Systems Analysis and Control

Matthew M. Peet
Illinois Institute of Technology

Lecture 19: Drawing Bode Plots, Part 1

In this Lecture, you will learn:

Drawing Bode Plots

- Drawing Rules

Simple Plots

- Constants
- Real Zeros

Review

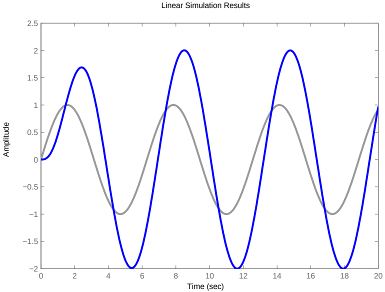
Recall from last lecture: Frequency Response

Input:

$$u(t) = M \sin(\omega t + \phi)$$

Output: Magnitude and Phase Shift

$$y(t) = |G(\imath\omega)| M \sin(\omega t + \phi + \angle G(\imath\omega))$$



Frequency Response to $\sin \omega t$ is given by $G(\imath\omega)$

Bode Plots

We know $G(\imath\omega)$ determines the frequency response.
How to plot this information?

- 1 independent Variable: ω
- 2 Dependent Variables: $Re(G(\imath\omega))$ and $Im(G(\imath\omega))$

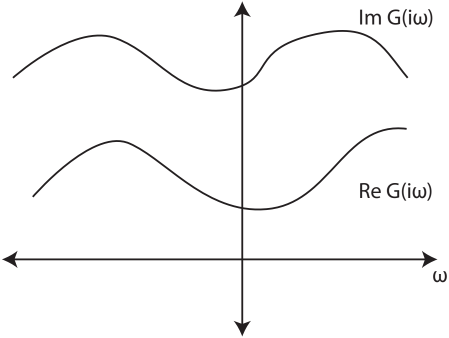


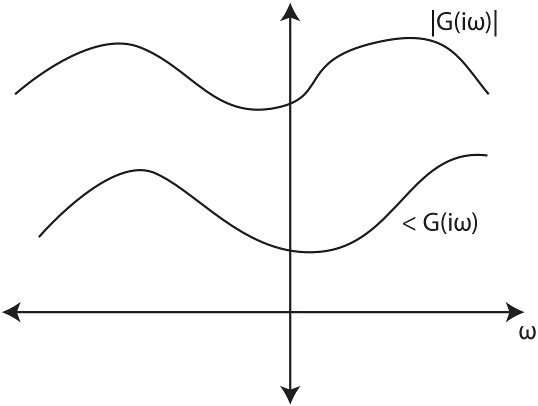
Figure: The Obvious Choice

Really 2 plots put together.

Bode Plots

An Alternative is to plot Polar Variables

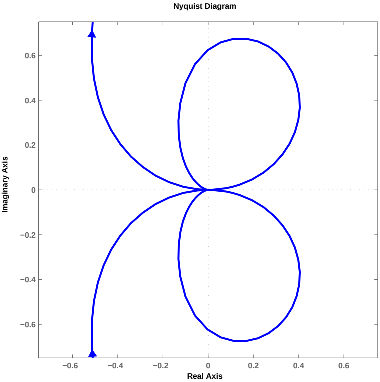
- 1 independent Variable: ω
- 2 Dependent Variables: $\angle G(\imath\omega)$ and $|G(\imath\omega)|$



- Advantage: All Information corresponds to physical data.
  - Can be found directly using a frequency sweep.

Bode Plots

If we only want a single plot we can use ω as a *parameter*.



A plot of $Re(G(\imath\omega))$ vs. $Im(G(\imath\omega))$ as a function of ω.

- Advantage: All Information in a single plot.
- AKA: Nyquist Plot

Bode Plots

We focus on Option 2.

Definition 1.

The Bode Plot is a pair of $\log$-$\log$ and semi-$\log$ plots:

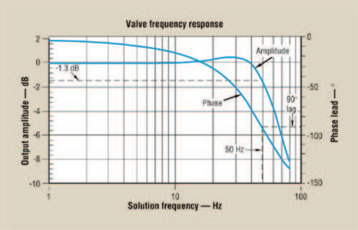
1. Magnitude Plot: $20\log_{10}|G(\imath\omega)|$ vs. $\log_{10}\omega$
2. Phase Plot: $\angle G(\imath\omega)$ vs. $\log_{10}\omega$

$20\log_{10}|G(\imath\omega)|$ is units of **Decibels (dB)**

- Used in Power and Circuits.
- $10\log_{10}|\cdot|$ in other fields.

