Spacecraft and Aircraft Dynamics

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Lecture 6: The Orbital Plane

Introduction

In this Lecture, you will learn:

The Orbital Plane

- Inclination
- Right Ascension
- Argument of Periapse

New Concepts

- The Earth-Centered Inertial reference frame
- The line of nodes

Practice

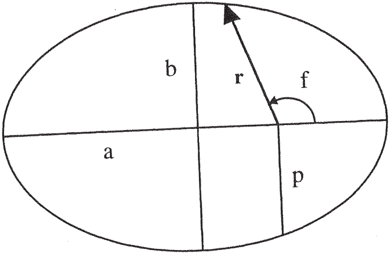
- How to construct all orbital elements from $\vec{r}$ and $\vec{v}$
- A Numerical Illustration

The Orbital Elements

2D orbits

So far, all orbits are parameterized by 3 parameters

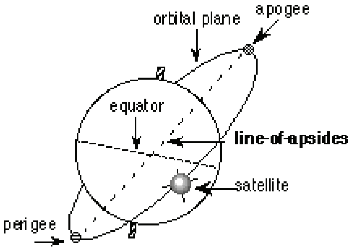
- semimajor axis, a
- eccentricity, e
- true anomaly, f



- a and e define the geometry of the orbit.
- f describes the position within the orbit (a proxy for time).

The Orbital Elements

Note: We have shown how to use a, e and f to find the scalars r and v.

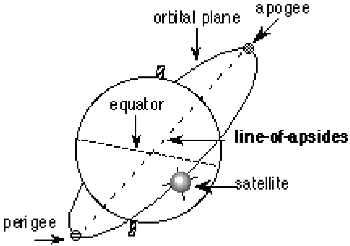


Question: How do we find the vectors $\vec{r}$ and $\vec{v}$?

Answer: We have to determine how the orbit is oriented in space.

- Orientation is determined by vectors $\vec{e}$ and $\vec{h}$.
- We need 3 new orbital elements
    - Orientation can be determined by 3 rotations.

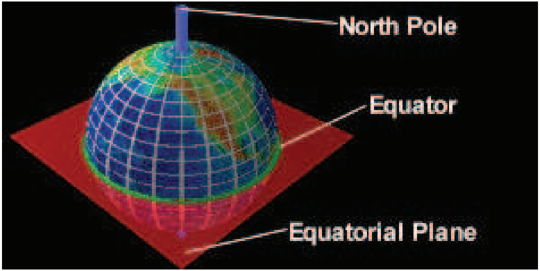
Question: How do we find the vectors $\vec{r}$ and $\vec{v}$?
Response: In which coordinate system??



- The origin is the center of the earth
- We need to define the $\hat{x}$, $\hat{y}$, and $\hat{z}$ vectors.

ECI: The Equatorial Plane

Defining the $\hat{z}$ vector



- The $\hat{z}$ vector is defined to be the vector parallel to the axis of rotation of the earth.
- Can apply to other planets
- Does not apply to Heliocentric Coordinates

Definition 1.

The **Equatorial Plane** is the set of vectors normal to the axis of rotation.

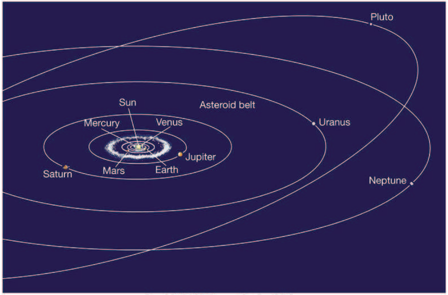
The rotation vector of the sun is unreliable.

- In heliocentric coordinates, the $\hat{z}$ vector is normal to the ecliptic plane.

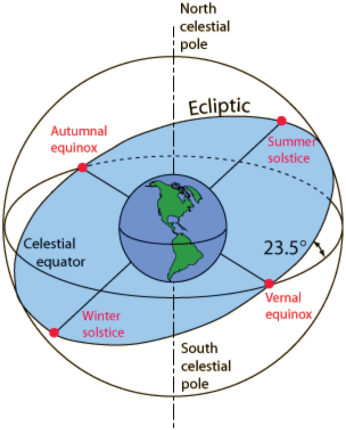
Definition 2.

