Nonlinear Systems Theory

Matthew M. Peet
Arizona State University

Lecture 02: Nonlinear Systems Theory

Overview

Our next goal is to extend LMI's and optimization to nonlinear systems analysis.

Today we will discuss

1. Nonlinear Systems Theory
   1.1 Existence and Uniqueness
   1.2 Contractions and Iterations
   1.3 Gronwall-Bellman Inequality
2. Stability Theory
   2.1 Lyapunov Stability
   2.2 Lyapunov's Direct Method
   2.3 A Collection of Converse Lyapunov Results

The purpose of this lecture is to show that Lyapunov stability can be solved **Exactly** via optimization of polynomials.

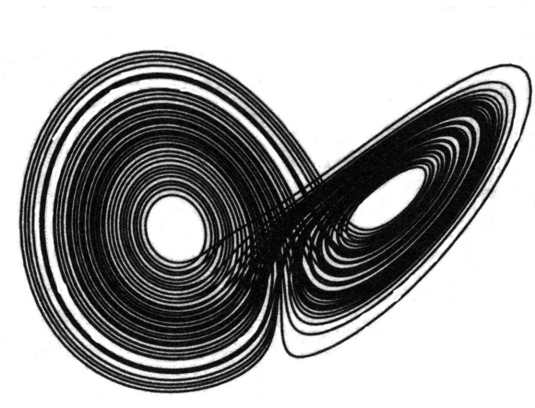
Consider: A System of Nonlinear Ordinary Differential Equations

$$\dot{x}(t) = f(x(t))$$

Problem: Stability
Given a specific polynomial $f : \mathbb{R}^n \to \mathbb{R}^n$,
find the largest $X \subset \mathbb{R}^n$
such that for any $x(0) \in X$,
$\lim_{t\to\infty} x(t) = 0$.

Nonlinear Dynamical Systems

Long-Range Weather Forecasting and the Lorentz Attractor



A model of atmospheric convection analyzed by E.N. Lorenz, Journal of Atmospheric Sciences, 1963.

$$\dot{x} = \sigma(y - x) \qquad \dot{y} = rx - y - xz \qquad \dot{z} = xy - bz$$

An oscillating circuit model:

$$\dot{y} = -x - (x^2 - 1)y$$
$$\dot{x} = y$$

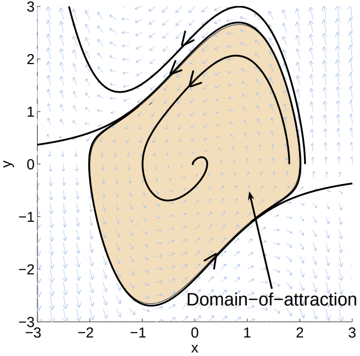


Figure : The van der Pol oscillator in reverse

Theorem 1 (Poincaré-Bendixson).

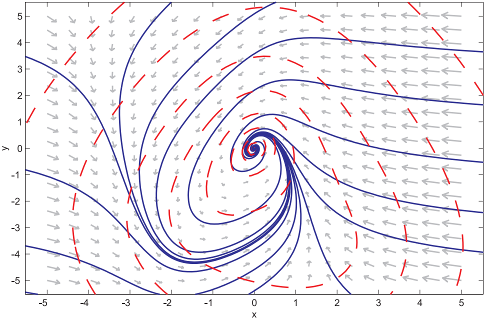
Invariant sets in $\mathbb{R}^2$ always contain a limit cycle or fixed point.

Stability of Ordinary Differential Equations

Consider

$$\dot{x}(t) = f(x(t))$$

with $x(0) \in \mathbb{R}^n$.



Theorem 2 (Lyapunov Stability).

Suppose there exists a continuous V and $\alpha, \beta, \gamma > 0$ where

$$\beta\|x\|^2 \le V(x) \le \alpha\|x\|^2$$
$$-\nabla V(x)^T f(x) \ge \gamma\|x\|^2$$

for all $x \in X$. Then any sub-level set of V in X is a **Domain of Attraction**.

The first question people ask is the Cauchy problem:
Autonomous System:

Definition 3.