Note that by $\log$, we mean $\log$ base 10 ($\log_{10}$)

- In Matlab, `log` means natural logarithm.

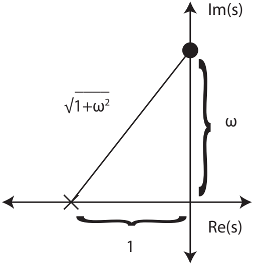


Lets do a simple pole

$$G(s) = \frac{1}{s+1}$$

We need

- Magnitude of $G(\imath\omega)$
- Phase of $G(\imath\omega)$



Recall that

$$|G(s)| = \frac{|s-z_1|\cdots|s-z_m|}{|s-p_1|\cdots|s-p_n|}$$

So that

$$|G(\imath\omega)| = \frac{1}{|\imath\omega+1|} = \frac{1}{\sqrt{1+\omega^2}}$$

Bode Plots

Example

How to Plot $|G(\imath\omega)| = \frac{1}{\sqrt{1+\omega^2}}$?
We are actually want to plot

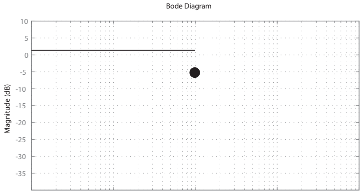
$$20\log|G(\imath\omega)| = 20\log\frac{1}{\sqrt{1+\omega^2}} = 20\log(1+\omega^2)^{-\frac{1}{2}} = -10\log(1+\omega^2)$$

Three Cases:

Case 1: $\omega << 1$

- Approximate $1+\omega^2 \cong 1$

$$\begin{aligned}20\log|G(\imath\omega)| &= -10\log(1+\omega^2)\\ &\cong -10\log 1 = 0\end{aligned}$$



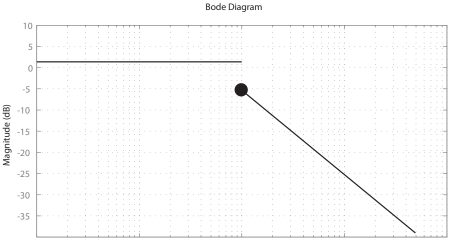
Case 2: $\omega = 1$

$$\begin{aligned}20\log|G(\imath\omega)| &= -10\log(1+\omega^2)\\ &= -10\log(1+\omega^2)\\ &= -3.01\end{aligned}$$

Case 3: $\omega >> 1$

- Approximate: $1+\omega^2 \cong \omega^2$

$$\begin{aligned}20\log|G(\imath\omega)| &= -10\log(1+\omega^2)\\ &\cong -10\log\omega^2\\ &= -20\log\omega\end{aligned}$$



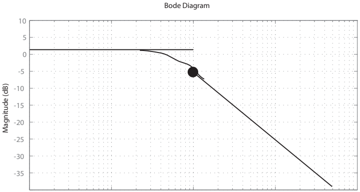
But we use a $\log-\log$ plot.

- x-axis is $x=\log\omega$
- y-axis is $y=20\log|G(\imath\omega)|=-20\log\omega=-20x$

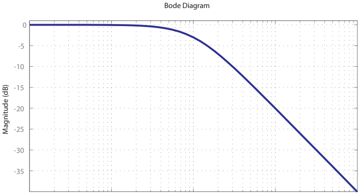
Conclusion: On the log-log plot, when $\omega >> 1$,

- Plot is Linear
- Slope is -20 dB/Decade!

Of course, we need to connect the dots.



