Spacecraft and Aircraft Dynamics

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Lecture 3: Elliptic Orbits

Introduction

In this Lecture, you will learn:

Geometry of the orbit

- How to use conservation of angular momentum and energy to derive geometric properties of the orbit
- The types of orbit
  - Elliptic
  - Parabolic
  - Hyperbolic

Different Representations

- How to convert between
  - Energy and Momentum
  - Position and velocity
  - geometric properties

Problem: Suppose we observe at perigee, $r_p = 15000km$ and at apogee, $r_a = 25000km$. At time t_0, we observe $r(t_0) = 20000km$. Determine $v(t_0)$.

If we don't care about motion of the planet, the 2-body problem is actually a 1-body problem.

$$\ddot{\vec{r}} = -\frac{\mu}{\|\vec{r}\|^3}\vec{r}$$

Last lecture, we showed that the angular momentum vector,

$$\vec{h} = \vec{r} \times \dot{\vec{r}}$$

is invarant. i.e. $\dot{\vec{h}} = 0$. Furthermore, the angular momentum vector is orthogonal to the orbital plane. That is

$$\vec{r} \times \vec{h} = 0$$

Definition 1.

The **Eccentricity Vector** can be defined as

$$\vec{e} = \frac{1}{\mu}\left(\dot{\vec{r}} \times \vec{h} - \mu\frac{\vec{r}}{\|\vec{r}\|}\right)$$

Now we show that the eccentricity vector is invariant. Since we have already shown $\dot{h} = 0$,

$$\dot{\vec{e}} = \left(\frac{1}{\mu}\ddot{\vec{r}} \times \vec{h} - \frac{\dot{\vec{r}}}{\|\vec{r}\|} + \frac{\vec{r}}{\|\vec{r}\|^3}\vec{r}^T\dot{\vec{r}}\right)$$

Now

$$\frac{1}{\mu}\ddot{\vec{r}} \times \vec{h} = -\frac{\mu}{\|\vec{r}\|^3}\vec{r} \times \vec{r} \times \dot{\vec{r}}$$

We use the triple cross identity

$$a \times b \times c = (a \cdot c)b - (a \cdot b)c$$

Eccentricity Vector

Applying the triple cross identity,

$$\begin{aligned}
\frac{1}{\mu}\ddot{\vec{r}} \times \vec{h} &= -\frac{1}{\|\vec{r}\|^3}(\vec{r}\cdot\dot{\vec{r}})\vec{r} - (\vec{r}\cdot\vec{r})\dot{\vec{r}} \\
&= -\frac{1}{\|\vec{r}\|^3}\left((\vec{r}\cdot\dot{\vec{r}})\vec{r} - (\vec{r}\cdot\vec{r})\dot{\vec{r}}\right) \\
&= -\frac{\vec{r}\cdot\dot{\vec{r}}}{\|\vec{r}\|^3}\vec{r} - \frac{1}{\|\vec{r}\|}\dot{\vec{r}}
\end{aligned}$$

Thus we conclude that $\dot{\vec{e}} = 0$.

Definition 2.

The eccentricity of an orbit is

$$e = \|\vec{e}\|$$

Eccentricity

The eccentricity vector has several important properties. Expand the dot product $\vec{r} \cdot (\vec{e} - \vec{e}) = 0$

$$\vec{r} \cdot \left(\dot{\vec{r}} \times \vec{h} - \mu \frac{\vec{r}}{\|\vec{r}\|} - \vec{e}\mu \right) = \vec{r} \cdot (\dot{\vec{r}} \times h) - \mu\|\vec{r}\| - \mu\vec{r} \cdot \vec{e} = 0$$

Now $\vec{r} \times \dot{\vec{r}} = \vec{h}$. Also, by definition $\vec{r}(t) \cdot \vec{e} = \|\vec{r}(t)\|\|\vec{e}\| \cos f(t)$, where f is the time-varying angle between $\vec{e}$ and $\vec{r}(t)$. Continuing,

$$\begin{aligned}
&\vec{r} \cdot (\dot{\vec{r}} \times h) - \mu\|\vec{r}\| - \mu\vec{r} \cdot \vec{e} \\
&= \|\vec{h}\|^2 - \mu\|\vec{r}\| - \mu\|\vec{r}\|\|\vec{e}\| \cos f \\
&= \|\vec{h}\|^2 - \mu\|\vec{r}\| \left(1 + \|\vec{e}\| \cos f\right) \\
&= 0
\end{aligned}$$

From which we get the important equation

$$\|\vec{r}(t)\| = \frac{\|\vec{h}\|^2}{\mu \left(1 + \|\vec{e}\| \cos f(t)\right)}$$

Is this a solution to the two-body problem?

