LMI Methods in Optimal and Robust Control

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Lecture 13: LMIs for Optimal Control and Quadratic Stability with Affine Polytopic and Interval Uncertainty

Types of Uncertainty

We will start with Time-Varying Parametric Uncertainty.

- **Unstructured, Dynamic, norm-bounded:**
$$\mathbf{\Delta} := \{\Delta \in \mathcal{L}(L_2) \,:\, \|\Delta\|_{H_\infty} < 1\}$$
- **Structured, Static, norm-bounded:**
$$\mathbf{\Delta} := \{\text{diag}(\delta_1, \cdots, \delta_K, \Delta_1, \cdots \Delta_N) \,:\, |\delta_i| < 1,\ \bar{\sigma}(\Delta_i) < 1\}$$
- **Structured, Dynamic, norm-bounded:**
$$\mathbf{\Delta} := \{\text{diag}(\Delta_1, \Delta_2, \cdots) \in \mathcal{L}(L_2) \,:\, \|\Delta_i\|_{H_\infty} < 1\}$$
- **Unstructured, Parametric, norm-bounded:**
$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \|\Delta\| \le 1\}$$
- **Parametric, Polytopic:**
$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \Delta = \sum_i \alpha_i H_i,\ \alpha_i \ge 0,\ \sum_i \alpha_i = 1\}$$
- **Parametric, Interval:**
$$\mathbf{\Delta} := \{\sum_i \Delta_i \delta_i \,:\, \delta_i \in [\delta_i^-, \delta_i^+]\}$$

Each of these can be Time-Varying or Time-Invariant!

Additive Affine Time-Varying Interval and Polytopic Uncertainty

Recall the system with Affine Time-Varying uncertainty (No Input).

$$\dot{x}(t) = (A_0 + \Delta A(t))x(t)$$

where

$$\Delta A(t) = A_1\delta_1(t) + \cdots + A_k\delta_k(t)$$

where $\delta(t)$ lies in either the intervals

$$\delta_i(t) \in [\delta_i^-, \delta_i^+]$$

or the simplex

$$\delta(t) \in \{\delta : \sum_i \alpha_i = 1,\ \alpha_i \geq 0\}$$

Definitions: Use Robust Stability for Static Uncertainty

Definition 1.

The system

$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$

is **Robustly Stable** over $\mathbf{\Delta}$ if $A_0 + \Delta$ is Hurwitz for all $\Delta \in \mathbf{\Delta}$.

Note that Robust Stability DOES NOT imply stability if $\Delta(t)$ is time-varying.

- It implies that for any $\Delta \in \mathbf{\Delta}$, there exists a $P(\Delta) > 0$ such that
$$(A + \Delta)^T P(\Delta) + P(\Delta)(A + \Delta) < 0 \quad \text{for all } \Delta \in \mathbf{\Delta}$$
- For a fixed Δ, this implies stability using Lyapunov function $V(x) = x^T P(\Delta)x$.
- Does not imply stability for TV Δ because if $V(x,t) = x^T P(\Delta(t))x$,

$$\begin{aligned}\frac{d}{dt}V(x(t),t) &= x(t)^T\Big((A + \Delta(t))^T P(\Delta(t)) + P(\Delta(t))(A + \Delta(t))\Big)x(t)\\ &\quad + x(t)^T\left(\frac{d}{dt}P(\Delta(t))\right)x(t)\\ &\le x(t)^T\left(\frac{d}{dt}P(\Delta(t))\right)x(t)\end{aligned}$$

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Robust Stability is necessary and sufficient for static uncertainty.

Definitions: Quadratic Stability for Dynamic Uncertainty

Definition 2.

The system

$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$

is **Quadratically Stable** over $\mathbf{\Delta}$ if there exists a $P > 0$ such that

$$(A + \Delta)^T P + P(A + \Delta) < 0 \quad \text{for all } \Delta \in \mathbf{\Delta}.$$

Quadratic Stability **Implies Stability** of trajectories for any $\Delta(t)$ with $\Delta(t) \in \mathbf{\Delta}$ for all $t \geq 0$.

- Use the Lyapunov function $V(x) = x^T P x$.

$$\frac{d}{dt}V(x(t)) = x(t)^T((A + \Delta(t))^T P + P(A + \Delta(t))x(t) < 0$$

Counterintuitive:

- Robust Stability does not imply stability!
- Stability does not imply quadratic stability!

Quadratic Stability is Conservative

Definition 3.