Compare to the Real Thing:



Bode Plots

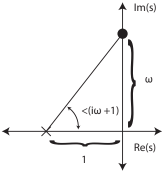
Example: Phase

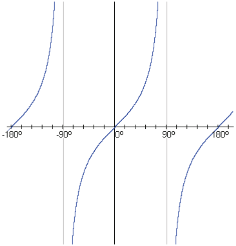
Now lets do the phase. Recall:

$$\angle G(s) = \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i)$$

In this case,

$$\begin{aligned} \angle G(\imath\omega) &= -\angle(\imath\omega + 1) \\ &= -\tan^{-1}(\omega) \end{aligned}$$



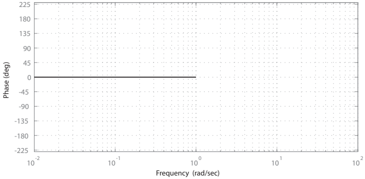


Graph of the Tangent Function

Again, 3 cases:

Case 1: $\omega << 1$

- $\tan(\angle G(\imath\omega)) \cong 0$
- $\tan(\angle G(\imath\omega)) \cong \angle G(\imath\omega) \cong 0$

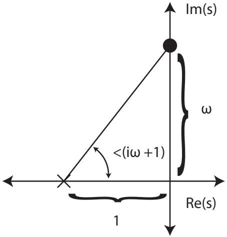


Bode Plots

Example: Phase

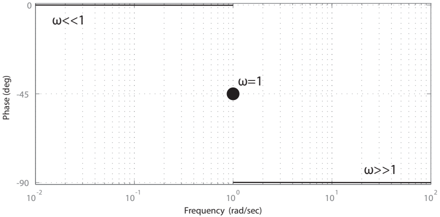
Case 2: $\omega = 1$

- $\tan(\angle G(\imath\omega)) = 1$
- $\angle G(\imath\omega) \cong 45°$

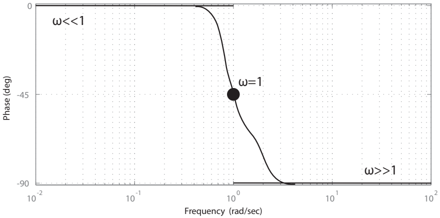


Case 3: $\omega >> 1$

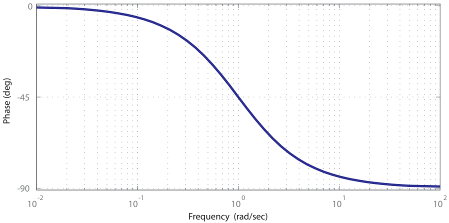
- $\tan(\angle G(\imath\omega)) \cong \frac{1}{0}$
- $\angle G(\imath\omega) \cong -90°$
- Fixed at $-90°$ for large ω!



We need to connect the dots somehow.



Compare to the real thing:



Bode Plots

Methodology

So far, drawing Bode Plots seems pretty intimidating.

- Solving $\tan^{-1}$
- dB and $\log$-plots
- Lots of trig

The process can be Greatly Simplified:

- Use a few simple rules.

Example: Suppose we have

$$G(s) = G_1(s)G_2(s)$$

Then

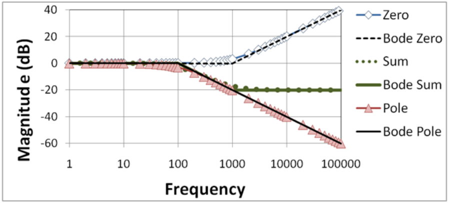
$$|G(\imath\omega)| = |G_1(\imath\omega)||G_2(\imath\omega)|$$

and

$$\log|G(\imath\omega)| = \log|G_1(\imath\omega)| + \log|G_2(\imath\omega)|$$

Rule # 1: Magnitude Plots Add in $\log$-space.
For $G(s) = G_1(s)G_2(s)$,

$$20 \log |G(\imath\omega)| = 20 \log |G_1(\imath\omega)| + 20 \log |G_2(\imath\omega)|$$



Decompose G into bite-size chunks:

$$G(s) = \frac{1}{s+3}(s+1)\frac{1}{s^2+3s+1} = G_1(s)G_2(s)G_3(s)$$

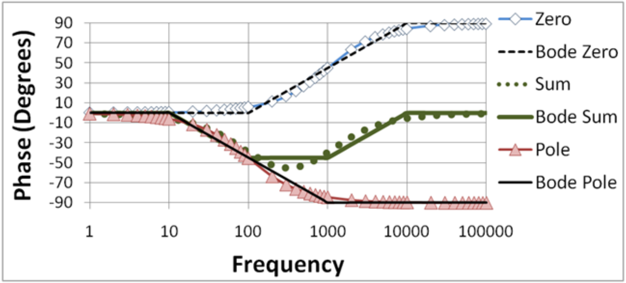
Bode Plots

Rule #2

Rule # 2: Phase Plots Add.

