Modern Control Systems

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Lecture 7: Controllability and Observability

State-Space

The standard state-space form is

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

State-space reflects an approach based on **internal dynamics** as opposed to input-output maps.

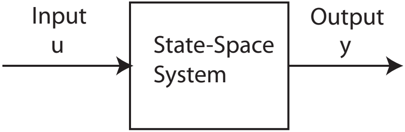
- For a given mapping, $G : u \mapsto y$, the choice of A, B, C, D is not unique.

Solving the Equations

Find the output given the input

State-Space:

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$



Basic Question: Given an input function, $u(t)$, what is the output?
Solution: Solve the differential Equation.

Example: The equation

$$\dot{x}(t) = ax(t), \qquad x(0) = x_0$$

has solution

$$x(t) = e^{at}x_0,$$

But we are interested in Matrices. Can we define the matrix exponential?

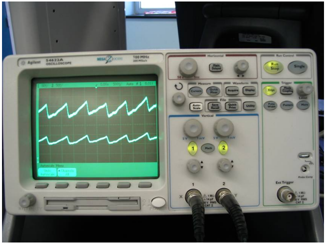
The Solution to State-Space

Ignore Inputs and Outputs:

The equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

has solution $\quad x(t) = e^{At}x_0$



The function e^{At} must satisfy the following

$$e^0 = I, \qquad \text{and} \qquad \frac{d}{dt}e^{At} = Ae^{At}$$

For scalars, the matrix exponential is defined as

$$e^a = 1 + a + a^2/2 + \cdots + \frac{a^n}{n!} + \cdots$$

We define the exponential for matrices is defined the same way as scalars

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots + \frac{1}{k!}A^k + \cdots$$

The matrix exponential has the following properties

- $e^0 = I$

$$e^0 = I + 0 + \frac{1}{2}0^2 + \frac{1}{6}0^3 + \cdots = I$$

- $e^{M^*} = \left(e^M\right)^*$

$$\begin{aligned} e^{M^*} &= I + M^* + \frac{1}{2}(M^*)^2 + \frac{1}{6}(M^*)^3 + \cdots + \frac{1}{k!}(M^*)^k + \cdots \\ &= I + M^* + (\frac{1}{2}M^2)^* + \left(\frac{1}{6}M^3\right)^* + \cdots + \left(\frac{1}{k!}M^k\right)^* + \cdots \\ &= \left(I + M + \frac{1}{2}M^2 + \frac{1}{6}M^3 + \cdots + \frac{1}{k!}M^k + \cdots\right)^* \end{aligned}$$

The Solution to State-Space

- $\frac{d}{dt}e^{At} = Ae^{At}$

$$\begin{aligned}\frac{d}{dt}e^{At} &= \frac{d}{dt}\left(I + (At) + \frac{1}{2}(At)^2 + \cdots + \frac{1}{k!}(At)^k + \cdots\right) \\ &= 0 + A + \frac{2}{2}A(At) + \frac{3}{6}A(At)^2 + \cdots + \frac{k}{k!}A(At)^k + \cdots \\ &= A\left(I + (At) + \frac{1}{2}(At)^2 + \cdots + \frac{1}{(k-1)!}(At)^{k-1} + \cdots\right)\end{aligned}$$

However,

$$e^{M+N} \neq e^M e^N$$

Unless, $MN = NM$.

Find the output given the input

The equation $$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

has solution $$x(t) = e^{At}x_0$$

Proof.

Let $x(t) = e^{At}x_0$, then

- $\dot{x}(t) = Ae^{At}x_0 = Ax(t)$.
- $x(t) = e^0 x_0 = x_0$

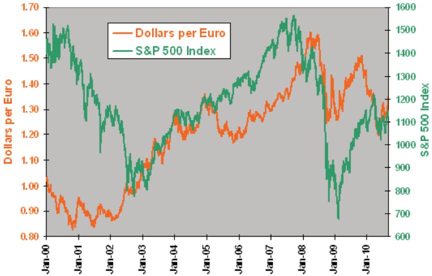
$\square$

What happens when we add an input instead of an initial condition?

Find the output given the input

State-Space:

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$



The equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = 0$$

has solution

$$x(t) = \int_0^t e^{A(t-s)} Bu(s) ds$$

Proof.

