Modern Control Systems

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Lecture 11: Stabilizability and Eigenvalue Assignment

Stabilizability

Stabilizability is weaker than controllability

Definition 1.

The pair (A, B) is stabilizable if for any $x(0) = x_0$, there exists a $u(t)$ such that $x(t) = \Gamma_t u$ satisfies

$$\lim_{t \to \infty} x(t) = 0$$

- Again, no restriction on $u(t)$.
- Weaker than controllability
  - **Controllability:** Can we drive the system to $x(T_f) = 0$?
  - **Stabilizability:** Only need to *Approach* $x = 0$.
- Stabilizable if uncontrollable subspace is naturally stable.

Stabilizability

Consider the system in Controllability Form.

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}B_1\\ 0\end{bmatrix}u(t)$$
$$x(0) = \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}$$

Note that

$$\dot{x}_2(t) = A_{22}x_2(t)$$

and so, we can solve explicitly

$$x_2(t) = e^{A_{22}t}x_2(0)$$

Clearly A_{22} must be Hurwitz if (A, B) is stabilizable.

- Necessary and Sufficient

PBH Test

Lemma 2.

The pair (A, B) is stabilizable if and only if A_{22} is Hurwitz.

This is an test for stabilizability, but requires conversion to controllability form.

- A more direct test is the PBH test

Theorem 3.

The pair (A, B) is

- **Stabilizable** *if and only if* $\operatorname{rank}\begin{bmatrix}\lambda I - A & B\end{bmatrix} = n$ *for all* $\lambda \in \mathbb{C}^+$
- **Controllable** *if and only if* $\operatorname{rank}\begin{bmatrix}\lambda I - A & B\end{bmatrix} = n$ *for all* $\lambda \in \mathbb{C}$

Note: We need only check the eigenvalues λ

PBH Test

Proof: Controllable if and only if $\operatorname{rank}\begin{bmatrix}\lambda I - A & B\end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$

Proof.

We will use proof by contradiction. $(\neg 2 \Rightarrow \neg 1)$. Suppose $\operatorname{rank}\begin{bmatrix}\lambda I - A & B\end{bmatrix} < n$.

- Thus $\dim\left(\text{Im}\begin{bmatrix}\lambda I - A & B\end{bmatrix}\right) < n$
- There exists an x such that $x^T\begin{bmatrix}\lambda I - A & B\end{bmatrix} = 0$.
- Thus $\lambda x^T = x^T A$ and $x^T B = 0$
- Thus $x^T A^2 = \lambda x^T A = \lambda^2 x^T$.
- Likewise $x^T A^k = \lambda^k x^T$.
- Thus

$$\begin{aligned} x^T C(A,B) = x^T\begin{bmatrix}B & AB & \cdots & A^{n-1}B\end{bmatrix} &= x^T\begin{bmatrix}B & \lambda B & \cdots & \lambda^{n-1}B\end{bmatrix} \\ &= \begin{bmatrix}0 & \cdots & 0\end{bmatrix} \end{aligned}$$

- Thus $\dim[\text{Im}C(A,B)] < n$, which means *Not Controllable*. $(\neg 2 \Rightarrow \neg 1)$.
- We conclude that controllable implies $\operatorname{rank}\begin{bmatrix}\lambda I - A & B\end{bmatrix} = n$.

PBH Test

Proof.

For the second part, we will also use proof by contradiction. $(\neg 1 \Rightarrow \neg 2)$.
Suppose (A, B) is not controllable. Then there exists an invertible T such that

$$TAT^{-1} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}, \qquad TB = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}$$

Now let λ be an eigenvalue of $\hat{A}_{22}^T$ with eigenvector $\hat{x}$. $\hat{A}_{22}^T \hat{x} = \lambda \hat{x}$. Thus $\hat{x}^T \hat{A}_{22} = \lambda \hat{x}^T$. Let

$$x = T^T \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}, \qquad \text{then} \quad x^T = \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T T$$