For $G(s) = G_1(s)G_2(s)$,

$$\angle G(\imath\omega) = \angle G_1(\imath\omega) + \angle G_2(\imath\omega)$$



Bode Plots

Approach

Our Approach is to Decompose $G(s)$ into simpler pieces.

- Plot the phase and magnitude of each component.
- Add up the plots.

Step 1: Decompose G into all its poles and zeros

$$G(s) = \frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

Then for magnitude

$$\begin{aligned}20\log|G(\imath\omega)| &= \sum_i 20\log|\imath\omega - z_i| + \sum_i 20\log\frac{1}{|\imath\omega - p_i|}\\ &= \sum_i 20\log|\imath\omega - z_i| - \sum_i 20\log|\imath\omega - p_i|\end{aligned}$$

And for phase:

$$\angle G(\imath\omega) = \sum_i \angle(\imath\omega - z_i) - \sum_i \angle(\imath\omega - p_i)$$

But how to plot $\angle(\imath\omega - z_i)$ and $20\log|\imath\omega - z_i|$?

Plotting Simple Terms

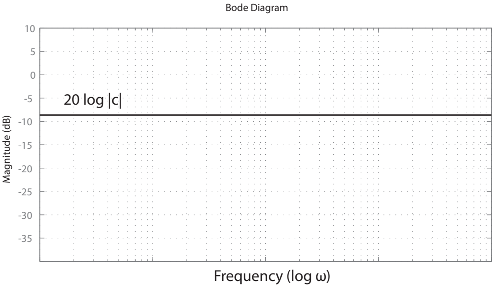
The Constant

Before rushing in, lets make sure we don't forget the constant term. If

$$G(s) = c\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

Magnitude: $G_1(s) = c$

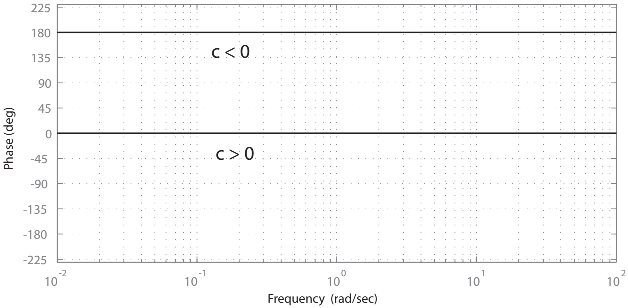
- $|G_1(\imath\omega)| = |c|$
- $20\log|G_1(\imath\omega)| = 20\log|c|$



Conclusion: Magnitude is Constant for all ω

Phase: $G_1(s) = c$

$$\angle G_1(\imath\omega) = \angle c = \begin{cases} 0^\circ & c > 0 \\ 180^\circ & c < 0 \end{cases}$$



Conclusion: phase is 0° if $c > 0$, otherwise 180°.

Lets start with a zero at the origin: $G_1(s) = s$.

Magnitude: $G_1(s) = s$

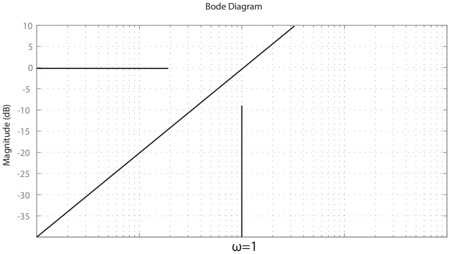
- $|G_1(\imath\omega)| = |\imath\omega| = |\omega|$
- $20\log|G_1(\imath\omega)| = 20\log|\omega|$

Our x-axis is $\log\omega$.