Check the solution:

$$\dot{x}(t) = e^0 Bu(t) + A \int_0^t e^{A(t-s)} Bu(s) ds$$
$$= Bu(t) + Ax(t)$$

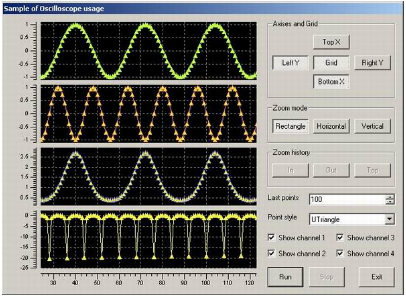
□

State-Space:

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$

Now that we have $x(t)$, finding $y(t)$ is easy

$$y(t) = Cx(t) + Du(t)$$
$$= \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t)$$

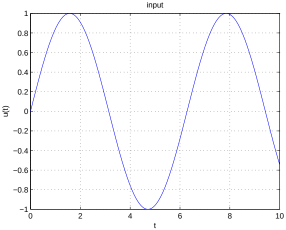


Conclusion: Given $u(t)$, only one integration is needed to find $y(t)$! Note that the state, x, doesn't appear!!

We have solved the problem.

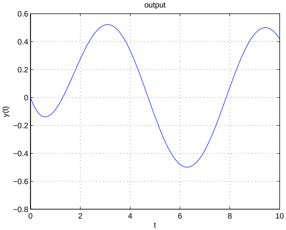
State-Space:

$$\dot{x} = -x(t) + u(t)$$
$$y(t) = x(t) - .5u(t) \qquad x(0) = 0$$
$$A = -1; \quad B = 1; \quad C = 1; \quad D = -.5$$



Solution:

$$\begin{aligned} y(t) &= \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t) \\ &= e^{-t}\int_0^t e^s \sin(s)ds - \frac{1}{2}\sin(t) \\ &= \frac{1}{2}e^{-t}\left(e^s(\sin s - \cos s)\big|_0^t\right) - \frac{1}{2}\sin(t) \\ &= \frac{1}{2}e^{-t}\left(e^t(\sin t - \cos t) + 1\right) - \frac{1}{2}\sin(t) \\ &= \frac{1}{2}\left(e^{-t} - \cos t\right) \end{aligned}$$



Stability

$$\dot{x}(t) = Ax(t)$$

There are several notions of stability.

- All notions are equivalent for linear systems.

Definition 1.

A differential equation is stable if any solution $x(t)$ satisfies

$$\lim_{t \to \infty} x(t) = 0$$

Stability

The unique solution has the form $x(t) = e^{At}x_0$.

$$\dot{x}(t) = Ax(t)$$

Question: Is it stable?
Suppose A is *diagonalizable*, so $A = T\Lambda T^{-1}$, so that

$$A^k = T\Lambda T^{-1}T\Lambda T^{-1} \cdots T\Lambda T^{-1} = T\Lambda^k T^{-1}$$

We conclude that

$$\begin{aligned} e^{At} &= \left(TT^{-1} + (T\Lambda T^{-1})t + \frac{1}{2}(T\Lambda^2 T^{-1})t^2 + \cdots + \frac{1}{k!}(T\Lambda^k T^{-1})t^k + \cdots\right) \\ &= T\left(I + (\Lambda t) + \frac{1}{2}(\Lambda t)^2 + \cdots + \frac{1}{k!}(\Lambda t)^k + \cdots\right)T^{-1} \\ &= Te^{\Lambda t}T^{-1} \end{aligned}$$

Stability

But we can see that $e^{\Lambda t}$ converges

$$e^{\Lambda t} = I + \Lambda + \cdots + \frac{1}{k!}\Lambda^k t^k + \cdots$$
$$= \begin{bmatrix} 1+\lambda_1+\cdots & & \\ & \ddots & \\ & & 1+\lambda_n+\cdots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

The solution is a linear combination of the functions of the form $e^{\lambda_i t}$.

- $\lim_{t\to\infty} e^{\lambda_i t} \to 0$ if and only if $\operatorname{Re}\lambda_i(A) < 0$ for all i.

Stability

Inconveniently, not all matrices are diagonalizable.

- However, all matrices are Jordan diagonalizable.
  - $A^k = TJ^kT^{-1}$, where J
- Hence $e^{At} = Te^{Jt}T^{-1}$

Consider a single Jordan block $J_i = \lambda_i I + N$.