The **Ecliptic Plane** is the orbital plane of the earth in motion about the sun.

- From the earth, the ecliptic plane is defined by the apparent motion of the sun about the earth.
  - Determined by the location of eclipses (hence the name).



The Ecliptic Plane



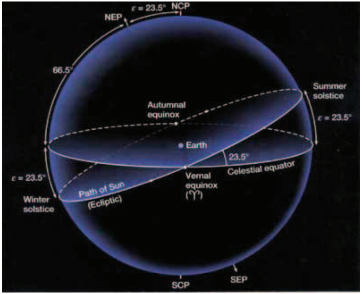
Definition 3.

The **Inclination to the Ecliptic** is the angle between the equatorial and ecliptic planes.

Currently, the inclination to the ecliptic is $23.5\,\mathrm{deg}$.

ECI: The First Point of Aries

- To define the ECI coordinate system, we define $\hat{x}$ axis in the equatorial plane.
- The $\hat{y}$ axis is then given by the right-hand rule.



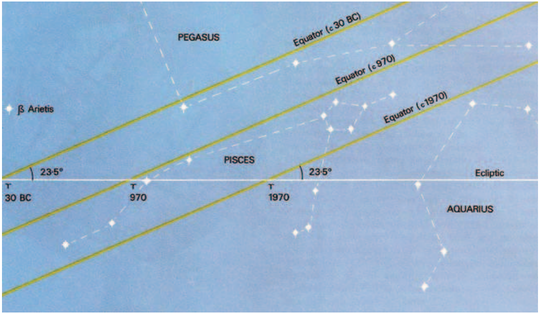
A convenient choice for $\hat{x}$ is the intersection of the equatorial and ecliptic planes. But there are two such points, at the two equinoxes (vernal and autumnal)

The **First Point of Aries** is the earth-sun vector at the vernal equinox.

Question Does the FPOA lie at the ascending or descending node?

ECI: The First Point in Aries

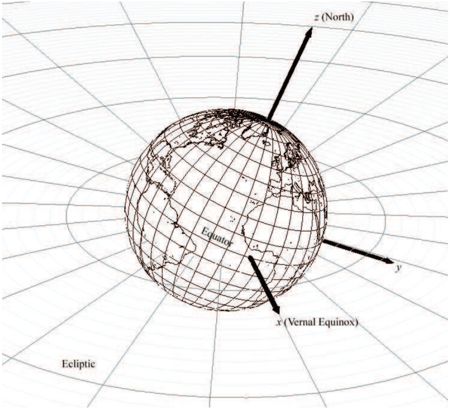
- The First Point of Aries is so named because this direction used to point towards the Constellation Aries.
- Precession of the earth's rotation vector means the FPOA now actually points toward Pisces.



- Since Motion of the FPOA is caused be precession, its motion is **Periodic**, not Secular.
    - The Period is about 26,000 years.
- The Coordinate System is not truly inertial.

Summary: The ECI frame

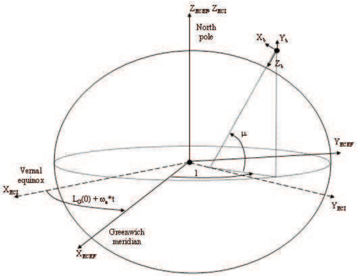
- $\hat{z}$ - North Pole
- $\hat{x}$ - FPOA
- $\hat{y}$ - Right Hand Rule



Because the FPOA migrates with time, positions given in ECI must be referenced to a year

- **J2000** - frame as defined at 12:00 TT on Jan 1, 2000.
- **TOD** - True of Data: date is listed explicitly.

Other Reference Frames



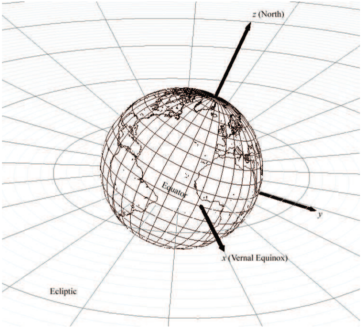
Note there are many other reference frames os interest

- Earth Centered Earth Fixed
- Topocentric Horizon
- Topocentric Equatorial
- Gaussian (Satellite Radial)
- Frenet System (Satellite Normal, Drag)
- Equinoctal

We will return to some of these frames when necessary.

