Spacecraft and Aircraft Dynamics

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Lecture 8: Impulsive Orbital Maneuvers

Introduction

In this Lecture, you will learn:

Coplanar Orbital Maneuvers

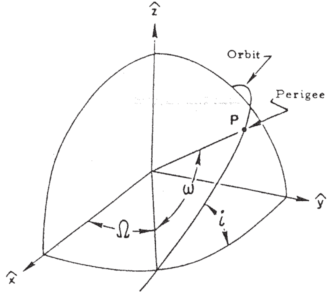
- Impulsive Maneuvers
  - Δv
- Single Burn Maneuvers
- Hohmann transfers
  - Elliptic
  - Circular

Numerical Problem: Suppose we are in a circular parking orbit at an altitude of 191.34km and we want to raise our altitude to 35,781km. Describe the required orbital maneuvers (time and Δv).

Changing Orbits

Suppose we have designed our ideal orbit.

- We have chosen a, e, i, Ω, ω
- For now, we don't care about f
  - Lambert's Problem
- Don't care about efficiency



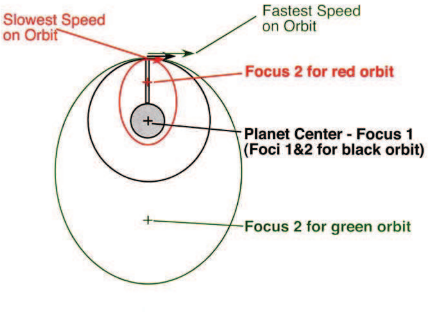
Question:

- Given an object with position, $\vec{r}$ and velocity $\vec{v}$.
- How to move the object into a desired orbit?

Unchanged, the object will remain in initial orbit indefinitely.

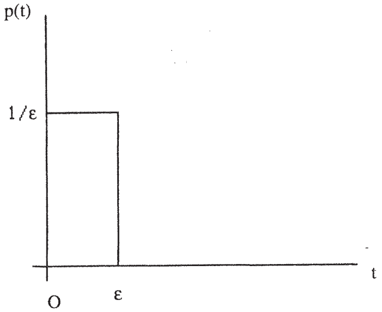
Impulsive Orbit Maneuvers

Orbit maneuvers are made through changes in velocity.



- $\vec{r}$ and $\vec{v}$ determine orbital elements.
- For fixed $\vec{r}$, changes in $\vec{v}$ map to changes in orbital elements.
  - Set of achievable orbits is limited.
  - Only 3 degrees of freedom.
  - Orbit must pass through $\vec{r}$.

Impulsive Orbit Maneuvers



Velocity change is caused by thrust.

- For constant thrust, F,

$$v(t) = v(0) + \frac{F}{m}\Delta t$$

- for a desired Δv, the time needed is

$$\Delta t = \frac{m\Delta v}{F}$$

The change in position is

$$\Delta \vec{r}(t) = \frac{m\Delta v^2}{F}$$

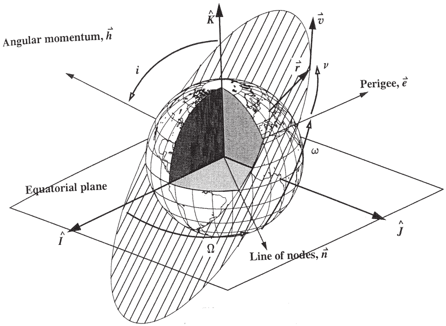
- For fixed Δv, if $\frac{m}{F}$ is small, the $\Delta \vec{r}$ is small
- We will assume $\Delta \vec{r} = 0$

The Δv Maneuver

Δv refers to the difference between the initial and final velocity vectors.

A Δv maneuver can:

- Raise/lower the apogee/perigee
- A change in inclination
- Escape
- Reduction/Increase in period
- Change in RAAN
- Begin a 2+ maneuver sequence of burns.
  - Creates a **Transfer Orbit**.