- Plot is Linear for all ω
- Slope is +20 dB/Decade!
- Need a point: $\omega = 1$

$$20\log|G_1(\imath\omega)||_{\omega=1} = 20\log 1 = 0$$

- Passes through 0dB at $\omega = 1$

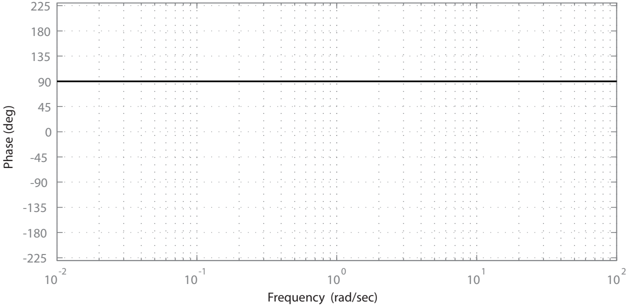


High Gain at High Frequency

- A pure zero means $u'(t)$
- The faster the input, The larger the output

Phase: $G_1(s) = s$

- $\angle G_1(\imath\omega) = \angle \imath\omega = 90^\circ$
- Always 90°!



Always 90° out of phase. Why?

What happens if there are multiple pure zeros

- Just what you would expect.

Magnitude: $G_1(s) = s^k$

- $|G_1(\imath\omega)| = |\imath\omega|^k = |\omega|^k$

$$20\log|G_1(\imath\omega)| = 20\log|\omega|^k$$
$$= 20k\log|\omega|$$

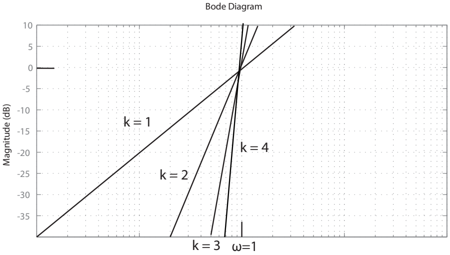
- Slope is +20k dB/Decade!

Need a Point

- At $\omega = 1$:

$$20\log|G_1(\imath\omega)||_{\omega=1} = 20k\log 1 = 0$$

- Still Passes through $0dB$ at $\omega = 1$

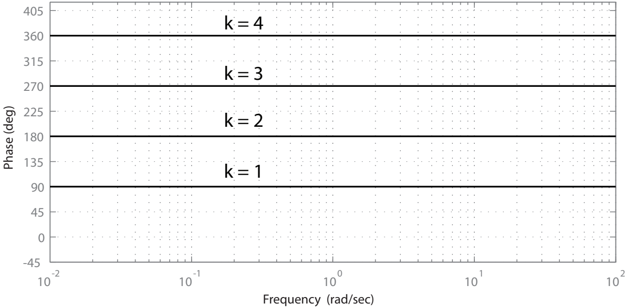


k pure zeros added together.

And phase for multiple pure zeros?

Phase: $G_1(s) = s^k$

- $\angle G_1(\imath\omega) = \angle(\imath\omega)^k = k\angle\imath\omega = 90^\circ k$
- Always $90^\circ k$



k pure zeros added together.

Plotting Simple Terms

Plotting Normal Zeros

A zero at the origin is a line with slope $+20^\circ/$Decade.

- What if the zero is not at the origin?
  - We did one example already ($\frac{1}{s+1}$).

Change of Format: to simplify steady-state response, we use

$$G_1(s) = (\tau s + 1)$$

- Pole is at $s = -\frac{1}{\tau}$
- Also put poles in this form

Rewrite $G(s)$: $(s+p) \to p(\frac{1}{p}s + 1)$.

$$\begin{aligned} G(s) &= k\frac{(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)} \\ &= k\frac{z_1\cdots z_m}{p_1\cdots p_n}\frac{(\frac{1}{z_1}s+1)\cdots(\frac{1}{z_m}s+1)}{(\frac{1}{p_1}s+1)\cdots(\frac{1}{p_n}s+1)} \\ &= c\frac{(\tau_{z1}s+1)\cdots(\tau_{zm}s+1)}{(\tau_{p1}s+1)\cdots(\tau_{pn}s+1)} \end{aligned}$$

Where

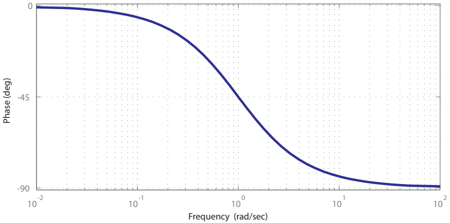
- $\tau_{zi} = \frac{1}{z_i}$
- $\tau_{pi} = \frac{1}{p_i}$
- $c = k\frac{z_1\cdots z_m}{p_1\cdots p_n}$