Solution to the Two-Body Problem

Examine the equation

$$\|\vec{r}(t)\| = \frac{\|\vec{h}\|^2}{\mu\left(1 + \|\vec{e}\| \cos f(t)\right)}$$

The **true anomaly**, f, is defined as the angle between $\vec{e}$ and $\vec{r}$. Since $\vec{e}$ is fixed, this equation gives the path on the orbital plane as f moves from $0\,\mathrm{deg}$ to $360\,\mathrm{deg}$ There are three cases

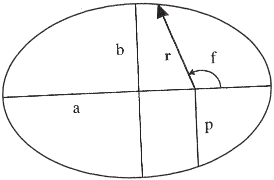


Figure: $\|\vec{e}\| < 1$

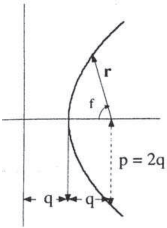


Figure: $\|\vec{e}\| = 1$

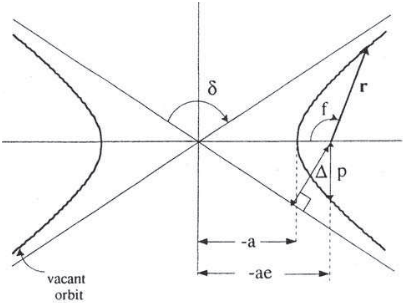


Figure: $\|\vec{e}\| > 1$

The case of Elliptic Orbits ($\|\vec{e}\| < 1$)

Lets use the simplified scalar equation

$$r(t) = \frac{h^2}{\mu \left(1 + e \cos f(t)\right)}$$

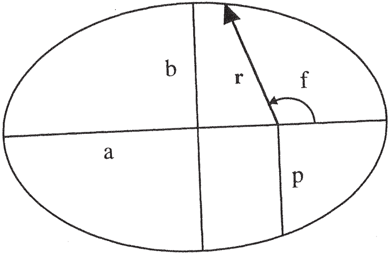
Consider several points.

Case 1 ($f = 0$): $\vec{r}$ is aligned with $\vec{e}$. The radius is *minimum*. Thus the radius at periapse is

$$r_p = \frac{h^2}{\mu(1 + e)}$$

The point of minimum radius is also referred to as

- *Perigee* for orbits around Earth.
- *Perihelion* for orbits around the Sun.
- *Perilune* for lunar orbits.



Thus the vector $\vec{e}$ always points toward the periapse. This orients the orbit in space. Note that $\|\vec{e}\| \neq r_p$.

The case of Elliptic Orbits ($\|\vec{e}\| < 1$)

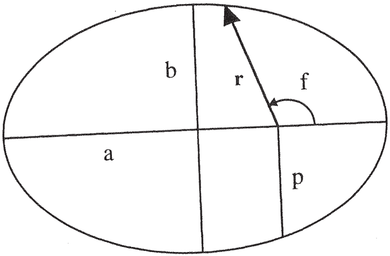
Case 2 ($f = \pi$):

- $\vec{r}$ is opposite to $\vec{e}$. Thus $\vec{r}$ points away from periapse.
- The radius now *maximum*. Thus the radius at *apoapse* is

$$r_a = \frac{h^2}{\mu(1-e)}$$

The point of maximum radius is also referred to as

- *Apogee* for orbits around Earth.
- *Aphelion* for orbits around the Sun.
- *Apolune* for lunar orbits.



Combine periapse and apoapse. we get the **semimajor axis** of the orbit:

$$a = \frac{r_p + r_a}{2} = \frac{h^2}{\mu}\frac{1}{1-e^2}$$

The case of Elliptic Orbits ($\|\vec{e}\| < 1$)

This relationship between r_p, r_a, a, and e can be simplified by eliminating h^2/μ.For example,

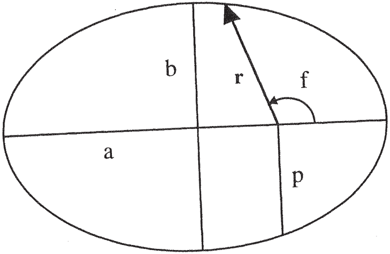
$$\frac{h^2}{\mu} = r_a(1-e) = r_p(1+e) = a(1-e^2)$$