Orbital Elements

Now that we have our coordinate system,



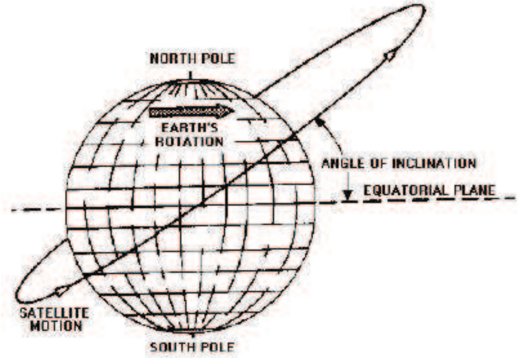
Question:, Suppose we are given $\vec{r}$ and $\vec{v}$ in the ECI frame. How to describe the orientation of the orbit?

Answer: 3 new orbital elements.

- Inclination
- Right Ascension
- Argument of Periapse

The Orbital Plane

Inclination, i

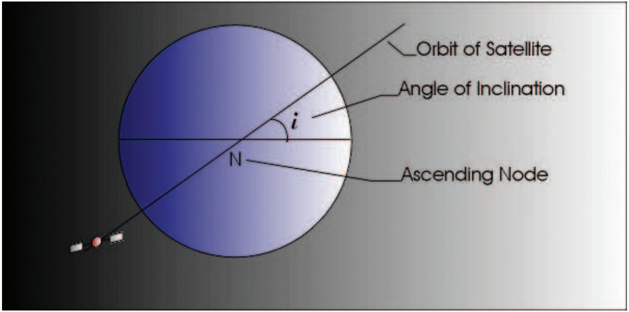


Angle the orbital plane makes with the reference plane.
The orbit is

- **Prograde** if $0 < i < 90°$.
- **Retrograde** if $90 < i < 180°$.

The Orbital Plane

Inclination, i

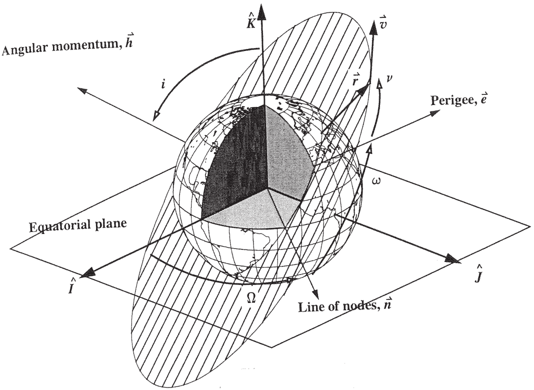


Inclination can be found from $\vec{h}$ as

$$\vec{h} \cdot \hat{z} = h \cos i.$$

- If $\vec{h}$ is defined in ECI, then $i = \cos^{-1} \frac{h_3}{h}$.
- No quadrant ambiguity because by definition, $i \leq 180\,\mathrm{deg}$

The Line Of Nodes



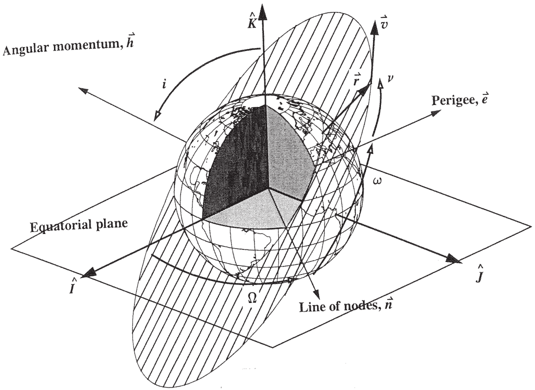
An important vector in defining the orbit is the line of nodes.

Definition 4.