The system
$$\dot{x}(t) = (A_0 + \Delta A(t))x(t)$$
is **Quadratically Stable** over $\boldsymbol{\Delta}$ if there exists a $P > 0$ such that
$$(A + \Delta A)^T P + P(A + \Delta A) < 0 \quad \text{for all } \Delta A \in \boldsymbol{\Delta}.$$

Quadratic Stability is CONSERVATIVE.

- There are Stable Systems which are not Quadratically Stable

$$\dot{x} = A(t)x,$$
$$A(t) = \delta_1(t) \begin{bmatrix} -100 & 0 \\ 0 & -1 \end{bmatrix} + \delta_2(t) \begin{bmatrix} 8 & -9 \\ 120 & -18 \end{bmatrix}, \qquad \delta_i \geq 0, \quad \delta_1 + \delta_2 = 1$$

- Use $V(x) = \max\{x^T P_1 x, x^T P_2 x\}$ where

$$P_1 = \begin{bmatrix} 14 & -1 \\ -1 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

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Quadratic Stability is sometimes referred to as an “infinite-dimensional LMI”

- Meaning it represents an infinite number of LMI constraints
- One for each possible value of $\Delta \in \mathbf{\Delta}$
- Also called a parameterized LMI
- Such LMIs are not tractable.
- For polytopic sets, the LMI can be made finite.

Enforcing an LMI on the entire Polytope

Making an infinite-dimensional LMI finite dimensional

Theorem 4 (LMIs on the Polytope).

The following are equivalent for any H, L_i, R_i.

$$H + \sum_i L_i \Delta R_i > 0 \qquad \text{for all } \Delta \in Co(\Delta_1, \cdots \Delta_k) \tag{1}$$

$$H + \sum_i L_i \Delta_j R_i > 0 \qquad \text{for all } \quad j = 1, \cdots, k \tag{2}$$

Proof.

To show $1 \Rightarrow 2$, note that $\Delta_j \in Co(\Delta_1, \cdots \Delta_k)$ for each j. Next, show $2 \Rightarrow 1$.

$$\begin{aligned} H + \sum_i L_i \Delta R_i &= H + \sum_i L_i \Big(\sum_j \alpha_i \Delta_j \Big) R_i \qquad \alpha_i \geq 0, \ \sum_j \alpha_j = 1 \\ &= \sum_j \alpha_j \Big(H + \sum_i L_i \Delta_j R_i \Big) \geq 0 \end{aligned}$$

□

An LMI for Polytopic Quadratic Stability

Definition 5.

The pair $(A + \Delta, \boldsymbol{\Delta})$ is **Quadratically Stable** over $\boldsymbol{\Delta}$ if there exists a $P > 0$ such that

$$(A + \Delta)^T P + P(A + \Delta) < 0 \quad \text{for all } \Delta \in \boldsymbol{\Delta}.$$

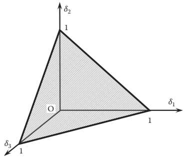
Theorem 6.

$(A + \Delta, \boldsymbol{\Delta})$ is quadratically stable over $\boldsymbol{\Delta} := Co(A_1, \cdots, A_k)$ if and only if there exists a $P > 0$ such that

$$(A + A_i)^T P + P(A + A_i) < 0 \quad \textit{for } i = 1, \cdots, k$$

The theorem says the LMI only needs to hold at the EXTREMAL POINTS or VERTICES of the polytope.

- In Fact, Quadratic Stability MUST be expressed as an LMI
- There is NO Ricatti Eqn. Equivalent.



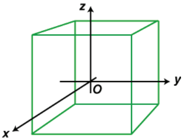
An LMI for Interval Quadratic Stability

Recall the system with Affine Time-Varying uncertainty.

$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$

where

$$\Delta(t) = A_1\delta_1(t) + \cdots + A_k\delta_k(t)$$



where $\delta_i(t) \in [-1, 1]$. Note: $\delta(t)$ lies in the *hypercube*.

Interval Stability is a Kind of Polytopic Uncertainty.

The vertices of the hypercube define the vertices of the uncertainty set

$$V := \left\{ A_0 + \sum_i A_i\delta_i,\ \delta_i \in \{-1, 1\} \right\}$$

Theorem 7 (Quadratic Stability using 2^k LMI constraints!).

$(A + \Delta, \boldsymbol{\Delta})$ *is quadratically stable over* $\boldsymbol{\Delta} := Co(V)$ *if and only if there exists a* $P > 0$ *such that*

$$\left(A_0 + \sum_i A_i\delta_i\right)^T P + P\left(A_0 + \sum_i A_i\delta_i\right) < 0 \quad \textit{for every } \delta \in \{-1, 1\}^k$$

An LMI for Quadratic Polytopic Stabilization

Controller Synthesis is a simple application of the previous theorem:

Theorem 8.