Thus we can solve for h^2/μ given

- r_a and e
- r_p and e
- a and e

Alternatively, we can solve directly

$$r_p = a(1-e)$$
$$r_a = a(1+e)$$



The case of Elliptic Orbits ($\|\vec{e}\| < 1$)

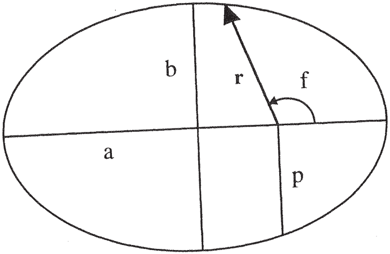
Case 3 ($f = \frac{\pi}{2}$):

- $\vec{r}$ is orthogonal to $\vec{e}$. $\vec{r}$ points up from the major axis, parallel to the minor axis.
- This length is called the *semi-latus rectum*, denoted by p.

$$p = \frac{h^2}{\mu}$$

Although difficult to measure, p is a useful quantity

- Directly represents h^2/μ.
- Also called "the parameter" of the ellipse.



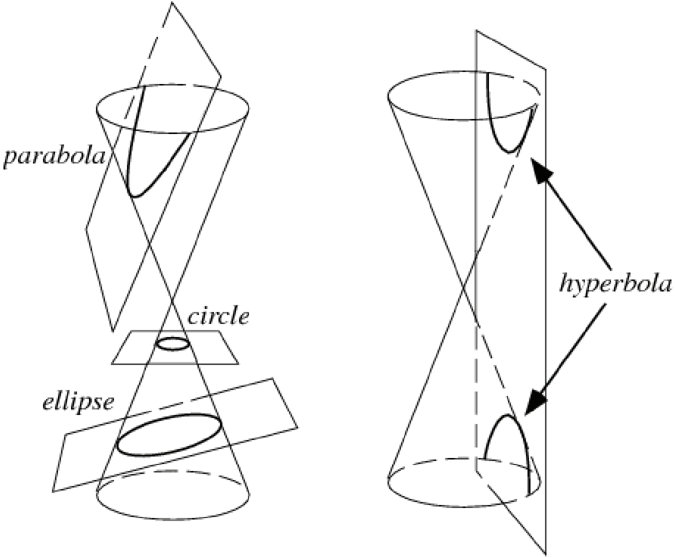
The orbit equation is simplified to

$$r(t) = \frac{p}{1 + e \cos f}$$

This establishes Kepler's First law: Elliptic Motion.

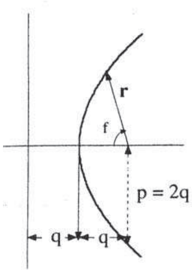
Elliptic Orbits

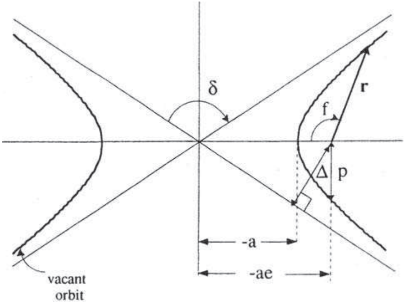
Conic Sections



The case of Parabolic/Hyperbolic Orbits ($\|\vec{e}\| \geq 1$)

$$r(t) = \frac{h^2}{\mu\left(1 + e\cos f(t)\right)}$$





Definition 3.

If $e = 1$, the orbit is parabolic.
$\delta = 180\,\text{deg}$.

Definition 4.

If $e > 1$, the orbit is hyperbolic.
$\delta < 180\,\text{deg}$

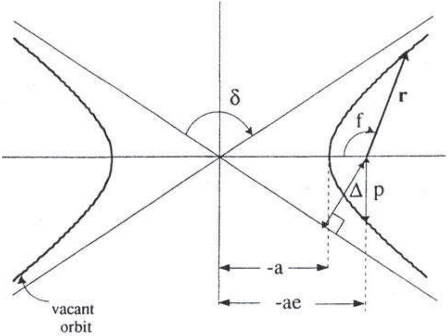
Hyperbolic Orbits ($\|\vec{e}\| > 1$)

Turning Angle

$$r(t) = \frac{h^2}{\mu\,(1 + e\cos f(t))}$$

If $e > 1$, then $\lim_t r(t) \to \infty$ when $1 + e\cos f = 0$.

