LMI Methods in Optimal and Robust Control

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Lecture 16: Optimization of Polynomials and an LMI for Global Lyapunov Stability

Optimization of Polynomials:

As Opposed to Polynomial Programming

Polynomial Programming (NOT CONVEX): n decision variables

$$\min_{x \in \mathbb{R}^n} \quad f(x)$$
$$g_i(x) \geq 0$$

- f and g_i must be convex for the problem to be convex.

Optimization of Polynomials: Lifting to a higher-dimensional space

$$\max_{g,\gamma} \quad \gamma$$
$$f(x) - \gamma = g(x) \qquad \text{for all} \quad x \in \mathbb{R}^n$$
$$g(x) \geq 0 \qquad \text{for all} \quad x \in \{x \in \mathbb{R}^n \ : \ h(x) \geq 0\}$$

- The decision variables are *functions* (e.g. g)
  - One constraint for *every possible value of* x.
- But how to parameterize functions????
- How to enforce an infinite number of constraints???
- Advantage: Problem is convex, even if f, g, h are not convex.

Optimization of Polynomials:

Some Examples: Matrix Copositivity

Of course, you already know some applications of Optimization of Polynomials

- Global Stability of Nonlinear Systems

$$V(x) > \epsilon x^2 \qquad \text{for all } y \in \mathbb{R}^n$$
$$\nabla V(x)^T f(x) < 0 \qquad \text{for all } y \in \mathbb{R}^n$$

Stability of Systems with Positive States: Not all states can be negative...

- Cell Populations/Concentrations
- Volume/Mass/Length

We want:

$$V(x) = x^T P x \geq 0 \qquad \text{for all} \qquad x \geq 0$$
$$\dot{V}(x) = x^T (A^T P + PA) x \leq 0 \qquad \text{for all} \qquad x \geq 0$$

- Matrix Copositivity (An NP-hard Problem)

Verify:

$$x^T P x \geq 0 \qquad \text{for all} \quad x \geq 0$$

Recall: Systems with Uncertainty

$$\dot{x}(t) = A(\delta)x(t) + B_1(\delta)w(t) + B_2(\delta)u(t)$$
$$y(t) = C(\delta)x(t) + D_{12}(\delta)u(t) + D_{11}(\delta)w(t)$$

Theorem 1.

There exists an $F(\delta)$ such that $\|\underline{S}(P(\delta), K(0,0,0,F(\delta)))\|_{H_\infty} \le \gamma$ for all $\delta \in \mathbf{\Delta}$ if there exist $Y > 0$ and $Z(\delta)$ such that

$$\begin{bmatrix} YA(\delta)^T + A(\delta)Y + Z(\delta)^T B_2(\delta)^T + B_2(\delta)Z(\delta) & *^T & *^T \\ B_1(\delta)^T & -\gamma I & *^T \\ C_1(\delta)Y + D_{12}(\delta)Z(\delta) & D_{11}(\delta) & -\gamma I \end{bmatrix} < 0 \quad \text{for all} \quad \delta \in \mathbf{\Delta}$$

Then $F(\delta) = Z(\delta)Y^{-1}$.

The Structured Singular Value, μ

Definition 2.

Given system $M \in \mathcal{L}(L_2)$ and set $\mathbf{\Delta}$ as above, we define the **Structured Singular Value** of $(M, \mathbf{\Delta})$ as

$$\mu(M, \mathbf{\Delta}) = \frac{1}{\inf_{\substack{\Delta \in \mathbf{\Delta} \\ I - M\Delta \text{ is singular}}} \|\Delta\|}$$

The system

$$\dot{x}(t) = A_0 x(t) + Mq(t), \qquad q(t) = \Delta(t)p(t),$$
$$p(t) = Nx(t) + Qq(t), \qquad \Delta \in \mathbf{\Delta}$$

Lower Bound for μ: $\mu \geq \gamma$ if there exists a $P(\delta)$ such that

$$P(\delta) \geq 0 \qquad \text{for all} \quad \delta$$

$$\dot{V} = x^T P(\delta)(A_0 x + Mq) + (A_0 x + Mq)^T P(\delta) x < \epsilon x^T x$$

for all x, q, δ such that

$$(x, q, \delta) \in \left\{ x, q, \delta \,:\, q = \operatorname{diag}(\delta_i)(Nx + Qq), \sum_i \delta_i^2 \leq \gamma^2 \right\}$$

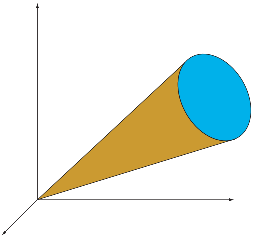
Overview

In this lecture, we will show how the LMI framework can be expanded dramatically to other forms of control problems.