- Convenient because $\lambda_i I$ and N commute.
  - Hence $e^{\lambda_i I + N} = e^{\lambda_i I} e^N$.

$$e^{J_i t} = e^{\lambda_i t + Nt} = e^{\lambda_i t} e^{Nt}$$

Conveniently $N^d = 0$, so the series expansion terminates

$$e^{Nt} = 1 + N + \frac{1}{2}N^2 + \cdots + \frac{1}{(k-1)!}N^{k-1}t^{k-1}$$

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \left[1 + N + \frac{1}{2}N^2 + \cdots + \frac{1}{(k-1)!}N^{k-1}t^{k-1}\right]$$

This time all terms have the form $t^i e^{\lambda t}$ for $i \leq k$

- $\lim_{t\to\infty} t^i e^{\lambda t} = 0$ if and only if $\operatorname{Re}\lambda < 0$

Stability

Now consider the general case $A = TJT^{-1}$ where

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_n \end{bmatrix}$$

Then

$$e^{At} = Te^{Jt}T^{-1} = T \begin{bmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_n t} \end{bmatrix} T^{-1}$$

e^{At} is entirely composed of terms of the form

$$\frac{e^{\lambda_i t} t^k}{k!}$$

We conclude that $\dot{x}(t) = Ax(t)$ is stable if and only if $\operatorname{Re} \lambda < 0$.

Stability

Definition 2.

A is **Hurwitz** if $\operatorname{Re}\lambda_i(A) < 0$ for all i.

$\dot{x}(t) = Ax(t)$ is stable if and only if A is Hurwitz.

Controllability

First add an input $u(t)$

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0$$

The solution is

$$x(t) = \int_0^t e^{A(t-s)} Bu(s) ds$$

Use Leibniz rule for differentiation of integrals

$$\begin{aligned}\dot{x}(t) &= e^{A(t-t)} Bu(t) + \int_0^t Ae^{A(t-s)} Bu(s) ds \\ &= Bu(t) + Ax(t)\end{aligned}$$

Controllability asks whether we can "control" the system states through appropriate choice of $u(t)$.

- Note that we do not care how $u(t)$ is chosen.

We start with a weaker definition

Controllability

Definition 3.

For a given (A, B), the **state x_f is Reachable** if for any fixed T_f, there exists a $u(t)$ such that

$$x_f = \int_0^{T_f} e^{A(T_f - s)} B u(s) ds$$

Definition 4.

The **system (A, B) is reachable** if any point $x_f \in \mathbb{R}^n$ is reachable.

For a fixed t, the set of reachable states is defined as

$$R_t := \{x \, : \, x = \int_0^{t} e^{A(t-s)} B u(s) ds \text{ for some function } u.\}$$

Controllability

The mapping $\Gamma : u \mapsto x_f$ is linear. Let $u = \alpha u_1 + \beta u_2$

$$\begin{aligned}\Gamma u &= \int_0^{T_f} e^{A(T_f - s)} B \left(\alpha u_1(s) + \beta u_2(s)\right) ds \\ &= \alpha \int_0^{T_f} e^{A(T_f-s)} B u_1(s) ds + \beta \int_0^{T_f} e^{A(T_f-s)} B u_2(s) ds \\ &= \alpha \Gamma u_1 + \beta \Gamma u_2\end{aligned}$$

Thus $R_t = \text{Image}(\Gamma)$.

- R_t is a subspace.

Definition 5.

For a given system (A, B), the **Controllability Matrix** is

$$C(A, B) := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

Controllability

Definition 6.

For a given (A, B), the **Controllable Subspace** is

$$C_{AB} = \text{Image}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

Definition 7.

The system (A, B) is **controllable** if

$$C_{AB} = \operatorname{Im} C(A, B) = \mathbb{R}^n$$

Question: How does R_t relate to C_{AB}?

Controllability

Definition 8.

The finite-time **Controllability Grammian** of pair (A, B) is

$$W_t := \int_0^t e^{As} B B^T e^{A^T s} ds$$

W_t is a positive semidefinite matrix.
The following relates these three concepts of controllability

Theorem 9.

For any $t \geq 0$,

$$R_t = C_{AB} = \textit{Image}\,(W_t)$$

or

$$\text{Image } \Gamma_t = \text{Image } C(A, B) = \text{Image } (W_t)$$

Controllability

The most important consequence is

- R_t does not depend on time!

If you can get there, you can get there arbitrarily fast.
This says nothing about how you get $u(t)$

- This $u(t)$ comes from the proof (and W_t)

We can test reachability of a point x by testing

$$x \in \text{Im}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

The system is controllable if $W_t > 0$. Summary

1. R_t is the set of reachable points
2. $C(A, B)$ is a fixed matrix, easily computable.
3. We need to find $u(t)$

Controllability

The following is a seminal result in state-space theory.

