Systems Analysis and Control

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Lecture 23: Drawing The Nyquist Plot

Overview

In this Lecture, you will learn:

Review of Nyquist

Drawing the Nyquist Plot

- Using the Bode Plot
- What happens at $r = \infty$
- Poles on the imaginary axis

Phase Margin and Gain Margin

- Reading Stability Margins off the Nyquist Plot

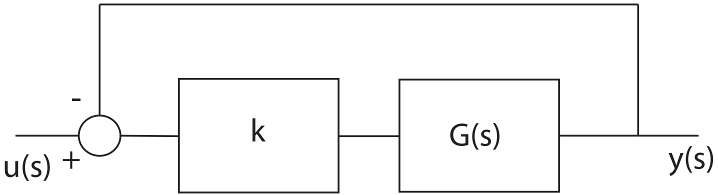
The closed loop is

$$\frac{kG(s)}{1+kG(s)}$$

We want to know when

$$1+kG(s)=0$$

Question: Does $\frac{1}{k}+G(s)$ have any zeros in the RHP?

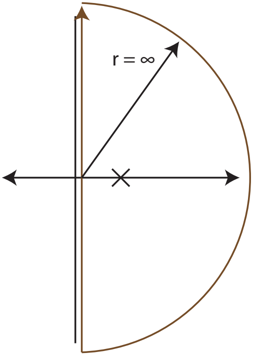


Definition 1.

The **Nyquist Contour**, $\mathcal{C}_N$ is a contour which contains the imaginary axis and encloses the right half-place. The Nyquist contour is clockwise.

A Clockwise Curve

- Starts at the origin.
- Travels along imaginary axis till $r = \infty$.
- At $r = \infty$, loops around clockwise.
- Returns to the origin along imaginary axis.



We want to know if

$$\frac{1}{k} + G(s)$$

has any zeros in the Nyquist Contour

Key Point: For a point on the mapped contour, $s^* = G(s)$,

$$\angle s^* = \angle G(s)$$

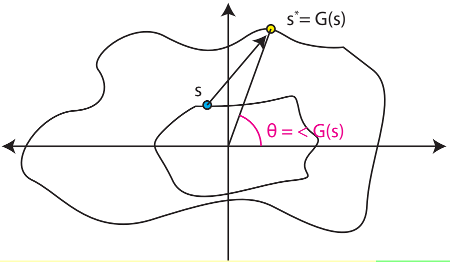
- We measure θ, not phase.

To measure the 360° resets in $\angle G(s)$

- We count the number of $+360^\circ$ resets in θ!
- We count the number of times $\mathcal{C}_G$ encircles the origin **Clockwise**.

The number of clockwise encirclements of 0 is

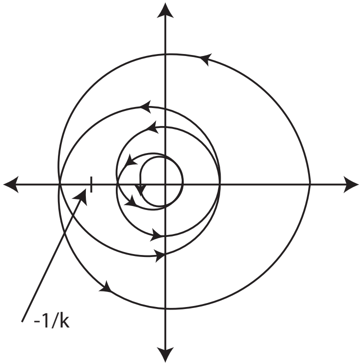
- The $\#_{poles} - \#_{zeros}$ in the RHP



The Nyquist Contour
Closed Loop

The number of unstable closed-loop poles is $N+P$, where

- N is the number of clockwise encirclements of $\frac{-1}{k}$.
- P is the number of unstable open-loop poles.



If we get our data from Bode, typically $P=0$

How to Plot the Nyquist Curve?

Plotting the Nyquist Diagram

Example

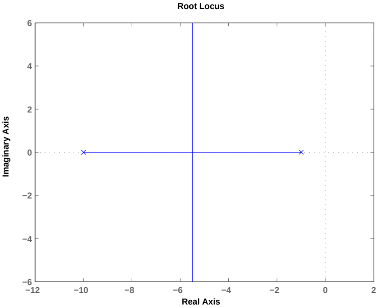
How are we to plot the Nyquist diagram for

$$G(s) = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

- $\tau_1 = 1$
- $\tau_2 = \frac{1}{10}$

First lets take a look at the root locus.

Obviously stable for any $k > 0$.