- The orbit is asymptotic at $f = \pm\cos^{-1}\frac{1}{e}$.
  - $f = -\cos^{-1}\frac{1}{e}$ is the incoming asymptote.
  - $f = \cos^{-1}\frac{1}{e}$ is the outgoing asymptote.
- The angle between incoming and outgoing asymptotes is the *Turning Angle*, δ.
  - $\delta = 2\cos^{-1}\frac{1}{e}$



When $f \geq \delta/2$ of $f \leq -\delta/2$, the orbit is fictional.

- Hyperbolic orbits do not repeat.

Hyperbolic Orbits ($\|\vec{e}\| > 1$)

Geometry

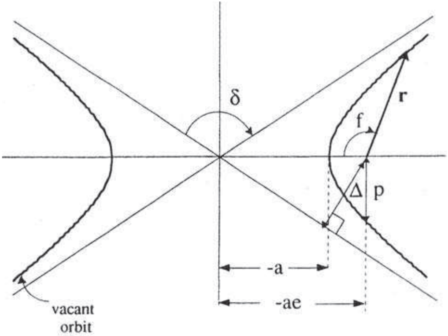
- When $f = 0$, we have closest approach.

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{p}{1+e}$$

- For hyperbolic orbits, $r_a = \infty$, we explicitly define semimajor axis as

$$a = \frac{p}{1-e^2}$$

so that $r_p = a(1-e)$. For hyperbolic orbits, this means $a < 0$.



For $f = \pi$, r has a minimum on the fictional orbit

$$r_a = \frac{p}{1+e} = a(1-e)$$

Parabolic Orbits ($\|\vec{e}\| = 1$)

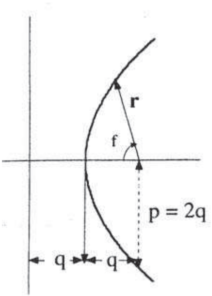
$$r(t) = \frac{h^2}{\mu\left(1 + \cos f(t)\right)}$$

Turning Angles: ($f \to \pm\pi$): If $e = 1$, then $\lim_t r(t) \to \infty$ as $\lim_t f(t) \to \pm\pi$.

- When $f = 0$, we have closest approach.

$$q = \frac{h^2}{2\mu} = \frac{p}{2}$$

- When $f \geq \pi$ of $f \leq -\pi$, the orbit is fictional.
  - Parabolic orbits do not repeat.
- The turning angle is $180\,\mathrm{deg}$.
- No "excess velocity"



Kepler's Second Law

Equal Areas in Equal Time

Kepler's Second Law

Conservation of Angular Momentum

We need to find an expression for dA/dt.

Consider an infinitesimal section of area:

$$dA = \frac{1}{2}r^2 df$$

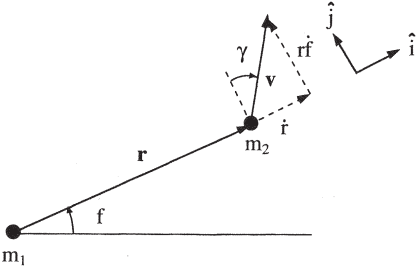
where we denote $r = \|\vec{r}\|$. To first order,

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{df}{dt}$$

- Note that $\dot{r} \neq \dot{\vec{r}}$. In the orbital plane,
$$\dot{\vec{r}} = \dot{r}\hat{i} + r\dot{f}\hat{j}$$

Now examine angular momentum

$$\vec{h} = \vec{r} \times \dot{\vec{r}} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} \dot{r} \\ r\dot{f} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r^2\dot{f} \end{bmatrix}$$



Thus from conservation of angular momentum, we have

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{f} = \frac{\|\vec{h}\|}{2} = constant$$

Which proves Kepler's Second Law

Kepler's Third Law

The Period of an Orbit

The period of an orbit is the time taken to sweep out the entire ellipse.