Theorem 10.

If

$$\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$$

then

$$A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots a_0 I = 0$$

Sketch.

The same principle as deriving the solution. Denote

$$\mathsf{char}_A(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0 = \det(sI - A)$$

Then if $A = T\lambda T^{-1}$

$$\mathsf{char}_A(A) = T\mathsf{char}_A(\Lambda)T^{-1} = T \begin{bmatrix} \mathsf{char}_A(\lambda_1) & & \\ & \ddots & \\ & & \mathsf{char}_A(\lambda_n) \end{bmatrix} T^{-1}$$

Controllability

Sketch.

But the λ_i are eigenvalues of A, so

$$\text{char}_A(\lambda) = \det(\lambda I - A) = 0$$

hence

$$\text{char}_A(A) = T \begin{bmatrix} \text{char}_A(\lambda_1) & & \\ & \ddots & \\ & & \text{char}_A(\lambda_n) \end{bmatrix} T^{-1} = T \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} T = 0$$

The same approach works for Jordan Blocks. □

Cayley-Hamilton says

$$A^n = -a_{n-1}A^{n-1} + \cdots + -a_0 I$$

thus $A^n \in \text{span}(A^{n-1}, \cdots, I)$

- This is unsurprising since A has n^2 dimensions but is formed by n bases.

Controllability

Proof: Show $R_t \subset C_{AB}$ for any $t \geq 0$. Expand

$$e^{At} = \left[I + At + \cdots + \frac{A^m t^m}{m!} + \cdots \right]$$

grouping by A^i,

$$e^{At} = \left[I\phi_0(t) + A_1\phi_1(t) + \cdots + A^{n-1}\phi_{n-1}(t) \right]$$

for scalar functions $\phi_i(t)$ due to Cayley-Hamilton

$$A^n = -a_{n-1}A^{n-1} + \cdots + -a_0 I$$

Because the ϕ_i are scalars,

$$\begin{aligned}\Gamma_t u &= \int_0^t e^{A(t-s)} Bu(s)ds \\ &= B\int_0^t \phi_0(t-s)u(s)ds + \cdots + A^{n-1}B\int_0^t \phi_{n-1}(t-s)u(s)ds\end{aligned}$$

Controllability

Let

$$y_i = \int_0^t \phi_i(t-s)u(s)ds,$$

then

$$\begin{aligned}\Gamma_t u &= By_0 + \cdots + A^{n-1}By_{n-1} \\ &= \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix} = C(A,B) \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix}\end{aligned}$$

Thus $\Gamma_t u \in \operatorname{Im} \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix}$. Therefore, $R_t \subset C_{AB}$.

2 new concepts: perp space

Definition 11.

The **Orthogonal Complement** of a subspace, $S \subset X$, is denoted

$$S^\perp := \left\{ x \in \mathbb{R}^n \; : \; \langle x, y \rangle = x^T y = 0 \qquad \text{for all } y \in S \right\}$$

Properties

- $\text{dim}(S^\perp) = n - \text{dim}(S)$
- For any $x \in \mathbb{R}^n$,

$$x = x_S + x_{S^\perp} \quad \text{for } x_S \in S \text{ and } x_{S^\perp} \in S^\perp$$

  - x_S and $x_{S^\perp}$ are unique.

Controllability

Definition 12.

The Projection operator P_S is defined by

$$x_S = Px$$

if $x_S \in S$ and $x - x_S \in S^\perp$.

Generalizes to any Hilbert space

Theorem 13.

For any $M \in \mathbb{R}^{n \times m}$, $\left[\mathit{Im}(M)\right]^\perp = \mathit{Ker}\left[M^T\right]$.

Proof.

We need to show $\left[\text{Im}(M)\right]^\perp \subset \text{Ker}\left[M^T\right]$ and $\text{Ker}\left[M^T\right] \subset \left[\text{Im}(M)\right]^\perp$.

- Suppose $x \in \left[\text{Im}(M)\right]^\perp$. If $x^T y = 0$ for any $y \in \text{Im}[M]$, then $x^T M z = 0$ for all z.
- Thus $z^T M^T x$ for all z. Let $z = M^T z$.
- Then $x^T M M^T x = \|M^T x\|^2 = 0$.

Controllability