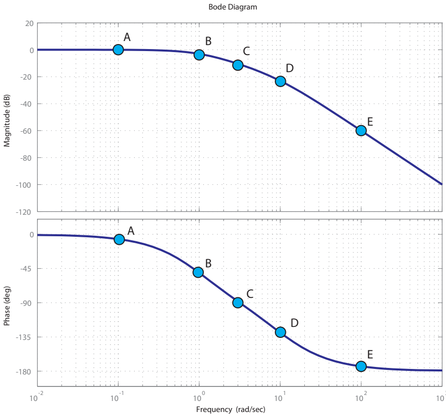
The Nyquist Plot

Bode Plot: Lets look at the Frequency Response.

The Bode plot can give us information on $|G|$ at different frequencies.

Point	ω	$\angle G$	$\|G\|$
A	.1	$0°$	1
B	1	$-45°$	.7
C	3	$-90°$	.3
D	10	$-135°$	.07
E	100	$-175°$	.001

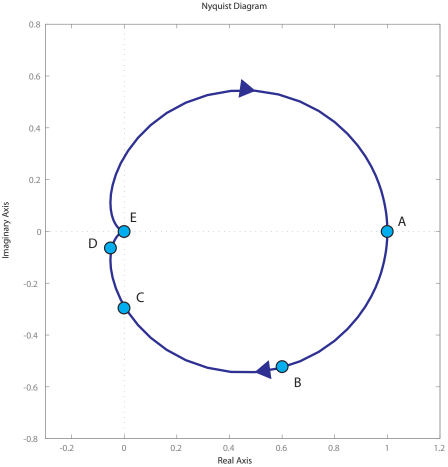
The last two columns give us points on the Nyquist diagram.



Plot the points from the Bode Diagram.

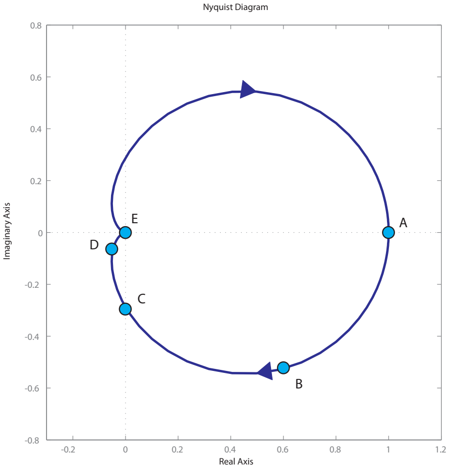
Point	ω	$\angle G$	$\lvert G\rvert$
A	$.1$	$0°$	1
B	1	$-45°$	$.7$
C	3	$-90°$	$.3$
D	10	$-135°$	$.07$
E	100	$-175°$	$.001$

We get



the upper half of the Nyquist diagram from symmetry.

The Nyquist Plot



There are no encirclements of $-\frac{1}{k}$.

- Stable for all $k > 0$.
- We already knew that from Root Locus.

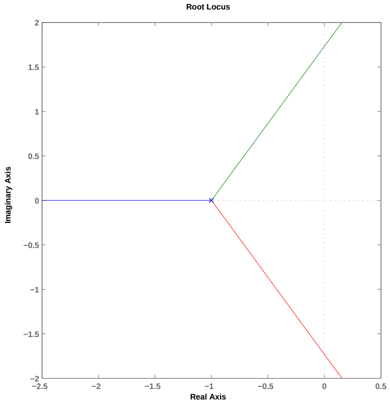
The Nyquist Plot

Example 2

$$G(s) = \frac{1}{(s+1)^3}$$

First lets take a look at the root locus.

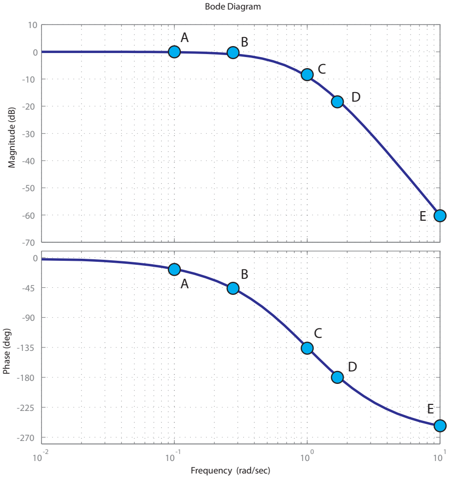
We expect instability for large k.



Bode Plot: Lets look at the Frequency Response.

The Bode plot can give us information on $|G|$ at different frequencies.

