ 0 & I & 0 \end{bmatrix} < 0,$$

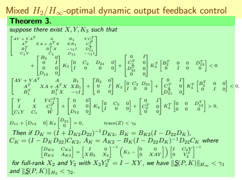
$$\begin{bmatrix} Y & I & YC_1^T \\ I & X & C_1^T \\ C_1Y & C_1 & W \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ D_{12} & 0 \end{bmatrix} K_3 \begin{bmatrix} 0 & C_2 & 0 \\ I & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ C_2^T & 0 \\ 0 & 0 \end{bmatrix} K_3^T \begin{bmatrix} 0 & 0 & D_{12}^T \\ 0 & 0 & 0 \end{bmatrix} > 0,$$

$$D_{11} + \begin{bmatrix} D_{12} & 0 \end{bmatrix} K_3 \begin{bmatrix} D_{21} \\ 0 \end{bmatrix} = 0, \qquad \operatorname{trace}(Z) < \gamma_2$$

Then if $D_K = (I + D_{K2}D_{22})^{-1}D_{K2}$, $B_K = B_{K2}(I - D_{22}D_K)$, $C_K = (I - D_K D_{22})C_{K2}$, $A_K = A_{K2} - B_K(I - D_{22}D_K)^{-1}D_{22}C_K$ *where*

$$\begin{bmatrix} D_{K2} & C_{K2} \\ B_{K2} & A_{K2} \end{bmatrix} = \begin{bmatrix} I & 0 \\ XB_2 & X_2 \end{bmatrix}^{-1} \left(K_3 - \begin{bmatrix} 0 & 0 \\ 0 & XAY \end{bmatrix} \right) \begin{bmatrix} I & C_2Y \\ 0 & Y_2^T \end{bmatrix}^{-1}$$

for full-rank X_2 *and* Y_2 *with* $X_2Y_2^T = I - XY$, *we have* $\|\underline{S}(P,K)\|_{H_\infty} < \gamma_1$ *and* $\|\underline{S}(P,K)\|_{H_2} < \gamma_2$.



The systems used for the H_∞ and H_2 norms can be different, but must use the same A, B_2, C_2, D_{22} matrices, since these appear in the variable substitution – i.e.

$$P_{H_2} = \left[\begin{array}{c||c|c} A & B_{1,H_2} & B_2 \\ \hline\hline C_{1,H_2} & D_{11,H_2} & D_{12,H_2} \\ \hline C_2 & D_{21,H_2} & D_{22} \end{array}\right]$$

and

$$P_{H_\infty} = \left[\begin{array}{c||c|c} A & B_{1,H_\infty} & B_2 \\ \hline\hline C_{1,H_\infty} & D_{11,H_\infty} & D_{12,H_\infty} \\ \hline C_2 & D_{21,H_\infty} & D_{22} \end{array}\right]$$

This allows us to select different disturbances and regulated outputs for each norm, and add filters to the H_2 norm, but not the H_∞ norm.

Example of Mixed H_∞-H_2 optimal control

Note the predicted gains in Mixed H_∞-H_2 optimal control are not *tight*.

$$A = \begin{bmatrix} 0 & 10 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & -5 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; C_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad D_{22} = 0;$$

$$B_{1,H_\infty} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C_{1,H_\infty} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad D_{11,H_\infty} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad D_{12,H_\infty} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad D_{21,H_\infty} = 1$$

$$B_{1,H_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C_{1,H_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad D_{11,H_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad D_{12,H_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad D_{21,H_2} = 1$$

bound/objective	$\|\cdot\|_{H_\infty}$	$\|\cdot\|_{H_2}$	$\|\cdot\|_{H_\infty} + \|\cdot\|_{H_2}$
Predicted $\|S(P,K)\|_{H_2}$	N/A	3.816	6.8877
Actual $\|S(P,K)\|_{H_2}$	∞	3.816	5.1625
Predicted $\|S(P,K)\|_{H_\infty}$	4.833	N/A	6.6551
Actual $\|S(P,K)\|_{H_\infty}$	4.833	5.138	6.4533