The **Line of Nodes** is the vector pointing to where the satellite crosses the equatorial plane from the southern to northern hemisphere.

$$\vec{n} = \hat{z} \times \vec{h}$$

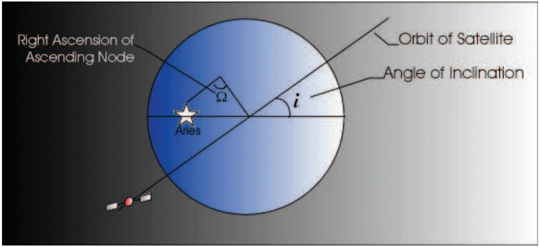
The Line Of Nodes



- Lies at the intersection of the equatorial and orbital planes.
- Points toward the Ascending Node.
- Zero for equatorial orbits ($i = 0$).

The Orbital Plane

Right Ascension of Ascending Node, Ω



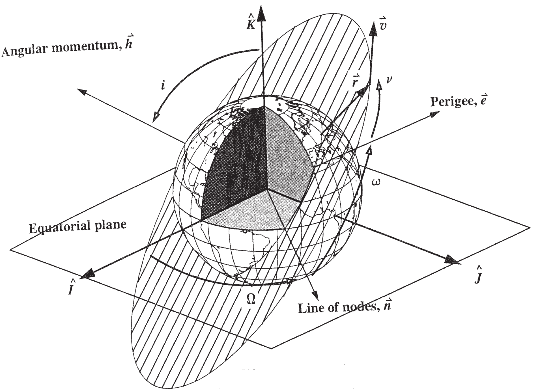
The Angle measured from reference direction, $\hat{x}$ in the reference plane to *ascending node*.

- Defined to be $0 \leq \Omega \leq 360$
- Undefined for equatorial orbits ($i = 0$).

RAAN can be found from the line of nodes as

$$\cos(\Omega) = \frac{\hat{x} \cdot \vec{n}}{\|\vec{n}\|}$$

Must resolve quadrant ambiguity.

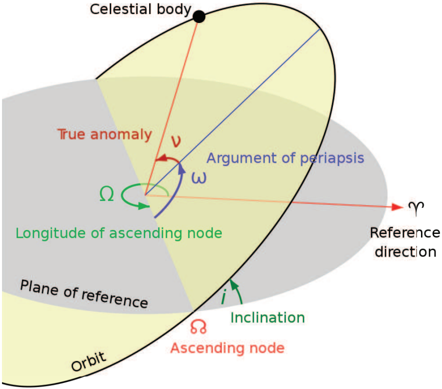


Quadrant Ambiguity: Calculators assume Ω is in quadrant 1 or 2. Correct as

$$\Omega = \begin{cases} \Omega & \hat{y} \cdot \vec{n} \geq 0 \\ 360 - \Omega & \hat{y} \cdot \vec{n} < 0 \end{cases}$$

Argument of Periapse, ω

- Undefined for *Circular* Orbits.
- define so $0 \leq \omega < 360\,\mathrm{deg}$



Definition 5.

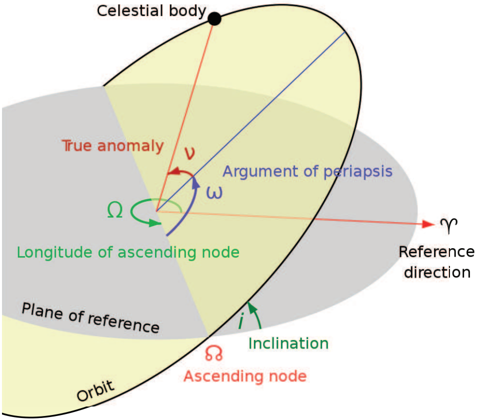
The **Argument of Periapse** is the angle from line of nodes to the point of periapse.

Argument of Periapse, ω

Can be calculated from

$$\cos(\omega) = \frac{\vec{n} \cdot \vec{e}}{\|\vec{n}\| e}$$

Must resolve quadrant ambiguity



Quadrant Ambiguity: Calculators assume ω is in quadrant 1 or 2. Correct as

$$\omega = \begin{cases} \omega & \hat{z} \cdot \vec{e} \geq 0 \\ 360 - \omega & \hat{z} \cdot \vec{e} < 0 \end{cases}$$

Summary: Visualization

Example: Finding Orbital Elements

Problem: Suppose we observe an object in the ECI frame at position

$$\vec{r} = \begin{bmatrix} 6524.8 \\ 6862.8 \\ 6448.3 \end{bmatrix} km \qquad \text{moving with velocity} \qquad \vec{v} = \begin{bmatrix} 4.901 \\ 5.534 \\ -1.976 \end{bmatrix} km/s$$

Determine the orbital elements.