Point	ω	$\angle G$	$\|G\|$
A	.1	$0°$	1
B	.28	$-45°$	.95
C	1	$-135°$	.35
D	1.8	$-180°$	.1
E	10	$-260°$	.001

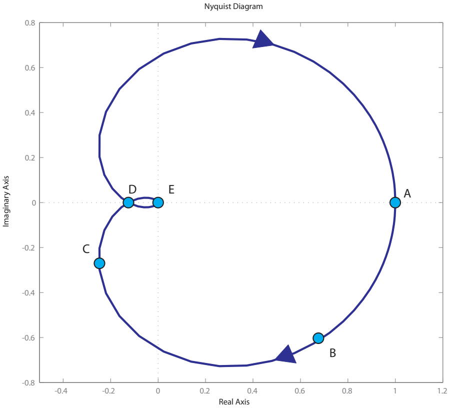


The Nyquist Plot

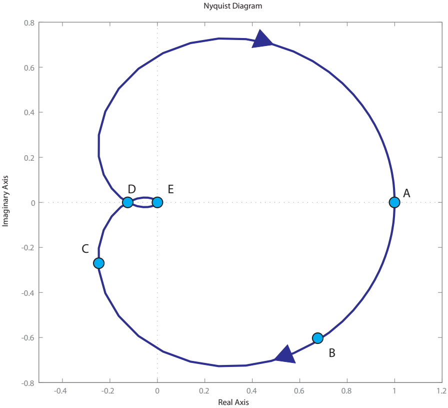
Plot the points from the Bode Diagram.

Point	ω	$\angle G$	$\|G\|$
A	$.1$	$0°$	1
B	$.28$	$-45°$	$.95$
C	1	$-135°$	$.35$
D	1.8	$-180°$	$.1$
E	10	$-260°$	$.001$

Point D is especially important.



The Nyquist Plot



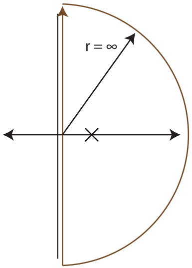
Point D: Two CW encirclements when $-\frac{1}{k} < -.1$ (N=2).

- Instability for $-\frac{1}{k} < -.1$
- Stable for $k < 10$.
- Could have used Routh Table.

The Nyquist Plot

Conclusion: We can use the Bode Plot to map the imaginary axis onto the Nyquist Diagram.

Question: What about the other part of the Nyquist contour at $r = \infty$?



Case 1: Strictly Proper.

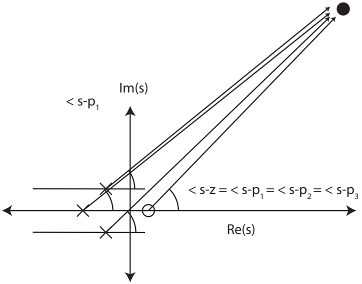
$$\lim_{s \to \infty} |G(s)| = 0$$

What happens at ∞ doesn't matter.

Case 2: Not Strictly Proper.

$$\lim_{s \to \infty} |G(s)| = c$$

Constant Magnitude at ∞.



The Nyquist Plot

Case 2: Not Strictly Proper.

- Angle to all poles and zeros is the same.
- Degree of $n(s)$ and $d(s)$ the same.
  - Number of Poles and Zeros the same.

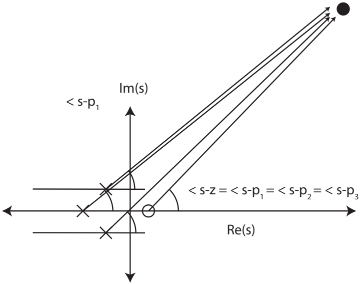
The total angle is

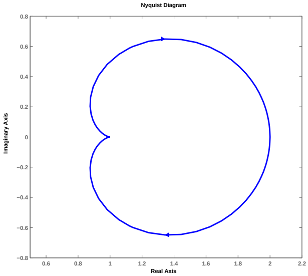
$$\angle G(s) = \sum_{i=1}^{n} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i)$$
$$= 0$$

The contour map at ∞ has

- Constant magnitude.
- Zero angle.

The infinite loop is mapped to a single point!
Either $(0,0)$ or $(c,0)$.