The system $\dot{x}(t) = f(x(t))$ is said to satisfy the Cauchy problem if there exists a continuous function $x : [0, t_f] \to \mathbb{R}^n$ such that $\dot{x}$ is defined and $\dot{x}(t) = f(x(t))$ for all $t \in [0, t_f]$

If f is continuous, the solution must be continuously differentiable.
Controlled Systems:

- For a controlled system, we have $\dot{x}(t) = f(x(t), u(t))$.
- At this point u is undefined, so for the Cauchy problem, we take $\dot{x}(t) = f(t, x(t))$
- In this lecture, we consider the autonomous system.
  - Including t complicates the analysis.
  - However, results are almost all the same.

There exist many systems for which no solution exists or for which a solution only exists over a finite time interval.

Even for something as simple as

$$\dot{x}(t) = x(t)^2 \qquad\qquad x(0) = x_0$$

has the solution

$$x(t) = \frac{x_0}{1 - x_0 t}$$

which clearly has escape time

$$t_e = \frac{1}{x_0}$$

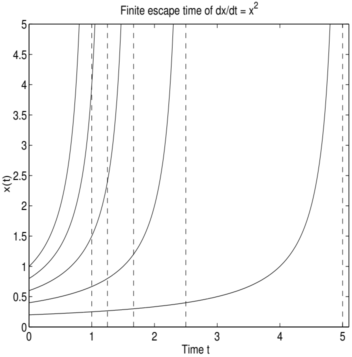


Figure : Simulation of $\dot{x} = x^2$ for several $x(0)$

A classical example of a system without a unique solution is

$$\dot{x}(t) = x(t)^{1/3} \qquad\qquad x(0) = 0$$

For the given initial condition, it is easy to verify that

$$x(t) = 0 \qquad \text{and} \qquad x(t) = \left(\frac{2t}{3}\right)^{3/2}$$

both satisfy the differential equation.

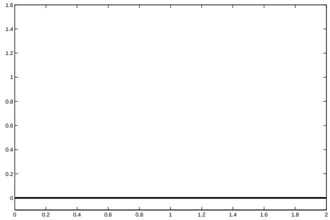


Figure : Matlab simulation of $\dot{x}(t) = x(t)^{1/3}$ with $x(0) = 0$

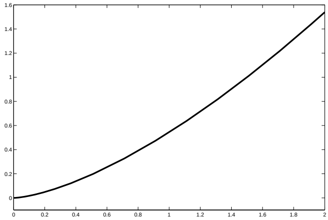


Figure : Matlab simulation of $\dot{x}(t) = x(t)^{1/3}$ with $x(0) = .000001$

Another Example of a system with several solutions is given by

$$\dot{x}(t) = \sqrt{x(t)} \qquad\qquad x(0) = 0$$

For the given initial condition, it is easy to verify that for any C,

$$x(t) = \begin{cases} \frac{(t-C)^2}{4} & t > C \\ 0 & t \le C \end{cases}$$

satisfies the differential equation.

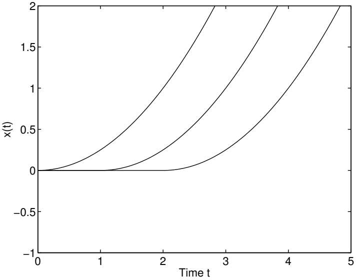
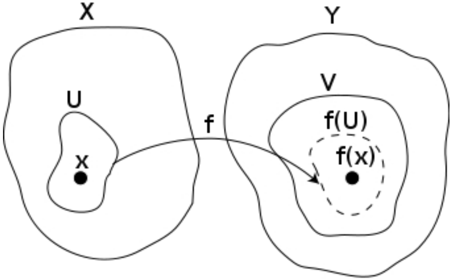


Figure : Several solutions of $\dot{x} = \sqrt{x}$

Definition 4.

For normed linear spaces X, Y, a function $f : X \to Y$ is said to be **continuous at the point** $x_0 \subset X$ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x - x_0\| < \delta$ implies $\|f(x) - f(x_0)\| < \epsilon$.



Definition 5.

For normed linear spaces X, Y, a function $f : A \subset X \to Y$ is said to be **continuous on** $B \subset A$ if it is continuous for any point $x_0 \in B$. A function is said to be simply **continuous** if $B = A$.