Then

$$\begin{aligned} x^T \begin{bmatrix} \lambda I - A & B \end{bmatrix} &= x^T T^{-1} \begin{bmatrix} \lambda T - TAT^{-1}T & TB \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T TT^{-1} \left[\lambda T - \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} T \quad \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} \right] \\ &= \left[\lambda \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T T - \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} T \quad \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} \right] \end{aligned}$$

PBH Test

Proof.

$$
\begin{aligned}
x^T \begin{bmatrix}\lambda I - A & B\end{bmatrix} &= \left[\lambda \begin{bmatrix}0\\ \hat{x}\end{bmatrix}^T T - \begin{bmatrix}0\\ \hat{x}\end{bmatrix}^T \begin{bmatrix}\hat{A}_{11} & \hat{A}_{12}\\ 0 & \hat{A}_{22}\end{bmatrix} T \quad \begin{bmatrix}0\\ \hat{x}\end{bmatrix}^T \begin{bmatrix}\hat{B}_1\\ 0\end{bmatrix}\right] \\
&= \begin{bmatrix}\begin{bmatrix}0 & \lambda \hat{x}^T\end{bmatrix} T - \begin{bmatrix}0 & \hat{x}^T \hat{A}_{22}\end{bmatrix} T & 0\end{bmatrix} \\
&= \begin{bmatrix}0 & \hat{x}^T \begin{bmatrix}\lambda I - \hat{A}_{22}\end{bmatrix} & 0\end{bmatrix} T = 0 \\
&= \begin{bmatrix}0 & \begin{bmatrix}\lambda I - \hat{A}_{22}^T\end{bmatrix} \hat{x} & 0\end{bmatrix} T = 0
\end{aligned}
$$

- Thus $x^T \begin{bmatrix}\lambda I - A & B\end{bmatrix} = 0$.
- Thus $\operatorname{rank} \begin{bmatrix}\lambda I - A & B\end{bmatrix} < n$.
- Finally $(\neg 1 \Rightarrow \neg 2)$.
- We conclude that $\operatorname{rank} \begin{bmatrix}\lambda I - A & B\end{bmatrix} = n$ implies controllability.

□

Single Input Controllability

Definition 4.

A **Companion Matrix** is any matrix of the form:

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & \cdots & & -a_{n-1} \end{bmatrix}$$

A companion matrix has the convenient property that

$$\det(sI - A) = \sum_{i=0}^{n-1} a_i s^i = a_0 + a_1 s + \cdots + a_{n-1} s^{n-1} + s^n$$

Single Input Controllability

Theorem 5.

Suppose (A, B) *is controllable.* $B \in \mathbb{R}^{n\times 1}$. *Then there exists an invertible* T *such that*

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & & \\ & & 0 & 1 \\ -a_0 & & & -a_{n-1} \end{bmatrix}, \qquad TB = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

This is **Controllable Canonical Form**

- Different from controllability form
- This is useful for reading off transfer functions

$$G(s) = C(sI - A)^{-1}B + D$$

which has a denominator

$$\det(sI - A) = a_0 + \cdots + a_{n-1}s^{n-1}$$

The problem of designing a controller

- We have touched on this problem in reachability
  - $u(t) = B^T e^{A(T_f - t)} T^{-1} z_f$
  - This controller is open-loop
- It assumes perfect knowledge of system and state.

Problems

- Prone to Errors, Disturbances, Errors in the Model

Solution

- Use continuous measurements of state to generate control

Static Full-State Feedback Assumes:

- We can directly and continuously measure the state $x(t)$
- Controller is a static linear function of the measurement

$$u(t) = Fx(t), \qquad F \in \mathbb{R}^{m \times n}$$

Eigenvalue Assignment

Static Full-State Feedback

State Equations: $u(t) = Fx(t)$

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ &= Ax(t) + BFx(t) \\ &= (A + BF)x(t)\end{aligned}$$

Stabilization: Find a matrix $F \in \mathbb{R}^{m \times n}$ such that

$$A + BF$$

is Hurwitz.

Eigenvalue Assignment: Given $\{\lambda_1, \cdots, \lambda_n\}$, find $F \in \mathbb{R}^{m \times n}$ such that

$$\lambda_i \in \text{eig}(A + BF) \qquad \text{for } i = 1, \cdots, n$$

is Hurwitz.