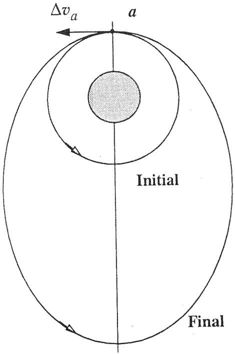


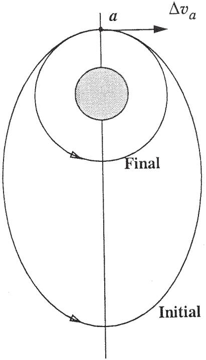
We'll start by talking about coplanar maneuvers.

Single Burn Coplanar Maneuvers

Definition 1.

Coplanar Maneuvers are those which do not alter i or Ω.





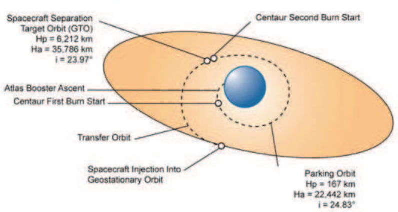
Example: Simple Tangential Burn

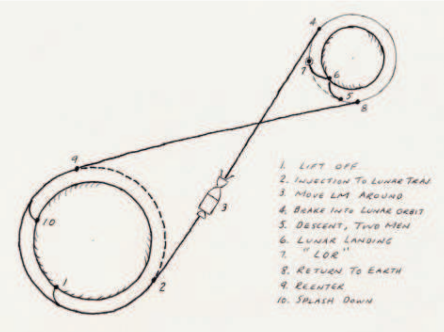
- For maximim efficiency, a burn must occur at 0° flight path angle
  - $\dot{r} = 0$
- Tangential burns can occur at perigee and apogee

Example: Parking Orbits

Suppose we launch from the surface of the earth.

- This creates an initial elliptic orbit which will re-enter.
- To circularize the orbit, we plan on using a burn at apogee.





Problem: We are given a and e of the initial elliptic orbit. Calculate the Δv required at apogee to circularize the orbit.

Example: Parking Orbits

Solution: At apogee, we have that

$$r_a = a(1+e)$$

From the vis-viva equation, we can calculate the velocity at apogee.

$$v_a = \sqrt{\mu\left(\frac{2}{r_a} - \frac{1}{a}\right)} = \sqrt{\frac{\mu}{a}\left(\frac{1-e}{(1+e)}\right)}$$

However, for a circular orbit at the same point, we calculate from vis-viva

$$v_c = \sqrt{\frac{\mu}{r_a}} = \sqrt{\frac{\mu}{a(1+e)}}$$

Therefore, the Δv required to circularize the orbit is

$$\Delta v = v_c - v_a = \sqrt{\frac{\mu}{a(1+e)}} - \sqrt{\frac{\mu}{a}\left(\frac{1-e}{(1+e)}\right)} = \frac{\mu}{a(1+e)}\left(1 - \sqrt{1-e}\right)$$

- It is unusual to launch directly into the desired orbit.
- Instead we use the parking orbit while waiting for more complicated orbital maneuvers.

Coplanar Two-Impulse Orbit Transfers

Most orbits cannot be achieved using a single burn.

Definition 2.

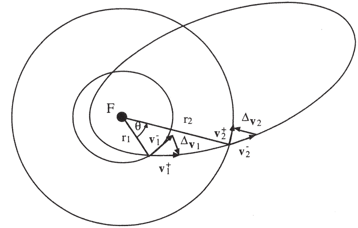
- The **Initial Orbit** is the orbit we want to leave.
- The **Target Orbit** is the orbit we want to achieve.
- The **Transfer Orbit** is an orbit which intersects both the initial orbit and target orbit.

Step 1: Design a transfer orbit (a,e,i, etc.).