Definition 6.

For normed linear spaces X, Y, a function $f : A \subset X \to Y$ is said to be **uniformly continuous on** $B \subset A$ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any points $x, y \in B$, $\|x - y\| < \delta$ implies $\|f(x) - f(y)\| < \epsilon$. A function is said to be simply **uniformly continuous** if $B = A$.

Lipschitz Continuity

A Quantitative Notion of Continuity

Definition 7.

We say the function f is **Lipschitz continuous** on X if there exists some $L > 0$ such that

$$\|f(x) - f(y)\| \leq L\|x - y\| \qquad \text{for any } x, y \in X.$$

The constant L is referred to as the Lipschitz constant for f on X.

Definition 8.

We say the function f is **Locally Lipschitz continuous** on X if for every $x \in X$, there exists a neighborhood, B of x such that f is Lipschitz continuous on B.

Definition 9.

We say the function f is **globally Lipschitz** if it is Lipschitz continuous on its entire domain.

It turns out that smoothness of the vector field is the critical factor.

- Not a **Necessary** condition, however.
- The Lipschitz constant, L, allows us to quantify the *roughness* of the vector field.

Theorem 10 (Simple).

Suppose $x_0 \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ *and there exist* L, r *such that for any* $x, y \in B(x_0, r)$,

$$\|f(x) - f(y)\| \le L\|x - y\|$$

and $\|f(x)\| \le c$. *Let* $b < \min\{\frac{1}{L}, \frac{r}{c}\}$. *Then there exists a unique differentiable map* $x \in \mathcal{C}[0, b]$, *such that* $x(0) = x_0$, $x(t) \in B(x_0, r)$ *and* $\dot{x}(t) = f(x(t))$.

Because the approach to its proof is so powerful, it is worth presenting the proof of the existence theorem.

Theorem 11 (Contraction Mapping Principle).

Let $(X, \|\cdot\|)$ be a complete normed space and let $P : X \to X$. Suppose there exists a $\rho < 1$ such that

$$\|Px - Py\| \le \rho\|x - y\| \qquad \textit{for all} \qquad x, y \in X.$$

Then there is a unique $x^* \in X$ such that $Px^* = x^*$. Furthermore for $y \in X$, define the sequence $\{x_i\}$ as $x_1 = y$ and $x_i = Px_{i-1}$ for $i > 2$. Then $\lim_{i\to\infty} x_i = x^*$.

Some Observations:

- Proof: Show that $P^k y$ is a Cauchy sequence for any $y \in X$.
- For a differentiable function P, P is a contraction if and only if $\|\dot{P}\| < 1$.
- In our case, X is the space of solutions. The contraction is

$$(Px)(t) = x_0 + \int_0^t f(x(s))ds$$

This contraction derives from the fundamental theorem of calculus.

Theorem 12 (Fundamental Theorem of Calculus).

Suppose $x \in \mathcal{C}$ and $f : M \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $M = [0, t_f] \subset \mathbb{R}$. Then the following are equivalent.

- *x is differentiable at any $t \in M$ and*

$$\dot{x}(t) = f(x(t)) \quad \text{at all } t \in M \tag{1}$$

$$x(0) = x_0 \tag{2}$$

- $$x(t) = x_0 + \int_0^t f(x(s))ds \quad \text{for all } t \in M$$

First recall what we are trying to prove:

Theorem 13 (Simple).

Suppose $x_0 \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ *and there exist* L, r *such that for any* $x, y \in B(x_0, r)$,

$$\|f(x) - f(y)\| \le L\|x - y\|$$

and $\|f(x)\| \le c$. *Let* $b < \min\{\frac{1}{L}, \frac{r}{c}\}$. *Then there exists a unique differentiable map* $x \in \mathcal{C}[0, b]$, *such that* $x(0) = x_0$, $x(t) \in B(x_0, r)$ *and* $\dot{x}(t) = f(x(t))$.

We will show that

$$(Px)(t) = x_0 + \int_0^t f(x(s))ds$$

is a contraction.

Proof of Existence Theorem

Proof.