1. Positivity of Polynomials
    1.1 Sum-of-Squares
2. Positivity of Polynomials on Semialgebraic sets
    2.1 Inference and Cones
    2.2 Positivstellensatz
3. Applications
    3.1 Nonlinear Analysis
    3.2 Robust Analysis and Synthesis
    3.3 Global optimization

A Generic Convex Optimization Problem:

$$\max_x \; bx$$
$$\text{subject to} \quad Ax \in C$$



The problem is *convex optimization* if

- C is a convex cone.
- b and A are affine.

Computational Tractability: Convex Optimization over C is tractable if

- The set membership test for $y \in C$ is in P (polynomial-time verifiable).
- The variable x is a finite dimensional vector (e.g. $\mathbb{R}^n$).

Optimization of Polynomials is Convex

The variables are finite-dimensional (if we bound the degree)

Convex Optimization of Functions: Variables $V \in \mathcal{C}[\mathbb{R}^n]$ and $\gamma \in \mathbb{R}$

$$\max_{V,\gamma} \gamma$$

subject to

$$V(x) - x^T x \geq 0 \qquad \forall x$$
$$\nabla V(x)^T f(x) + \gamma x^T x \leq 0 \qquad \forall x$$

V is the decision variable (infinite-dimensional)

- How to make it finite-dimensional???

The set of polynomials is an infinite-dimensional (but *Countable*) vector space.

- It is **Finite Dimensional** if we bound the degree
- All finite-dimensional vector spaces are equivalent!

But we need a way to parameterize this space...

To Begin: How do we Parameterize Polynomials???

A Parametrization consists of a **basis** and a **set of parameters** (coordinates)

- The set of polynomials is an infinite-dimensional vector space.
- It is **Finite Dimensional** if we bound the degree
  - The monomials are a simple basis for the space of polynomials

Definition 3.

Define $Z_d(x)$ to be the vector of monomial bases of degree d or less.

e.g., if $x \in \mathbb{R}^2$, then the vector of basis functions is

$$Z_2(x_1, x_2)^T = \begin{bmatrix} 1 & x_1 & x_2 & x_1 x_2 & x_1^2 & x_2^2 \end{bmatrix}$$

and

$$Z_4(x_1)^T = \begin{bmatrix} 1 & x_1 & x_1^2 & x_2^3 & x_1^4 \end{bmatrix}$$

Linear Representation

- Any polynomial of degree d can be represented with a vector $c \in \mathbb{R}^m$
$$p(x) = c^T Z_d(x)$$
- c is the vector of *parameters* (decision variables).
- $Z_d(x)$ doesn't change (fixed).

$$2x_1^2 + 6x_1x_2 + 4x_2 + 1 = \begin{bmatrix} 1 & 0 & 4 & 6 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2 \end{bmatrix}^T$$

Optimization of Polynomials is Convex

The variables are finite-dimensional (if we bound the degree)

Convex Optimization of Functions: Variables $V \in \mathbb{R}[x]$ and $\gamma \in \mathbb{R}$

$$\max_{V,\gamma} \gamma$$

subject to

$$V(x) - x^T x \geq 0 \qquad \forall x$$
$$\nabla V(x)^T f(x) + \gamma x^T x \leq 0 \qquad \forall x$$

Now use the polynomial parametrization $V(x) = c^T Z(x)$

- Now c is the decision variable.

Convex Optimization of Polynomials: Variables $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$

$$\max_{c,\gamma} \gamma$$

subject to

$$c^T Z(x) - x^T x \geq 0 \qquad \forall x$$
$$c^T \nabla Z(x) f(x) + \gamma x^T x \leq 0 \qquad \forall x$$

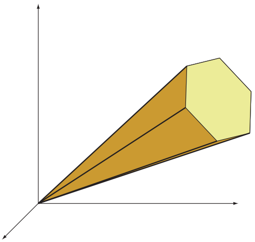
Can LMIs be used for Optimization of Polynomials???