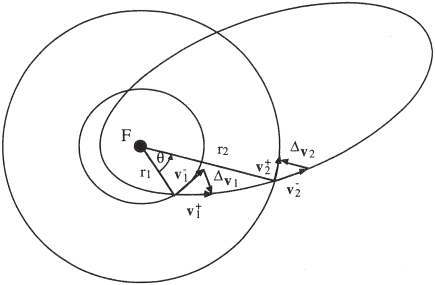
Step 2: Calculate $\vec{v}_{tr,1}$ at the point of intersection with initial orbit.

Step 3: Calculate initial burn to maneuver into transfer orbit.

$$\Delta v_1 = \vec{v}_{tr,1} - \vec{v}_{init}$$



Coplanar Two-Impulse Orbit Transfers



Step 4: Calculate $\vec{v}_{tr,2}$ at the point of intersection with target orbit.

Step 5: Calculate velocity of the target orbit, $\vec{v}_{fin}$, at the point of intersection with transfer orbit.

Step 6: Calculate the final burn to maneuver into target orbit.

$$\Delta v_2 = \vec{v}_{fin} - \vec{v}_{tr,2}$$

Transfer Orbits

There are many orbits which intersect both the initial and target orbits.

However, there are some constraints.

Consider

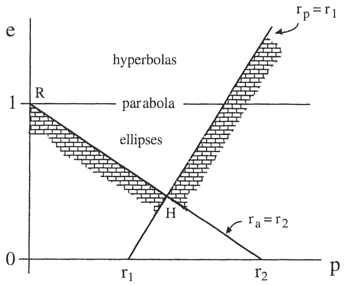
- Circular initial orbit of radius r_2
- Circular target orbit of radius r_1

Obviously, the transfer orbit must satisfy

$$r_p = \frac{p}{1+e} \le r_1$$

and

$$r_a = \frac{p}{1-e} \ge r_2$$



Transfer Orbits in Fixed Time

Lambert's Problem

Occasionally, we want to arrive at

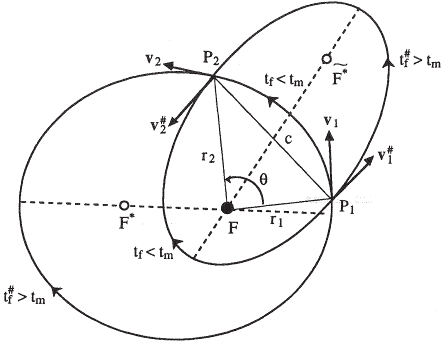
- A certain point in the target orbit, $\vec{r}_2$
- at a certain time, t_f

Finding the necessary transfer orbit is **Lambert's Problem**.

Primary Applications are:

- Targeting
- Rendez-vous

We are skipping the section on Lambert's problem.



Transfer Costs

The cost of a transfer orbit can be calculated using kinetic energy arguments

$$E_{cost} = \frac{\|\Delta v_1\|^2 + \|\Delta v_2\|^2}{2}$$

Of course, this doesn't tell us how good the transfer orbit is.

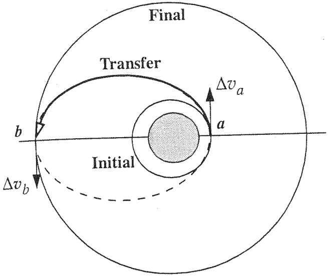
- The energy difference between 2 orbits must come from somewhere.

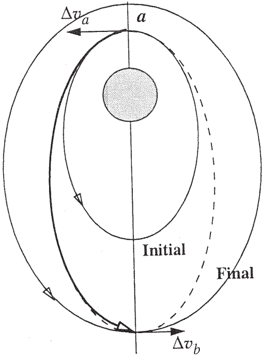
$$\Delta E_{\min} = -\frac{\mu}{2a_2} + \frac{\mu}{2a_1}$$

- The closer E_{cost} is to $E_{\min}$, the more efficient the transfer
- More on this effect later

The Hohmann Transfer

The Hohmann transfer is the energy-optimal two burn maneuver between any two coaxial elliptic orbits.