There exists a K *such that*

$$\dot{x}(t) = (A + \Delta_A + (B + \Delta_B)K)x(t)$$

is **quadratically stable** *for* $(\Delta_A, \Delta_B) \in Co((A_1, B_2), \cdots, (A_k, B_k))$ *if and only if there exists some* $P > 0$ *and* Z *such that*

$$(A + A_i)P + P(A + A_i)^T + (B + B_i)Z + Z^T(B + B_i)^T < 0 \qquad \text{for } i = 1, \cdots k.$$

with $K = ZP^{-1}$.

Note that here the controller doesn't depend on Δ!

- If you want K to depend on Δ, the problem is harder.
- But this would require sensing Δ in real-time.

An LMI for Quadratic D-Stabilization

Lemma 9 (An LMI for Quadratic D-Stabilization).

Suppose there exists $X > 0$ *and* Z *such that*

$$\begin{bmatrix} -rP & AP+BZ \\ (AP+BZ)^T & -rP \end{bmatrix} + \begin{bmatrix} 0 & A_iP+B_iZ \\ (A_iP+B_iZ)^T & 0 \end{bmatrix} < 0,$$

$$AP+BZ+(AP+BZ)^T + A_iP+B_iZ+(A_iP+B_iZ)^T + 2\alpha P < 0, \quad \textit{and}$$

$$\begin{bmatrix} AP+BZ+(AP+BZ)^T & c(AP+BZ-(AP+BZ)^T) \\ c((AP+BZ)^T-(AP+BZ)) & AP+BZ+(AP+BZ)^T \end{bmatrix}$$

$$+ \begin{bmatrix} A_iP+B_iZ+(A_iP+B_iZ)^T & c(A_iP+B_iZ-(A_iP+B_iZ)^T) \\ c((A_iP+B_iZ)^T-(A_iP+B_iZ)) & A_iP+B_iZ+(A_iP+B_iZ)^T \end{bmatrix} < 0$$

for $i = 1, \cdots, k$*. Then if* $K = ZP^{-1}$*, the pole locations,* $z \in \mathbb{C}$ *of* $A(\Delta) + B(\Delta)K$ *satisfy* $|x| \leq r$*,* $\operatorname{Re} x \leq -\alpha$ *and* $z + z^* \leq -c|z - z^*|$ *for all* $\Delta \in Co(\Delta_1, \cdots, \Delta_k)$*.*

Of course, if Δ is time-varying, eigenvalues are meaningless.

An LMI for Quadratic Polytopic H_∞-Optimal State-Feedback Control

Recall the closed-loop in state feedback is:

$$\underline{S}(P,K) = \left[\begin{array}{c|c} A + B_2 F & B_1 \\ \hline C_1 + D_{12}F & D_{11} \end{array}\right]$$

Now add uncertainty to system matrices A, B_1, B_2, C_1, D_{12} and D_{11}.

Theorem 10.

There exists an F such that $\|\underline{S}(P(\Delta), K(0,0,0,F))\|_{H_\infty} \le \gamma$ for all $\Delta \in Co(\Delta_1, \cdots \Delta_k)$ if there exist $Y > 0$ and Z such that

$$\begin{bmatrix} Y(A + A_i)^T + (A + A_i)Y + Z^T(B_2 + B_{2,i})^T + (B_2 + B_{2,i})Z & *^T & *^T \\ (B_1 + B_{1,i})^T & -\gamma I & *^T \\ (C_1 + C_{1,i})Y + (D_{12} + D_{12,i})Z & D_{11} + D_{11,i} & -\gamma I \end{bmatrix} < 0 \;\; i = 1, \cdots, k$$

Then $F = ZY^{-1}$.

$$\underline{S}(P(\Delta),K) = \left[\begin{array}{c|c} A + B_2 F & B_1 \\ \hline C_1 + D_{12}F & D_{11} \end{array}\right] + \Delta \qquad \Delta \in Co(\Delta_1, \cdots \Delta_k)$$

$$\Delta_i = \left[\begin{array}{c|c} A_i + B_{2,i} F & B_{1,i} \\ \hline C_{1,i} + D_{12,i}F & D_{11,i} \end{array}\right]$$

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$$\begin{bmatrix} Y(A+A_i)^T + (A+A_i)Y + Z^T(B_2+B_{2,i})^T + (B_2+B_{2,i})Z & *^T & *^T \\ (B_1+B_{1,i})^T & -\gamma I & *^T \\ (C_1+C_{1,i})Y + (D_{12}+D_{12,i})Z & D_{11}+D_{11,i} & -\gamma I \end{bmatrix} < 0 \quad i = 1, \cdots, k$$