Optimization of Polynomials is NP-Hard!!!

Problem: Use a finite number of variables:

$$\max\ b^T x$$

$$\text{subject to } A_0(y) + \sum_i^n x_i A_i(y) \succeq 0 \quad \forall y$$



The A_i are matrices of polynomials in y. e.g. Using multi-index notation,

$$A_i(y) = \sum_\alpha A_{i,\alpha}\, y^\alpha$$

The FEASIBLITY TEST is Computationally Intractable

The problem: "Is $p(x) \geq 0$ for all $x \in \mathbb{R}^n$?" (i.e. "$p \in \mathbb{R}^+[x]$?") is NP-hard.

How to Find Lyapunov Functions (LF)?

We know a LF by its Properties

What makes a LF a Lyapunov Function?

Property 1: Positivity

- The Lyapunov function must be positive (metrics are positive).
- Also $V(0) = 0$ if 0 is an equilibrium.

Property 2: Negativity along Trajectories

- A Lyapunov Function decreases monotonically in time.
  - A longer trajectory is always “bigger” than a shorter one.
  - As time progresses, the trajectory gets shorter.
  - Hence the LF is always decreasing.

Thus: If a function has properties 1) and 2), it is a Lyapunov Function

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Note: For Linear Systems, we can restrict the search to quadratic LFs.

Property 3: Quadratic (e.g. $x^T P x$)

- If the system is Linear, the solution map is Linear
- Then if the metric is Quadratic, the Lyapunov function is Quadratic.
  - The Composition of Linear and Quadratic Functions is Quadratic

Lyapunov Functions for Linear ODEs

$$\dot{x}(t) = Ax(t)$$

if $x \in \mathbb{R}^n$, then any quadratic function has the form:

$$V(x)_{[1]} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}^T}_{x^T} \underbrace{\begin{bmatrix} P_{11} & P_{21} & P_{31} & P_{41} & P_{51} \\ P_{21} & P_{22} & P_{32} & P_{42} & P_{52} \\ P_{31} & P_{32} & P_{33} & P_{43} & P_{53} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{54} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} \end{bmatrix}}_{P \geq 0} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} \geq 0$$

P3 - Quadratic: The 15 *variables* P_{ij} parameterize all possible quadratic functions on $x \in \mathbb{R}^5$.
P1 - Positive: $V(x) > 0$ for all $x \neq 0$ if and only if P has all positive eigenvalues.

Lyapunov Functions for Nonlinear ODEs

$$\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\end{bmatrix} = f(x(t), y(t))$$

Lets try a quadratic function of the form:

$$V(x,y)_{[2]} = \underbrace{\begin{bmatrix}x\\ y\\ xy\\ x^2\\ y^2\end{bmatrix}^T}_{Z(x,y)^T} \underbrace{\begin{bmatrix}P_{11} & P_{21} & P_{31} & P_{41} & P_{51}\\ P_{21} & P_{22} & P_{32} & P_{42} & P_{52}\\ P_{31} & P_{32} & P_{33} & P_{43} & P_{53}\\ P_{41} & P_{42} & P_{43} & P_{44} & P_{54}\\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55}\end{bmatrix}}_{P \geq 0} \underbrace{\begin{bmatrix}x\\ y\\ xy\\ x^2\\ y^2\end{bmatrix}}_{Z(x,y)} \geq 0$$

P3 - Pseudo-Quadratic: The 15 *variables* P_{ij} parameterize positive polynomial functions $x, y \in \mathbb{R}^2$ of degree $d \leq 4$.
P1 - Positive: $V(x) > 0$ for all $x \neq 0$ if P has all positive eigenvalues.

Can LMIs be used to Optimize Positive Polynomials???

Show Me the LMI!

Basic Idea: If there exists a Positive Matrix $P \geq 0$ such that

$$V(x) = Z_d(x)^T P Z_d(x)$$

Positive Matrices ($P \geq 0$) have square roots!

$$P = Q^T Q$$

Hence

$$\begin{aligned} V(x) &= Z_d(x)^T Q^T Q Z_d(x) = (Q Z_d(x))^T (Q Z_d(x)) \\ &= h(x)^T h(x) \geq 0 \end{aligned}$$

Conclusion:

$$V(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^n$$

if there exists a $P \geq 0$ such that

$$V(x) = Z_d(x)^T P Z_d(x)$$

- Such a function is called Sum-of-Squares (SOS), denoted $V \in \Sigma_s$.
- This is an LMI! Equality constraints relate the coefficients of V (decision variables) to the elements of P (more decision variables).