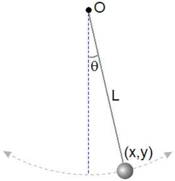




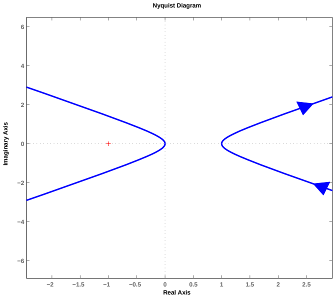
The Nyquist Plot

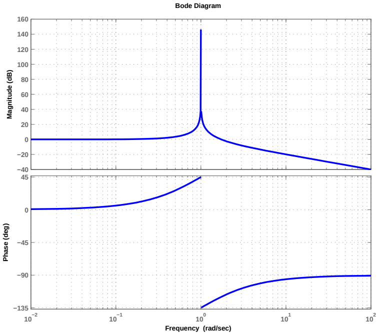
Another Problem: Recall the non-inverted pendulum with PD feedback.

$$G(s) = \frac{s+1}{s^2 + \sqrt{\frac{g}{l}}}$$



Magnitude goes to ∞ at $\omega = \sqrt{\frac{g}{l}}$.
Question How do we plot the Nyquist Diagram?



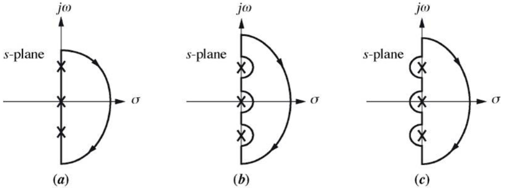


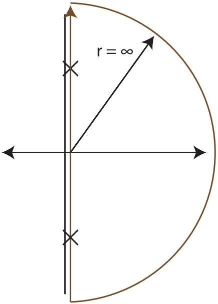
The Nyquist Plot

Problem: The Nyquist Contour passes through a pole.

Because of the pole, the *argument principle* is invalid.

What to do?





We Modify the Nyquist Contour.

- We detour around the poles.
- Can detour to the right or left.

If we detour to the left, then the poles count as unstable open loop poles.

- P=2

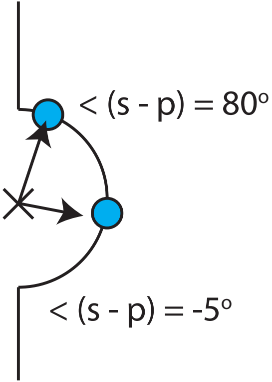
Assume we detour to the right.

- P=0

The Nyquist Plot

Look at the detours at small radius.

- Obviously, magnitude $\to \infty$
- Before the Detour, the phase from the pole is
$$-\angle(s-p) = 90^\circ$$
- In the middle of the Detour, the phase from the pole is
$$-\angle(s-p) = 0^\circ$$
- At the end of the Detour, the phase from the pole is
$$-\angle(s-p) = -90^\circ$$



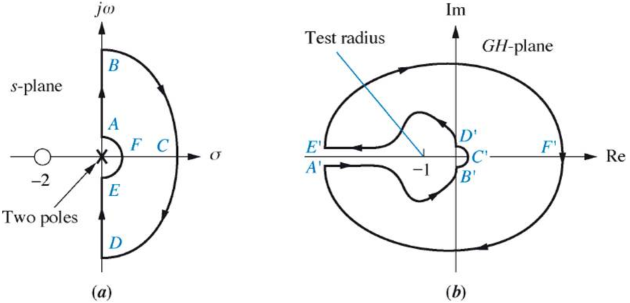
The total phase change through the detour is -180°.

- Corresponds to a CW loop at large radius.
- If there are two or more poles, there is a -180 loop for each pole.

The Nyquist Plot

Look at the following example:

$$G(s) = \frac{s+2}{s^2}$$

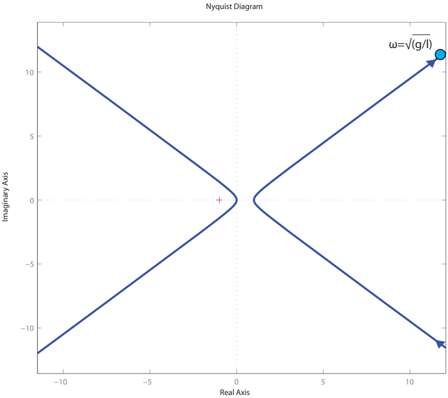


There are 2 poles at the origin.

- At $\omega = 0$,
  - $\angle G(0) = -180^\circ$
  - $|G(0)| = \infty$
- 2 poles means -360° loop at $\omega = 0$

The Nyquist Plot

Lets re-examine the pendulum problem with derivative feedback.