Then $F = ZY^{-1}$.

$$\underline{S}(P(\Delta),K) = \left[\begin{array}{c|c} A+B_2F & B_1 \\ \hline C_1+D_{12}F & D_{11}\end{array}\right] + \Delta \qquad \Delta \in Co(\Delta_1, \cdots \Delta_k)$$

$$\Delta_i = \left[\begin{array}{c|c} A_i+B_{2,i}F & B_{1,i} \\ \hline C_{1,i}+D_{12,i}F & D_{11,i}\end{array}\right]$$

In this case, the uncertain system is:

$$\dot{x}(t) = (A + \Delta A(t))x(t) + (B_1 + \Delta B_1(t))w(t) + (B_2 + \Delta B_2(t))u(t)$$
$$y(t) = (C_1 + \Delta C_1(t))x(t) + (D_{11} + \Delta D_{11}(t))w(t) + (D_{12} + \Delta D_{12}(t))u(t)$$

where

$$\begin{aligned}
\Delta A(t) &= A_1\delta_1(t) + \cdots + A_k\delta_k(t) \\
\Delta B_1(t) &= B_{1,1}\delta_1(t) + \cdots + B_{1,k}\delta_k(t) \\
\Delta B_2(t) &= B_{2,1}\delta_1(t) + \cdots + B_{2,k}\delta_k(t) \\
\Delta C_1(t) &= C_{1,1}\delta_1(t) + \cdots + C_{1,k}\delta_k(t) \\
\Delta D_{11}(t) &= D_{11,1}\delta_1(t) + \cdots + D_{11,k}\delta_k(t) \\
\Delta D_{12}(t) &= D_{12,1}\delta_1(t) + \cdots + D_{12,k}\delta_k(t)
\end{aligned}$$

An LMI for Quadratic Polytopic H_2-Optimal State-Feedback Control

Similarly

Theorem 11.

There exists an F such that $\|\underline{S}(P(\Delta), K(0,0,0,F))\|_{H_2}^2 \le \gamma$ for all $\Delta \in Co(\Delta_1, \cdots \Delta_k)$ if there exist $X > 0$ and Z such that

$$\begin{bmatrix} AX + B_2 Z + XA^T + Z^T B_2^T & B_1 \\ B_1^T & -I \end{bmatrix} + \begin{bmatrix} A_i X + B_{2,i} Z + XA_i^T + Z^T B_{2,i}^T & B_{1,i} \\ B_{1,i}^T & 0 \end{bmatrix} < 0 \qquad i = 1, \cdots, k$$

$$\begin{bmatrix} X & (C_1 X + D_{12} Z)^T \\ C_1 X + D_{12} Z & W \end{bmatrix} + \begin{bmatrix} 0 & (C_{1,i} X + D_{12,i} Z)^T \\ C_{1,i} X + D_{12,i} Z & 0 \end{bmatrix} > 0 \qquad i = 1, \cdots, k$$

$$\text{Trace} W < \gamma$$

Then $F = ZY^{-1}$.

Similar Steps can be taken for robust estimator design, using the LMIs in Duan.

- However, I am not aware of a robust version of the general optimal output feedback LMI for polytopic uncertainty.

An LMI for Quadratic Schur Stabilization

State Equations: Let $u(k) = Fx(k)$ In this case, the uncertain system is:

$$\begin{aligned} x(k+1) &= (A + \Delta A(k))x(k) + (B + \Delta B(k))u(k) \\ &= (A + \Delta A(k) + BF + \Delta B(k)F)x(k) \end{aligned}$$

where

$$\begin{aligned} \Delta A(k) &= A_1\delta_1(t) + \cdots + A_m\delta_m(k) \\ \Delta B(k) &= B_1\delta_1(k) + \cdots + B_m\delta_m(k) \end{aligned}$$

Theorem 12.

There exists a F such that

$$x_{k+1} = (A + \Delta_A + (B + \Delta_B)F)x_k$$

is **quadratically stable** *for $(\Delta_A, \Delta_B) \in Co((A_1, B_2), \cdots, (A_k, B_k))$ if and only if there exists some $X > 0$ and Z such that*

$$\begin{bmatrix} X & AX + BZ \\ (AX + BZ)^T & X \end{bmatrix} + \begin{bmatrix} 0 & A_iX + B_iZ \\ (A_iX + B_iZ)^T & 0 \end{bmatrix} > 0 \quad \textit{for } i = 1, \cdots m.$$

In this case, we have $F = ZP^{-1}$.