Definition 4.

A Polynomial, f, is called Positive SemiDefinite (PSD) if

$$f(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n$$

The Primary Problem: How to enforce the constraint $f(x) \geq 0$ for all x?

Easy Proof: Certificate of Infeasibility

- A Proof that f is NOT PSD.
- i.e. To show that

$$f(x) \geq 0 \qquad \text{for all } x \in \mathbb{R}^n$$

  is FALSE, we need only find a point x with $f(x) < 0$.

Complicated Proof: It is much harder to identify a **Certificate of Feasibility**

- A Proof that f is PSD.

Global Positivity Certificates (Proofs and Counterexamples)

Question: How does one prove that $f(x)$ is positive semidefinite?

What Kind of Functions do we Know are PSD?

- Any squared function is positive.
- The sum of squared forms is PSD
- The product of squared forms is PSD
- The ratio of squared forms is PSD

So $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ if

$$V(x) = \prod_k \frac{\sum_i f_{ik}(x)^2}{\sum_j h_{jk}(x)^2}.$$

But is any PSD polynomial the sum, product, or ratio of squared polynomials?

- An old Question....

Definition 5.

A polynomial, $p(x) \in \mathbb{R}[x]$ is a **Sum-of-Squares (SOS)**, denoted $p \in \Sigma_s$ if there exist polynomials $g_i(x) \in \mathbb{R}[x]$ such that

$$p(x) = \sum_{i=1}^{k} g_i(x)^2.$$

David Hilbert created a famous list of 23 then-unsolved mathematical problems in 1900.

- Only 10 have been fully resolved.
- The 17th problem has been resolved.

"Given a multivariate polynomial that takes only non-negative values over the reals, can it be represented as a sum of squares of rational functions?" -D. Hilbert, 1900

Hilbert's 17th was resolved in the affirmative by E. Artin in 1927.

- Any PSD polynomial is the sum, product and ratio of squared polynomials.
- If $p(x) \geq 0$ for all $x \in \mathbb{R}^n$, then

$$p(x) = \frac{g(x)}{h(x)}$$

  where $g, h \in \Sigma_s$.
- If p is positive definite, then we can assume $h(x) = (\sum_i x_i^2)^d$ for some d. That is,

$$(x_1^2 + \cdots + x_n^2)^d p(x) \in \Sigma_s$$

- If we can't find a SOS representation (certificate) for $p(x)$, we can try $(\sum_i x_i^2)^d p(x)$ for higher powers of d.

Of course this doesn't answer the question of how we find SOS representations.

Quadratic Parameterization of Polynomials

Quadratic Representation

- Alternative to Linear Parametrization, a polynomial of degree d can be represented by a matrix $M \in \mathbb{S}^m$ as
$$p(x) = Z_d(x)^T M Z_d(x)$$
- However, now the problem may be under-determined
$$\begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} M_1 & M_2 & M_3 \\ M_2 & M_4 & M_5 \\ M_3 & M_5 & M_6 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}$$
$$= M_1 x^4 + 2M_2 x^3 y + (2M_3 + M_4) x^2 y^2 + 2M_5 x y^3 + M_6 y^4$$

Thus, there are infinitely many quadratic representations of p. For the polynomial
$$f(x) = 4x^4 + 4x^3 y - 7x^2 y^2 - 2xy^3 + 10y^4,$$
we can use the alternative solution
$$4x^4 + 4x^3 y - 7x^2 y^2 - 2xy^3 + 10y^4$$
$$= M_1 x^4 + 2M_2 x^3 y + (2M_3 + M_4) x^2 y^2 + 2M_5 x y^3 + M_6 y^4$$

Polynomial Representation - Quadratic

For the polynomial

$$f(x) = 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4,$$

we require

$$\begin{aligned} &4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4 \\ &\qquad = M_1x^4 + 2M_2x^3y + (2M_3 + M_4)x^2y^2 + 2M_5xy^3 + M_6y^4 \end{aligned}$$

Constraint Format:

$$M_1 = 4; \quad 2M_2 = 4; \quad 2M_3 + M_4 = -7; \quad 2M_5 = -2; \quad 10 = M_6.$$

An underdetermined system of linear equations (6 variables, 5 equations).