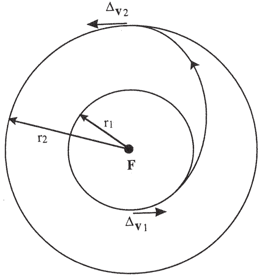




- Proposed by Hohmann (1925)
  - Why?
- Proven for circular orbits by Lawden (1952)
- Proven for coaxial ellipses by Thompson (1986)

The Hohmann Transfer

We will first consider the circular case.



Theorem 3 (The Hohmann Conjecture).

The energy-optimal transfer orbit between two circular orbits of radii r_1 and r_2 is an elliptic orbit with

$$r_p = r_1 \qquad \textit{and} \qquad r_a = r_2$$

This yields the orbital elements of the transfer orbit (a, e) as

$$a = \frac{r_1 + r_2}{2} \qquad \text{and} \qquad e = 1 - \frac{r_p}{a} = \frac{r_2 - r_1}{r_2 + r_1}$$

The Hohmann Transfer

To calculate the required Δv_1 and Δv_2, the initial velocity is the velocity of a circular orbit of radius r_1

$$v_{init} = \sqrt{\frac{\mu}{r_1}}$$

The required initial velocity is that of the transfer orbit at perigee. From the vis-viva equation,

$$v_{trans,p} = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} = \sqrt{2\mu}\sqrt{\frac{1}{r_1} - \frac{1}{r_1 + r_2}} = \sqrt{2\mu \frac{r_2}{r_1(r_1 + r_2)}}$$

So the initial Δv_1 is

$$\Delta v_1 = v_{trans,p} - v_{init} = \sqrt{2\mu \frac{r_2}{r_1(r_1 + r_2)}} - \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{(r_1 + r_2)}} - 1\right)$$

The velocity of the transfer orbit at apogee is

$$v_{trans,a} = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a}} = \sqrt{2\mu \frac{r_1}{r_2(r_1 + r_2)}}$$

The Hohmann Transfer

The required velocity for a circular orbit at apogee is

$$v_{fin} = \sqrt{\frac{\mu}{r_2}}$$

So the final Δv_2 is

$$\Delta v_2 = v_{fin} - v_{trans,a} = \sqrt{\frac{\mu}{r_2}} - \sqrt{2\mu \frac{r_1}{r_2(r_1 + r_2)}} = \sqrt{\frac{\mu}{r_2}} \left(1 - \sqrt{\frac{2r_1}{(r_1 + r_2)}}\right)$$

Thus we conclude to raise a circular orbit from radius r_1 to radius r_2, we use

$$\Delta v_1 = \sqrt{\frac{\mu}{r_1}} \left(\sqrt{\frac{2r_2}{(r_1 + r_2)}} - 1\right)$$

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}} \left(1 - \sqrt{\frac{2r_1}{(r_1 + r_2)}}\right)$$

Hohmann Transfer Illustration

The Hohmann Transfer

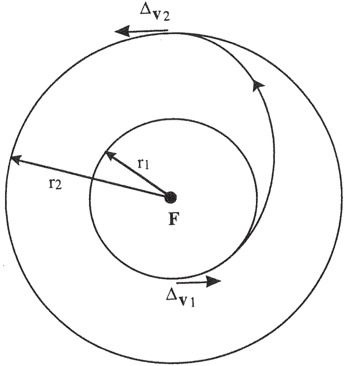
Transfer Time

The Hohmann transfer is optimal

- Only for impulsive transfers
  - Continuous Thrust is not considered
- Only for **two** impulse transfers
  - A three impulse transfer can be better
  - Bi-elliptics are better

The transfer time is simply half the period of the orbit. Hence

$$\Delta t = \frac{\tau}{2} = \pi\sqrt{\frac{a^3}{\mu}}$$
$$= \pi\sqrt{\frac{(r_1 + r_2)^3}{2\mu}}$$



The Hohmann transfer is also the *Maximum Time 2-impulse Transfer*.