Note: A solution to the eigenvalue assignment problem will also solve the stabilization problem.

Question: Is eigenvalue assignment actually harder?

Eigenvalue Assignment

Single-Input Case

Theorem 6.

Suppose $B \in \mathbb{R}^{n\times 1}$. Eigenvalues of $A+BF$ are freely assignable if and only if (A,B) is controllable.

Proof.

1. There exists a T such that

$$\hat{A} = TAT^{-1} = \begin{bmatrix} 0 & I \\ -a_0 & \begin{bmatrix} -a_1 & \cdots & -a_{n-1} \end{bmatrix} \end{bmatrix} \qquad \hat{B} = TB = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

2. Define $\hat{F} = \begin{bmatrix} \hat{f}_0 & \cdots & \hat{f}_{n-1} \end{bmatrix} \in \mathbb{R}^{1\times n}$. Then

$$\hat{B}\hat{F} = \begin{bmatrix} 0 & 0 \\ \hat{f}_0 & \begin{bmatrix} \hat{f}_1 & \cdots & \hat{f}_{n-1} \end{bmatrix} \end{bmatrix}$$

□

Proof.

- Then
$$\hat{B}\hat{F} = \begin{bmatrix} 0 & 0 \\ \hat{f}_0 & \begin{bmatrix} \hat{f}_1 & \cdots & \hat{f}_{n-1} \end{bmatrix} \end{bmatrix}$$
$$\hat{A} + \hat{B}\hat{F} = \begin{bmatrix} O & I \\ -a_0 + \hat{f}_0 & \begin{bmatrix} -a_1 + \hat{f}_1 & \cdots & -a_{n-1} + \hat{f}_{n-1} \end{bmatrix} \end{bmatrix}$$
- This has the characteristic equation
$$\det\left(sI - (\hat{A} + \hat{B}\hat{F})\right) = s^n + (\hat{f}_{n-1} - a_{n-1})s^{n-1} + \cdots + (\hat{f}_0 - a_0)$$
- Suppose we want eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$. Then define b_i as
$$p(s) = (s - \lambda_1) \cdots (s - \lambda_n) = s^n + b_{n-1}s^{n-1} + \cdots + b_0$$
- Choose $\hat{f}_i = a_i - b_i$.
- Now let $F = \hat{F}T$. Then $A + BF = T^{-1}(\hat{A} + \hat{B}\hat{F})T$

□

Eigenvalue Assignment

Single-Input Case

Proof.

- Then
$$\det\left(sI-(A+BF)\right)=\det\left(T\left(sI-(\hat{A}+\hat{B}\hat{F})\right)T^{-1}\right)$$
$$=\det\left(sI-(\hat{A}+\hat{B}\hat{F})\right)$$
$$=(s-\lambda_1)\cdots(s-\lambda_n)$$
- Hence $A+BF$ has eigenvalues $\{\lambda_1,\cdots,\lambda_n\}$.

□

Suppose we want the eigenvalues $\{\lambda_1,\cdots,\lambda_n\}$.

1. Find the b_i
2. Choose $\hat{f}_i=a_i-b_i$.
3. Then use $F=\begin{bmatrix}\hat{f}_0 & \cdots & \hat{f}_{n-1}\end{bmatrix}T$.

Conclusion: For Single-Input, controllability implies eigenvalue assignability.

- Requires conversion to controllable canonical form
- Matlab command `acker`.

Eigenvalue Assignment

Multiple-Input Case

The multi-input case is harder

Lemma 7.

If (A, B) is controllable, then for any $x_0 \neq 0$, there exists a sequence $\{u_0, u_1, \cdots, u_{n-2}\}$ such that span$\{x_0, x_1, \cdots, x_{n-1}\} = \mathbb{R}^n$, where

$$x_{k+1} = Ax_k + Bu_k \qquad \text{for } k = 0, \cdots, n-1$$

Proof.

For $1 \Rightarrow 2$, we again use proof by contradiction. We show $(\neg 2 \Rightarrow \neg 1)$.