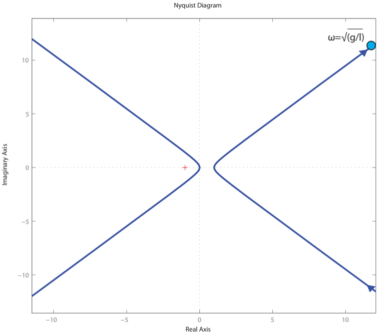


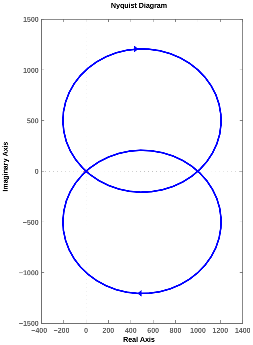
Now we can figure out what goes on at ∞.

- There is a $-180°$ loop at each $\omega = \sqrt{\frac{g}{l}}$.

The Nyquist Plot

Conclusion: The loops connect in a non-obvious way!





For $0 < \frac{-1}{k} < 1$, we have $N = 1$

Recall the definitions of Gain Margin.

Definition 2.

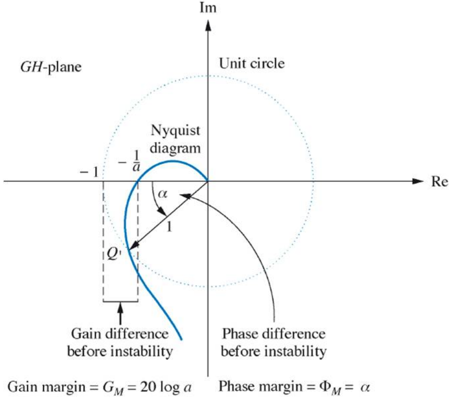
The **Gain Margin**, $K_m = 1/|G(\imath\omega)|$ when $\angle G(\imath\omega) = 180°$

Let K_m is the maximum stable gain in closed loop.

- $K_m G(s)$ is unstable in closed loop
- Sometimes expressed in dB

It is easy to find the maximum stable gain from the Nyquist Plot.

- Find the point $\frac{-1}{K_m}$ which destabilizes



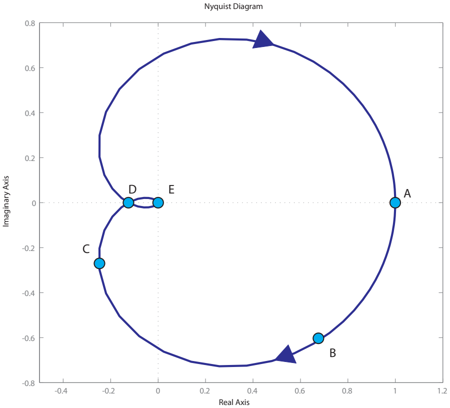
Stability Margins
Example

Recall

$$G(s) = \frac{1}{(s+1)^3}$$

Stability: Stable for $k < 10$.

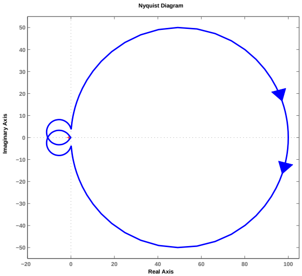
$$K_m = 10 \quad \text{or} \quad 20dB.$$

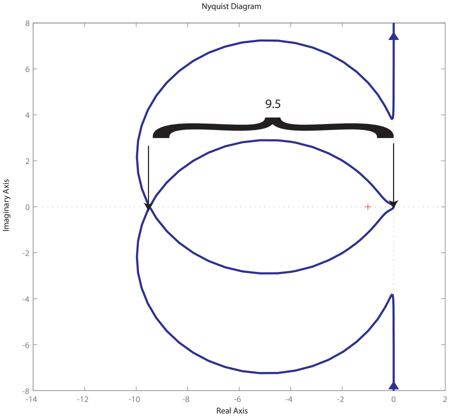


Suspension System with integral feedback

There is a pole at the origin.

- CW loop at ∞.





Conclusion: Stable for $\frac{-1}{k} > -9.5$.