For given x_0, define the space $\mathcal{B} = \{x \in \mathcal{C}[0,b] : x(0) = x_0, x(t) \in B(x_0, r)\}$ with norm $\sup_{t\in[0,b]} \|x(t)\|$ which is complete. Define the map P as

$$Px(t) = x_0 + \int_0^t f(x(s))ds$$

We first show that P maps $\mathcal{B}$ to $\mathcal{B}$. Suppose $x \in \mathcal{B}$. To show $Px \in \mathcal{B}$, we first show that Px a continuous function of t.

$$\|Px(t_2) - Px(t_1)\| = \|\int_{t_1}^{t_2} f(x(s))ds\| \le \int_{t_1}^{t_2} \|f(x(s))\|ds \le c(t_2 - t_1)$$

Thus Px is continuous. Clearly $Px(0) = x_0$. Now we show $x(t) \in B(x_0, r)$.

$$\begin{aligned}\sup_{t\in[0,b]} \|Px(t) - x_0\| &= \sup_{t\in[0,b]} \|\int_0^t f(x(s))ds\| \\ &\le \int_0^b \|f(x(s))\|ds \\ &\le bc < r\end{aligned}$$

Proof of Existence Theorem

Proof.

Now we have shown that $P : \mathcal{B} \to \mathcal{B}$. To prove existence and uniqueness, we show that Φ is a contraction.

$$\begin{aligned}\|Px - Py\| &= \sup_{t\in[0,b]} \|\int_0^t f(x(s)) - f(y(s))ds\| \\ &\le \sup_{t\in[0,b]} (\int_0^t \|f(x(s)) - f(y(s))\|ds) \le \int_0^b \|f(x(s)) - f(y(s))\|ds \\ &\le L\int_0^b \|x(s) - y(s)\|ds \le Lb\|x - y\|\end{aligned}$$

Thus, since $Lb < 1$, the map is a contraction with a unique fixed point $x \in \mathcal{B}$ such that

$$x(t) = x_0 + \int_0^t f(x(s))ds$$

By the fundamental theorem of calculus, this means that x is a differentiable function such that for $t \in [0, b]$

$$\dot{x} = f(x(t))$$

This proof is particularly important because it provides a way of actually **constructing** the solution.

Picard-Lindelöf Iteration:

- From the proof, unique solution of $Px^* = x^*$ is a solution of $\dot{x}^* = f(x^*)$, where
$$(Px)(t) = x_0 + \int_0^t f(x(s))ds$$
- From the contraction mapping theorem, the solution $Px^* = x^*$ can be found as
$$x^* = \lim_{k \to \infty} P^k z \qquad \text{for any } z \in B$$

Extension of Existence Theorem

Note that this existence theorem only guarantees existence on the interval

$$t \in \left[0, \frac{1}{L}\right] \qquad \text{or} \qquad t \in \left[0, \frac{r}{c}\right]$$

Where

- r is the size of the neighborhood near x_0
- L is a Lipschitz constant for f in the neighborhood of x_0
- c is a bound for f in the neighborhood of x_0

Note further that this theorem only gives a solution for a particular initial condition x_0

- It does not imply existence of the *Solution Map*

However, convergence of the solution map can also be proven.

Illustration of Picard Iteration

This is a plot of Picard iterations for the solution map of $\dot{x} = -x^3$.

$$z(t,x) = 0; \qquad Pz(t,x) = x; \qquad P^2 z(t,x) = x - tx^3;$$
$$P^3 z(t,x) = x - tx^3 + 3t^2x^5 - 3t^3x^7 + t^4x^9$$

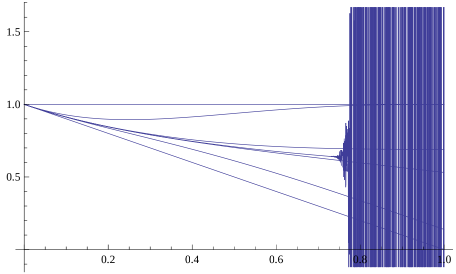


Figure : The solution for $x_0 = 1$

Convergence is only guaranteed on interval $t \in [0, .333]$.

Extension of Existence Theorem

Theorem 14 (Extension Theorem).