- Suppose that for any x_0, and any $\{u_0, u_1, \cdots, u_{n-2}\}$, span$\{x_0, \cdots, x_{n-1}\} \neq \mathbb{R}^n$. Then there exists some y such that $y^T x_k = 0$ for any $k = 0, \cdots, n-1$. We can solve explicitly for x_k:

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} B u_j$$

Proof.

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} B u_j$$

- Let $k = n-1$, and $x_0 = Bu_{n-1}$ for some u_{n-1}. Then for any u

$$y^T x_{n-1} = y^T \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u_{n-1} \\ u_0 \\ \vdots \\ u_{n-2} \end{bmatrix} = y^T C(A,B) u = 0$$

- Therefore, image$(C(A,B)) \neq \mathbb{R}^n$. Hence (A,B) is not controllable. This proves the lemma.

□

Lemma 8.

Suppose (A, B) is controllable. Then for any nonzero column, $B_1 \in \mathbb{R}^n$, of B, there exists a $F_1 \in \mathbb{R}^{m\times n}$ such that $(A + BF_1, B_1)$ is controllable

Proof.

Suppose (A, B) is controllable. Let $x_0 = B_1$ and apply the previous Lemma to find some input $u_0, \cdots, u_{n-2}$ such that span$\{x_0, \cdots x_{n-1}\} = \mathbb{R}^n$ where

$$x_{k+1} = Ax_k + Bu_k$$

Let $T = \begin{bmatrix} x_0 & \cdots & x_{n-1} \end{bmatrix}$. Then T is invertible. Let

$$F_1 = \begin{bmatrix} u_0 & \cdots & u_{n-2} \end{bmatrix} T^{-1} = UT^{-1}$$

- This implies $F_1 T = U$ and hence $F_1 x_i = u_i$ for $i = 0, \cdots, n-1$.
- Now expand

$$x_{k+1} = Ax_k + Bu_k = Ax_k + BF_1 x_k = [A + BF_1]x_k$$

Proof.

$$x_{k+1} = Ax_k + Bu_k = Ax_k + BF_1 x_k = [A + BF_1]x_k$$

Which means that $x_k = [A + BF_1]^k x_0$. However, since $x_0 = B_1$, we have

$$\begin{aligned} T &= \begin{bmatrix} x_0 & \cdots & x_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} B_1 & \cdots & (A + BF_1)^{n-1} B_1 \end{bmatrix} \\ &= C(A + BF_1, B_1) \end{aligned}$$

- Since T is invertible, $C(A + BF_1, B_1)$ is full rank and hence $(A + BF_1, B_1)$ is controllable.

□

Eigenvalue Assignment

Multiple-Input Case

Theorem 9.

The eigenvalues of $A + BF$ are freely assignable if and only if (A, B) is controllable.

Proof.

The "only if" direction is clear. Suppose (A, B) is controllable and we want eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$. Let B_1 be the first column of B.

- By Lemma, there exists a F_1 such that $(A + BF_1, B_1)$ is controllable.
- By other Lemma, since the $(A + BF_1, B_1)$ is controllable, the eigenvalues of $(A + BF_1, B_1)$ are assignable. This we can find a F_2 such that $A + BF_1 + B_1 F_2$ has eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$.
- Choose $F = F_1 + \begin{bmatrix} F_2 \\ 0 \end{bmatrix}$. Then
$$A + BF = A + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \left[F_1 + \begin{bmatrix} F_2 \\ 0 \end{bmatrix} \right] = A + BF_1 + B_1 F_2$$
has the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$.

□

Theorem 10.

The eigenvalues of $A+BF$ are freely assignable if and only if (A,B) is controllable.

Note that the proof was not very constructive: Need to find F_1 and F_2... **2**

Matlab Commands

- `K=acker(A,B,p)` for 1-D
- `K=place(A,B,p)` for n-D. p is the vector of pole locations.

Theorem 11.

If (A,B) is stabilizable, then there exists a F such that $A+BF$ is Hurwitz.

Proof.

Apply the previous result to the controllability form. □

Conclusion: If (A,B) is stabilizable, then it can be stabilized using *only* static state feedback. $u(t)=Kx(t)$.