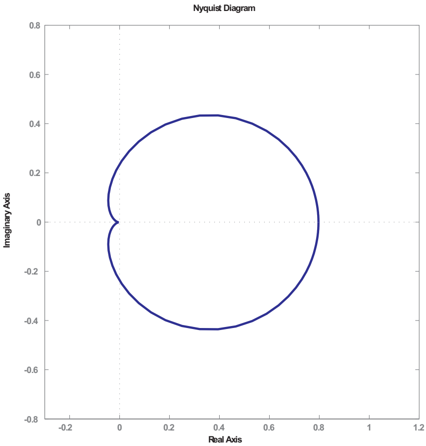
- Stable for $k < .105$

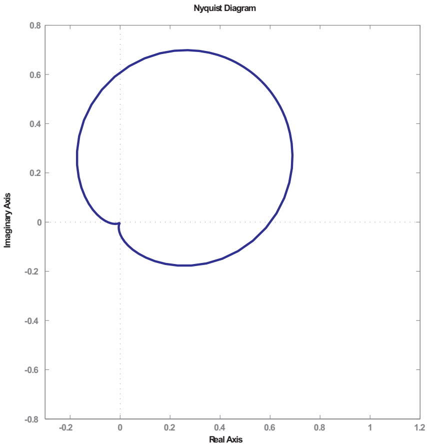
$$K_m = .105 \quad \text{or} \quad -19.5dB$$

Stability Margins

Question: What is the effect of a phase change on the Nyquist Diagram.

- A shift in phase changes the angle of all points.
- A Rotation about the origin.
- Will we rotate into instability?

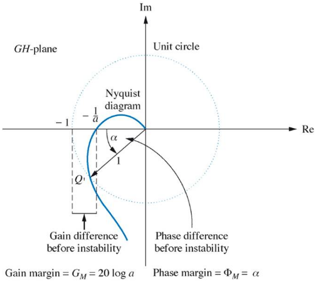


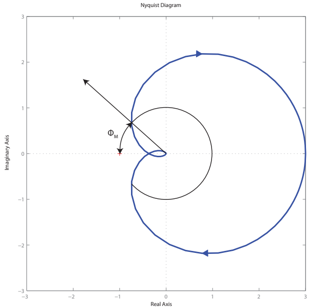


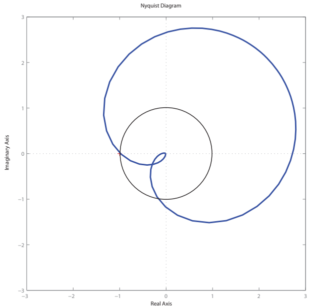
Recall the definitions of Phase Margin.

Definition 3.

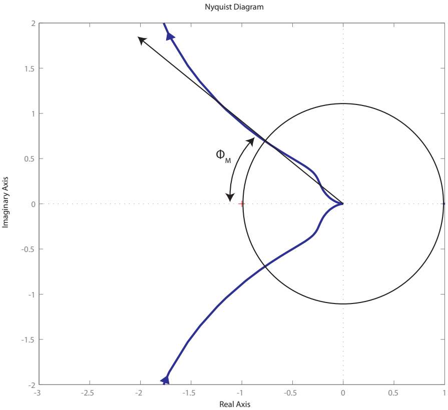
The **Phase Margin**, Φ_M is the uniform phase change required to destabilize the system under unitary feedback.







The Suspension Problem

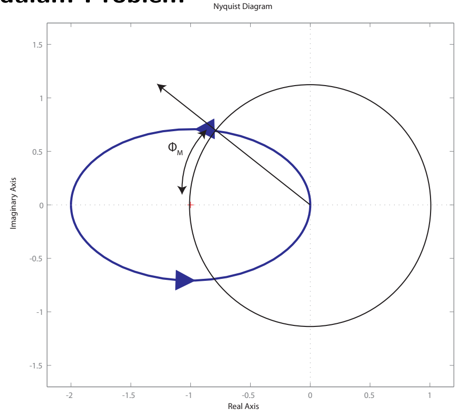


Looking at the intersection with the circle:

- Phase Margin: $\Phi_M \cong 40°$

Gain Margin is infinite.

The Inverted Pendulum Problem



Even though open-loop is unstable, we can still find the phase margin:

- Phase Margin: $\Phi_M \cong 35°$

Gain Margin is technically undefined because open loop is unstable.

- There is a minimum gain, not a maximum.

Summary

What have we learned today?

Review of Nyquist

Drawing the Nyquist Plot

- Using the Bode Plot
- What happens at $r = \infty$
- Poles on the imaginary axis

Phase Margin and Gain Margin

- Reading Stability Margins off the Nyquist Plot

Next Lecture: Controller Design in the Frequency Domain