Solution: Although not necessary, as per your homework, lets first convert to canonical units ($1ER = 6378.14km$, $1TU = 806.3s$).

$$\vec{r}' = \frac{\vec{r}}{6378.14km} = \begin{bmatrix} 1.023 \\ 1.076 \\ 1.011 \end{bmatrix}$$

$$\vec{v}' = \vec{r}\frac{806.8s}{6378.14km} = \begin{bmatrix} .62 \\ .7 \\ -.25 \end{bmatrix}$$

First, lets construct angular momentum, $\vec{h}$, the line of nodes, $\vec{n}$ and the eccentricity vector, $\vec{e}$.

Example: Finding Orbital Elements

Continued

We construct $\vec{h}$, $\vec{n}$ and $\vec{e}$.

$$\vec{h} = \vec{r} \times \vec{v} = \begin{bmatrix} -.9767 \\ .882 \\ .049 \end{bmatrix} \frac{ER^2}{TU}$$

Since $\vec{r}$ and $\vec{v}$ are in ECI coordinates,

$$\vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \vec{h} = \begin{bmatrix} -.882 \\ -.9767 \\ 0 \end{bmatrix} \frac{ER^2}{TU}.$$

$$\vec{e} = \frac{1}{\mu} \vec{v} \times \vec{h} - \frac{\vec{r}}{r} = \begin{bmatrix} -.315 \\ -.385 \\ .668 \end{bmatrix}$$

where recall $\mu = 1$ in canonical units.

Now we begin solving for orbital elements.

$$e = \|\vec{e}\| = .8328$$

Use energy to calculate a.

$$E = \frac{v^2}{2} - \frac{\mu}{r} = -.088$$
$$a = -\frac{\mu}{2E} = 5.664ER$$
$$p = \frac{h^2}{\mu} = 1.735ER$$

We can now calculate our three new orbital elements as indicated. Start with inclination

$$i = \cos^{-1}\left(\frac{\vec{h}}{h} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = 87.9\,\mathrm{deg}$$

No quadrant ambiguity by definition.

Example: Finding Orbital Elements
Continued

Continue with RAAN, we want the angle between $\hat{x}$ and $\vec{n}$.

$$\Omega = \cos^{-1}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\frac{\vec{n}}{\|\vec{n}\|}\right) = \pm 132.10\,\mathrm{deg}$$

Because $\cos$ has quadrant ambiguity, we must check the quadrant. Specifically, we need the sign of

$$\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\vec{n} = -.9767 < 0$$

Therefore, $\vec{n}$ is in the *third* quadrant, and we need to correct

$$\Omega = 360 - 132.10 = 227.9\,\mathrm{deg}$$

Example: Finding Orbital Elements
Continued

Next, the argument of perigee is the angle between $\vec{e}$ and $\vec{n}$.

$$\omega = \cos^{-1}\left(\frac{\vec{n}\cdot\vec{e}}{\|\vec{n}\|e}\right) = \pm 53.4\,\text{deg}$$

We resolve the quadrant ambiguity be checking

$$\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\vec{e} = .668 > 0$$

so we are in the right quadrant

$$\omega = 53.4\,\text{deg}$$

Example: Finding Orbital Elements

Continued

Finally, we solve for true anomaly. But this is simply the angle between $\vec{r}$ and $\vec{e}$, so we can use

$$f = \cos^{-1}\left(\frac{\vec{r}\cdot\vec{e}}{re}\right) = \pm 92.3\,\mathrm{deg}$$

We resolve the quadrant ambiguity by checking

$$\vec{r}\cdot\vec{v} > 0$$

So we are in the right quadrant

$$f = 92.3\,\mathrm{deg}$$

Summary

This Lecture you have learned:

The Orbital Plane

- Inclination
- Right Ascension
- Argument of Periapse

New Concepts

- The Earth-Centered Inertial reference frame
- The line of nodes

Practice

- How to construct all orbital elements from $\vec{r}$ and $\vec{v}$
- A Numerical Illustration