Assume z_i and p_i are *Real*.

$$G(s) = c\frac{(\tau_{z1}s+1)\cdots(\tau_{zm}s+1)}{(\tau_{p1}s+1)\cdots(\tau_{pn}s+1)}$$

The advantage of this form is that steady-state response to a step is

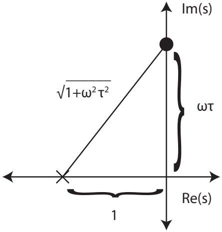
$$y_{ss} = \lim_{s\to 0} G(s) = G(0) = c$$



Low Frequency Response is given by the constant term, c.

$$G_1(s) = (\tau s + 1)$$

$$|G_1(\imath\omega)| = |\imath\omega\tau + 1| = \sqrt{1 + \tau^2\omega^2}$$



Magnitude:

$$20\log|G_1(\imath\omega)| = 20\log(1+\omega^2\tau^2)^{\frac{1}{2}} = 10\log(1+\omega^2\tau^2)$$

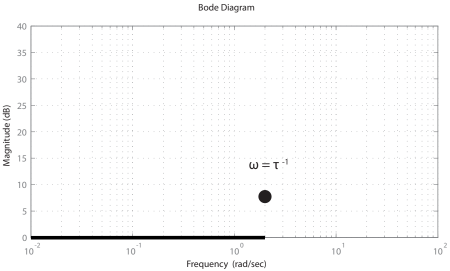
Case 1: $\omega\tau << 1$

- Approximate $1 + \omega^2\tau^2 \cong 1$

$$20\log|G(\imath\omega)| = 20\log(1+\omega^2\tau^2)$$
$$\cong 20\log 1 = 0$$

Case 2: $\omega\tau = 1$

$$20\log|G(\imath\omega)| = 10\log(1+\omega^2\tau^2)$$
$$= 10\log 2 = 3.01$$



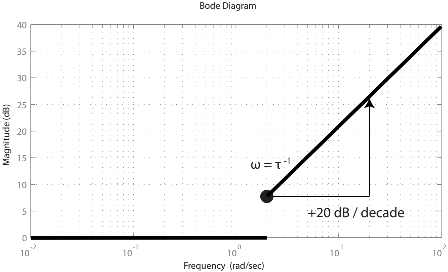
Bode Plots

Example

Case 3: $\omega\tau >> 1$

- Approximate $1 + \omega^2\tau^2 \cong \omega^2\tau^2$

$$\begin{aligned} 20\log|G(\imath\omega)| &= 20\log\sqrt{1+\omega^2\tau^2} \\ &\cong 10\log\omega^2\tau^2 \\ &= 20\log\omega\tau \\ &= 20\log\omega + 20\log\tau \end{aligned}$$

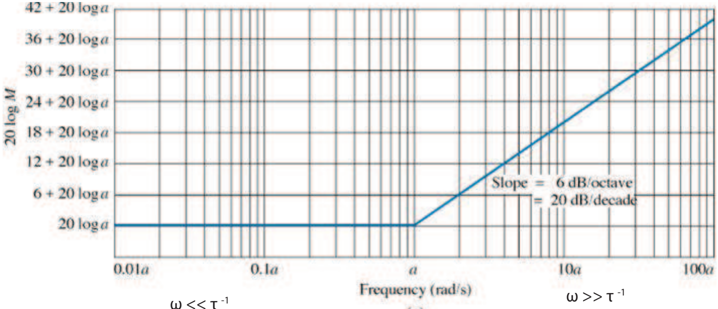


Conclusion: When $\omega >> 1$,

- Plot is Linear
- Slope is +20 dB/Decade!
- inflection at $\omega = \frac{1}{\tau}$

Compare this to the magnitude plot of

$$G_1(s) = s + a$$



(a)

This is why we use the format $G_1(s) = \tau s + 1$

- We want 0dB (no gain) at low frequency.

Summary

What have we learned today?

Drawing Bode Plots

- Drawing Rules

Simple Plots

- Constants
- Real Zeros

Next Lecture: More Bode Plotting