Fortunately, the area of an ellipse is known

$$A_{ellipse} = \pi ab$$

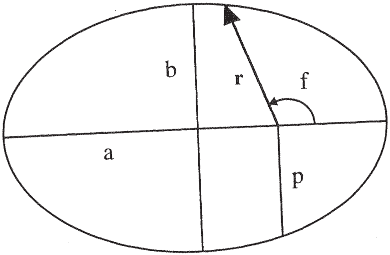
where b is the semi-minor axis of the ellipse.

$$b = a\sqrt{1-e^2}$$

Therefore $dA = \frac{h}{2}dt$ implies

$$T_{orbit} = \frac{2A_{ellipse}}{h} = \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{\mu a(1-e^2)}}$$

$$= 2\pi\sqrt{\frac{a^3}{\mu}}$$



Which proves Kepler's second Law

$$\frac{T^2}{a^3} = \frac{4\pi^2}{\mu} = constant$$

For Kepler, μ was μ_{sun}.

Energy Methods

Now, lets talk about energy in terms of geometry. Recall

$$E = \frac{1}{2}v^2 - \frac{\mu}{r}$$

- But what is the energy associated with a geometry?
- We already know angular momentum in terms of geometry

$$p = \frac{h^2}{\mu}, \qquad \text{implies} \qquad h = \sqrt{p\mu}$$

Consider r and v at periapse. Then $r_p = a(1-e)$ and $r_p v_p = h$, so

$$E = \frac{1}{2}v_p^2 - \frac{\mu}{r_p} \qquad = \frac{1}{2}\frac{p\mu}{a^2(1-e)^2} - \frac{\mu}{a(1-e)}$$

$$= \frac{1}{2}\frac{\mu a(1-e^2)}{a^2(1-e)^2} - \frac{\mu}{a(1-e)} \qquad = \mu\frac{\frac{1}{2}(1+e)-1}{a(1-e)} = -\frac{\mu}{2a}$$

Therefore, the total energy of an orbit is

$$E = -\frac{\mu}{2a}$$

Energy Methods

Vis-Viva Equation

$$E = -\frac{\mu}{2a}$$

Note for an elliptic orbit, the total energy is negative, as expected.

This yields the famous relationship

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

Also yields the "Excess Velocity":

$$v_{excess} = \lim_{r \to \infty} v(r) = \sqrt{-\frac{\mu}{a}}$$

Only real when $a \leq 0$.

Example

Problem: Suppose we observe at perigee, $r_p = 15000km$ and at apogee, $r_a = 25000km$. At time t_0, we observe $r(t_0) = 20000km$. Determine $v(t_0)$.
Solution: We first find the total energy, E from

$$E = -\frac{\mu}{2a}$$

where a can be found from

$$a = \frac{r_p + r_a}{2} = 20000km$$

Therefore

$$E = -\frac{\mu}{2a} = -9.96$$

Now we use

$$E = v^2/2 - \mu/r = -9.96$$

to find

$$v = \sqrt{2\left((E + \frac{\mu}{r}\right)} = \sqrt{2\left(-9.96 + 19.93\right)} = 4.464km/s$$

Example

Problem: Suppose we want to create a satellite which always maintains the same position above the earth. Find a and e for such an orbit.

Solution: The earth rotates 363.25 times a year (one day comes from motion about the sun). Thus the earth rotates once every 23 hours, 56 minutes and 4 seconds. Thus the period of the satellite must be $\tau = 86164s$.

From Kepler's Third Law, we have

$$\begin{aligned} a &= \left(\frac{\mu\tau^2}{4\pi^2}\right)^{1/3} \\ &= 42,164km \end{aligned}$$

Since the rotation rate of the earth is constant, we want $e = 0$.

Summary

Some Important Scalar Relations

$$r(t) = \frac{h^2}{\mu\left(1 + e\cos f(t)\right)}$$

$$r_p = a(1-e)$$

$$r_a = a(1+e)$$

$$p = a(1-e^2) = \frac{h^2}{\mu}$$

$$b = a\sqrt{1-e^2}$$

$$E = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$h = r_p v_p = r_a v_a \cong r^2 \dot{f}$$

Summary

Some Important Vector Relations

$$
\begin{aligned}
\dot{\vec{e}} &= 0 \\
\dot{\vec{h}} &= 0 \\
\vec{h} &= \vec{r} \times \dot{\vec{r}} \\
\vec{e} &= \frac{1}{\mu}\left(\dot{\vec{r}} \times \vec{h} - \mu \frac{\vec{r}}{\|\vec{r}\|}\right) \\
\vec{r}(t) \times \vec{h} &= 0 \\
\vec{e} \times \vec{h} &= 0 \\
\dot{\vec{r}} &= \dot{r}\hat{i} + r\dot{f}\hat{j}
\end{aligned}
$$