- This yields a family of quadratic representations, parameterized by λ as

$$4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4 = \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 4 & 2 & -\lambda \\ 2 & -7 + 2\lambda & -1 \\ -\lambda & -1 & 10 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}$$

which holds for any $\lambda \in \mathbb{R}$

Positive Matrix Representation of SOS

Sufficiency

Quadratic Form:

$$p(x) = Z_d(x)^T M Z_d(x)$$

Consider the case where the matrix M is positive semidefinite.

Suppose: $p(x) = Z_d(x)^T M Z_d(x)$ where $M > 0$.

- Any positive semidefinite matrix, $M \geq 0$ has a square root $M = PP^T$

Hence

$$p(x) = Z_d(x)^T M Z_d(x) = Z_d(x)^T P P^T Z_d(x).$$

Which yields

$$p(x) = \sum_i \left(\sum_j P_{i,j} Z_{d,j}(x) \right)^2$$

which makes $p \in \Sigma_s$ an SOS polynomial.

Positive Matrix Representation of SOS

Necessity

Moreover: Any SOS polynomial has a quadratic rep. with a PSD matrix.

Suppose: $p(x) = \sum_i g_i(x)^2$ is degree $2d$ (g_i are degree d).

- Each $g_i(x)$ has a linear representation in the monomials.

$$g_i(x) = c_i^T Z_d(x)$$

- Hence

$$p(x) = \sum_i g_i(x)^2 = \sum_i Z_d(x) c_i c_i^T Z_d(x) = Z_d(x) \left(\sum_i c_i c_i^T \right) Z_d(x)$$

- Each matrix $c_i c_i^T \geq 0$. Hence $Q = \sum_i c_i c_i^T \geq 0$.
- We conclude that if $p \in \Sigma_s$, there is a $Q \geq 0$ with $p(x) = Z_d(x) Q Z_d(x)$.

Lemma 6.

Suppose M is polynomial of degree $2d$. $M \in \Sigma_s$ if and only if there exists some $Q \succeq 0$ such that

$$M(x) = Z_d(x)^T Q Z_d(x).$$

Sum-of-Squares

Thus we can express the search for a SOS certificate of positivity as an LMI.

Take the numerical example

$$4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$$

The question of an SOS representation is equivalent to

$$\text{Find} \quad M = \begin{bmatrix} M_1 & M_2 & M_3 \\ M_2 & M_4 & M_5 \\ M_3 & M_5 & M_6 \end{bmatrix} \geq 0 \quad \text{such that}$$

$$M_1 = 4; \quad 2M_2 = 4; \quad 2M_3 + M_4 = -7; \quad 2M_5 = -2; \quad M_6 = 10.$$

In fact, this is feasible for

$$M = \begin{bmatrix} 4 & 2 & -6 \\ 2 & 5 & -1 \\ -6 & -1 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$

We can use this solution to construct an SOS certificate of positivity.

$$
\begin{aligned}
4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4 &= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 4 & 2 & -6 \\ 2 & 5 & -1 \\ -6 & -1 & 10 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \\
&= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \\
&= \begin{bmatrix} 2xy + y^2 \\ 2x^2 + xy + 3y^2 \end{bmatrix}^T \begin{bmatrix} 2xy + y^2 \\ 2x^2 + xy + 3y^2 \end{bmatrix} \\
&= (2xy + y^2)^2 + (2x^2 + xy + 3y^2)^2
\end{aligned}
$$

Quadratic Representation: (Using Matrix $M \in \mathbb{R}^{p\times p}$):

$$p(x) = Z_d(x)^T M Z_d(x)$$

Linear Representation: (Using Vector $c \in \mathbb{R}^q$)

$$q(x) = c^T Z_{2d}(x)$$

To constrain $p(x) = q(x)$, we write $[Z_d]_i = x^{\alpha_i}$, $[Z_{2d}]_j = x^{\beta_j}$ and reformulate

$$p(x) = Z_d(x)^T M Z_d(x) = \sum_{i,j} M_{i,j} x^{\alpha_i + \alpha_j} = \mathsf{vec}(M)^T A Z_{2d}(x)$$

where $A \in \mathbb{R}^{p^2 \times q}$ is defined as

$$A_{i,j} = \begin{cases} 1 & \text{if } \alpha_{\mathsf{mod}(i,p)} + \alpha_{\lfloor i \rfloor_p + 1} = \beta_j \\ 0 & \text{otherwise} \end{cases}$$