- Always a tradeoff between time and efficiency
- Bielliptic Transfers extend this tradeoff.

Numerical Example

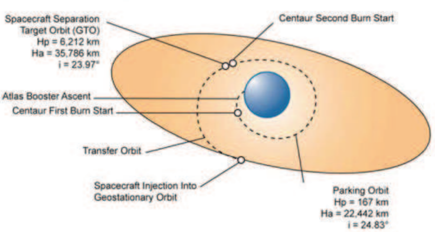
Problem: Suppose we are in a circular parking orbit at an altitude of 191.34km and we want to raise our altitude to 35,781km. Describe the required orbital maneuvers (time and Δv).

Solution: We will use a Hohmann transfer between circular orbits of

$r_1 = 191.35km + 1ER = 1.03ER$ and $r_2 = 35781km + 1ER = 6.61ER$

The initial velocity is

$$v_i = \sqrt{\frac{\mu}{r_1}} = .985\frac{ER}{TU}$$



The transfer ellipse has $a = \frac{r_1+r_2}{2} = 3.82ER$. The velocity at perigee is

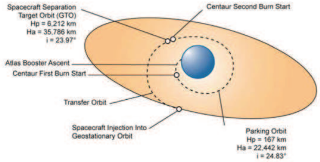
$$v_{trans,1} = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} = 1.296\frac{ER}{TU}$$

Thus the initial Δv is $\Delta v_1 = 1.296 - .985 = .315\frac{ER}{TU}$.

Numerical Example

The velocity at apogee is

$$v_{trans,1} = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a}} = .202\frac{ER}{TU}$$



However, the required velocity for a circular orbit at radius r_2 is

$$v_f = \sqrt{\frac{\mu}{r_2}} = .389\frac{ER}{TU}$$

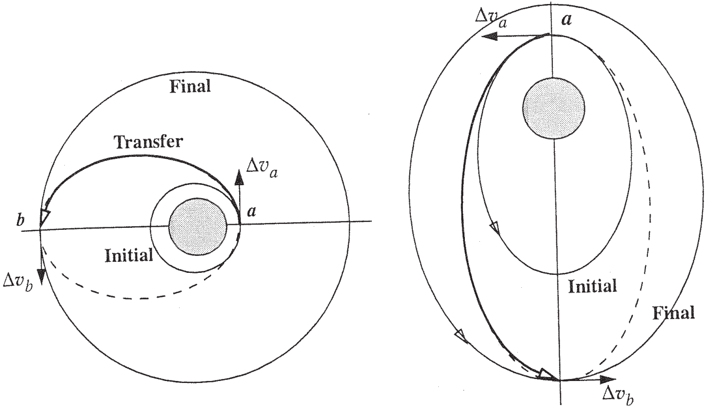
Thus the final Δv is $\Delta v_2 = .389 - .202 = .182\frac{ER}{TU}$. The second Δv maneuver should be made at time

$$t_{fin} = \pi\sqrt{\frac{a^3}{\mu}} = 23.45TU = 5.256hr$$

The total Δv budget is $.497ER/TU$.

The Elliptic Hohmann Transfer

The Hohmann transfer is also energy optimal for coaxial elliptic orbits.



The only ambiguity is whether to make the initial burn at perigee or apogee.

- Need to check both cases
- Often better to make initial burn at perigee
  - Due to Oberth Effect

Summary

This Lecture you have learned:

Coplanar Orbital Maneuvers

- Impulsive Maneuvers
  - Δv
- Single Burn Maneuvers
- Hohmann transfers
  - Elliptic
  - Circular

Next Lecture: Oberth Effect, Bi-elliptics, Out-of-plane maneuvers.