For a given set W and r, define the set $W_r := \{x : \|x - y\| \le r, \, y \in W\}$. Suppose that there exists a domain D and $K > 0$ such that $\|f(t,x) - f(t,y)\| \le K\|x - y\|$ for all $x, y \in D \subset \mathbb{R}^n$ and $t > 0$. Suppose there exists a compact set W and $r > 0$ such that $W_r \subset D$. Furthermore suppose that it has been proven that for $x_0 \in W$, any solution to

$$\dot{x}(t) = f(t,x), \quad x(0) = x_0$$

must lie entirely in W. Then, for $x_0 \in W$, there exists a unique solution x with $x(0) = x_0$ such that x lies entirely in W.

Illustration of Extended Picard Iteration

Picard iteration can also be used with the extension theorem

- Final time of previous Picard iterate is used to seed next Picard iterate.

Definition 15.

Suppose that the solution map ϕ exists on $t \in [0, \infty]$ and $\|\phi(t,x)\| \le K\|x\|$ for any $x \in B_r$. Suppose that f has Lipschitz factor L on B_{4Kr} and is bounded on B_{4Kr} with bound Q. Given $T < \min\{\frac{2Kr}{Q}, \frac{1}{L}\}$, let $z = 0$ and define

$$G_0^k(t,x) := (P^k z)(t,x)$$

and for $i > 0$, define the functions G_i recursively as

$$G_{i+1}^k(t,x) := (P^k z)(t, G_i^k(T,x)).$$

Define the concatenation of the G_i^k as

$$G^k(t,x) := G_i^k(t - iT, x) \quad \forall \quad t \in [iT, iT+T] \quad \text{and } i = 1, \cdots, \infty.$$

Illustration of Extended Picard Iteration

We take the previous approximation to the solution map and extend it.

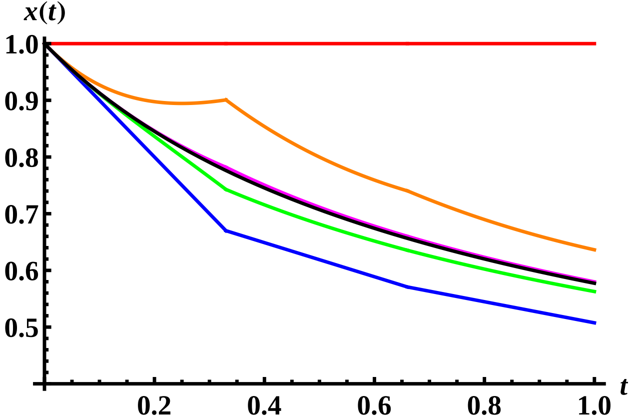


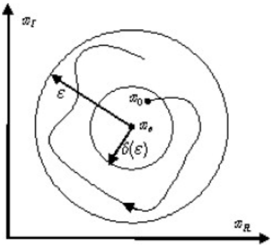
Figure : The Solution map ϕ and the functions G_i^k for $k = 1, 2, 3, 4, 5$ and $i = 1, 2, 3$ for the system $\dot{x}(t) = -x(t)^3$. The interval of convergence of the Picard Iteration is $T = \frac{1}{3}$.

Whenever you are trying to prove stability, *Please* define your notion of stability!

We denote the set of bounded continuous functions by
$\bar{\mathcal{C}} := \{x \in \mathcal{C} : \|x(t)\| \le r, r \ge 0\}$ with norm $\|x\| = \sup_t \|x(t)\|$.

Definition 16.

The system is **locally Lyapunov stable** on D where D contains an open neighborhood of the origin if it defines a unique map $\Phi : D \to \bar{\mathcal{C}}$ which is continuous at the origin.



The system is locally Lyapunov stable on D if for any $\epsilon > 0$, there exists a $\delta(\epsilon)$ such that for $\|x(0)\| \le \delta(\epsilon)$, $x(0) \subset D$ we have $\|x(t)\| \le \epsilon$ for all $t \ge 0$

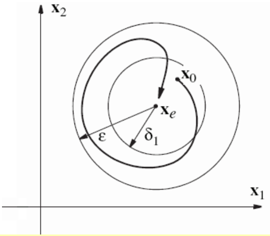
Definition 17.