This then implies that

$$Z_d(x)^T M Z_d(x) = \mathsf{vec}(M)^T A Z_{2d}(x)$$

Hence if we constrain $c = \mathsf{vec}(M)^T A$, this is equivalent to $p(x) = q(x)$

Summarizing, e.g., for Lyapunov stability, we have variables $M > 0, Q > 0$ with the constraint

$$-\text{vec}(M)^T A = \text{vec}(Q)^T AB$$

Feasibility implies stability since

$$\begin{aligned} V(x) &= Z(x)^T Q Z(x) \geq 0 \\ \dot{V}(x) &= \text{vec}(Q)^T A \nabla Z_{2d}(x) \\ &= \text{vec}(Q)^T AB Z_{2d}(x) \\ &= -\text{vec}(M)^T A Z_{2d}(x) \\ &= -Z(x)^T M Z(x) \geq 0 \end{aligned}$$

YALMIP has SOS functionality
Link: YALMIP SOS Manual

To test whether

$$4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$$

is a positive polynomial, we use:

```
> sdpvar x y
> p = 4*x^4 + 4*x^3*y - 7*x^2*y^2 - 2*x*y^3 + 10*y^4;
> F=[];
> F=[F;sos(p)];
> solvesos(F);
```

To retrieve the SOS decomposition, we use

```
> sdisplay(p)
> ans =
>     '1.7960*x^2 - 3.0699*y^2 + 0.6468*x*y'
>     ' - 0.6961*x^2 - 0.7208*y^2 - 1.4882*x*y'
>     '0.5383*x^2 + 0.2377*y^2 - 0.3669*x*y'
```

In this class, we will use instead SOSTOOLS
Link: SOSTOOLS Website

To test whether

$$4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$$

is a positive polynomial, we use:

```
> pvar x y
>  p = 4*x^4 + 4*x^3*y - 7*x^2*y^2 - 2*x*y^3 + 10*y^4;
> prog=sosprogram([x y]);
> prog=sosineq(prog,p);
> prog=sossolve(prog);
```

SOS Programming:

Numerical Example

This also works for matrix-valued polynomials.

$$M(y,z) = \begin{bmatrix} (y^2+1)z^2 & yz \\ yz & y^4+y^2-2y+1 \end{bmatrix}$$

$$\begin{bmatrix} (y^2+1)z^2 & yz \\ yz & y^4+y^2-2y+1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix}$$

$$= \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix}$$

$$= \begin{bmatrix} yz & 1-y \\ z & y^2 \end{bmatrix}^T \begin{bmatrix} yz & 1-y \\ z & y^2 \end{bmatrix} \in \Sigma_s$$

SOS Programming:

Numerical Example

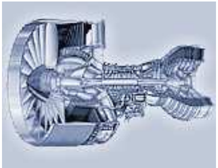
This also works for matrix-valued polynomials.

$$M(y,z) = \begin{bmatrix} (y^2+1)z^2 & yz \\ yz & y^4+y^2-2y+1 \end{bmatrix}$$

SOSTOOLS Code: Matrix Positivity

```
> pvar x y
> M = [(y^2 + 1) * z^2  y * z; y * z  y^4 + y^2 - 2 * y + 1];
> prog=sosprogram([y z]);
> prog=sosmatrixineq(prog,M);
> prog=sossolve(prog);
```

An Example of Global Stability Analysis



A controlled model of a jet engine (Derived from Moore-Greitzer).

$$\dot{x} = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3$$

$$\dot{y} = 3x - y$$

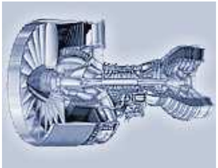
SOSTOOLS Code: Global Stability

```
> pvar x y
> f = [-y - 1.5*x^2 - .5*x^3; 3*x - y];
> prog=sosprogram([x y]);
> Z=monomials([x,y],0:2);
> [prog,V]=sossosvar(prog,Z);
> V = V + .0001*(x^4 + y^4);
> prog=soseq(prog,subs(V,[x; y],[0; 0]));
> nablaV=[diff(V,x);diff(V,y)];
> prog=sosineq(prog,-nablaV'*f);
> prog=sossolve(prog);
> Vn=sosgetsol(prog,V)
```

Finds a Lyapunov Function of degree 4.

An Example of Global Stability Analysis

A controlled model of a jet engine (Derived from Moore-Greitzer).



$$\dot{x} = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3$$

$$\dot{y} = 3x - y$$

YALMIP Code: Global Stability

```
> sdpvar x y
> f = [-y - 1.5*x^2 - .5*x^3; 3*x - y];
> [V,Vc]=polynomial([x y],4);
> F=[Vc(1)==0];
> F = [F; sos(V - .00001*(x^2 + y^2))];
> nablaV=jacobian(V,[x y]);
> F=[F;sos(-nablaV*f)];
> solvesos(F,[],[],[Vc])
```

Finds a Lyapunov Function of degree 4.

- Going forward, we will use mostly SOSTOOLS

SOSOPT and DelayTOOLS

There is a third relatively new Parser called SOSOPT

Link: SOSOPT Website

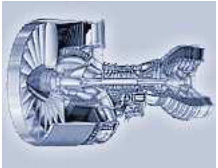
And I can plug my own mini-toolbox version of SOSTOOLS:

Link: DelayTOOLS Website

- However, I don't expect you to need this toolbox for this class.

An Example of Global Stability Analysis

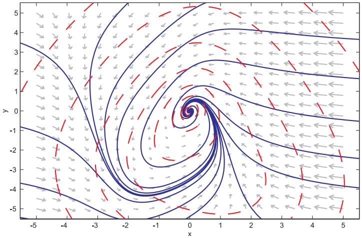
A controlled model of a jet engine (Derived from Moore-Greitzer).



$$\dot{x} = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3$$
$$\dot{y} = 3x - y$$

This is feasible with

$$V(x) = 4.5819x^2 - 1.5786xy + 1.7834y^2 - 0.12739x^3 + 2.5189x^2y - 0.34069xy^2 + 0.61188y^3 + 0.47537x^4 - 0.052424x^3y + 0.44289x^2y^2 + 0.090723y^4$$



Summary of the SOS Conditions

Proposition 1.

Suppose: *$p(x) = Z_d(x)^T Q Z_d(x)$ for some $Q > 0$. Then $p(x) \geq 0$ for all $x \in \mathbb{R}^n$*

Refinement 1: Suppose $Z_d(x)^T P Z_d(x) p(x) = Z_d(x)^T Q Z_d(x)$ for some $Q, P > 0$. Then $p(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Refinement 2: Suppose $(\sum_i x_i^2)^q p(x) = Z_d(x)^T Q Z_d(x)$ for some $P > 0$, $q \in \mathbb{N}$. Then $p(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Ignore these Refinements

- SOS by itself is sufficient. The refinements are Necessary and Sufficient.
- Almost never necessary in practice...

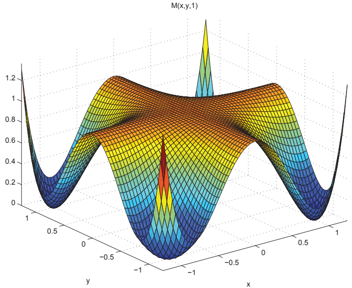
Problems with SOS

Unfortunately, a Sum-of-Squares representation is not necessary for positivity.

- Artin included ratios of squares.

Counterexample: The Motzkin Polynomial

$$M(x,y) = x^2y^4 + x^4y^2 + 1 - 3x^2y^2$$



However, $(x^2 + y^2 + 1)M(x,y)$ is a Sum-of-Squares.

$$\begin{aligned}(x^2 + y^2 + 1)M(x,y) &= (x^2y - y)^2 + (xy^3 - x)^2 + (x^2y^2 - 1)^2 \\ &\quad + \frac{1}{4}(xy^3 - x^3y)^2 + \frac{3}{4}(xy^3 + x^3y - 2xy)^2\end{aligned}$$