The system is **globally Lyapunov stable** if it defines a unique map $\Phi : \mathbb{R}^n \to \bar{\mathcal{C}}$ which is continuous at the origin.

We define the subspace of bounded continuous functions which tend to the origin by $G := \{x \in \bar{\mathcal{C}} : \lim_{t\to\infty} x(t) = 0\}$ with norm $\|x\| = \sup_t \|x(t)\|$.

Definition 18.

The system is **locally asymptotically stable** on D where D contains an open neighborhood of the origin if it defines a map $\Phi : D \to G$ which is continuous at the origin.



Stability Definitions

Definition 19.

The system is **globally asymptotically stable** if it defines a map $\Phi : \mathbb{R}^n \to G$ which is continuous at the origin.

Definition 20.

The system is **locally exponentially stable** on D if it defines a map $\Phi : D \to G$ where

$$\|(\Phi x)(t)\| \leq Ke^{-\gamma t}\|x\|$$

for some positive constants $K, \gamma > 0$ and any $x \in D$.

Definition 21.

The system is **globally exponentially stable** if it defines a map $\Phi : \mathbb{R}^n \to G$ where

$$\|(\Phi x)(t)\| \leq Ke^{-\gamma t}\|x\|$$

for some positive constants $K, \gamma > 0$ and any $x \in \mathbb{R}^n$.

Lyapunov Theorem

$$\dot{x} = f(x), \qquad f(0) = 0$$

Theorem 22.

Let $V : D \to R$ be a continuously differentiable function such that

$$\begin{aligned} V(0) &= 0 \\ V(x) &> 0 \quad \text{for } x \in D,\, x \neq 0 \\ \nabla V(x)^T f(x) &\leq 0 \quad \text{for } x \in D. \end{aligned}$$

- *Then $\dot{x} = f(x)$ is well-posed and locally Lyapunov stable on the largest sublevel set of V contained in D.*
- *Furthermore, if $\nabla V(x) < 0$ for $x \in D$, $x \neq 0$, then $\dot{x} = f(x)$ is locally asymptotically stable on the largest sublevel set of V contained in D.*

Lyapunov Theorem

Sublevel Set: For a given Lyapunov function V and positive constant γ, we denote the set $V_\gamma = \{x : V(x) \le \gamma\}$.

Proof.

Existence: Denote the largest bounded sublevel set of V contained in the interior of D by V_{γ^*}. Because $\dot{V}(x(t)) = \nabla V(x(t))^T f(x(t)) \le 0$ is continuous, if $x(0) \in V_{\gamma^*}$, then $x(t) \in V_{\gamma^*}$ for all $t \ge 0$. Therefore since f is continuous and V_{γ^*} is compact, by the extension theorem, there is a unique solution for any initial condition $x(0) \in V_{\gamma^*}$.

Lyapunov Stability: Given any $\epsilon' > 0$, choose $e < e'$ with $B(\epsilon) \subset V_{\gamma^*}$, choose γ_i such that $V_{\gamma_i} \subset B(\epsilon)$. Now, choose $\delta > 0$ such that $B(\delta) \subset V_{\gamma_i}$. Then $B(\delta) \subset V_{\gamma_i} \subset B(\epsilon)$ and hence if $x(0) \in B(\delta)$, we have $x(0) \in V_{\gamma_i} \subset B(\epsilon) \subset B(\epsilon')$.

Asymptotic Stability:

- V monotone decreasing implies $\lim_{t\to} V(x(t)) = 0$.
- $V(x) = 0$ implies $x = 0$.
- Proof omitted.

□

Theorem 23.

Suppose there exists a continuously differentiable function V and constants $c_1, c_2, c_3 >$ and radius $r > 0$ such that the following holds for all $x \in B(r)$.

$$c_1\|x\|^p \leq V(x) \leq c_2\|x\|^p$$
$$\nabla V(x)^T f(x) \leq -c_3\|x\|^p$$

Then $\dot{x} = f(x)$ is exponentially stable on any ball contained in the largest sublevel set contained in $B(r)$.

Exponential Stability allows a quantitative prediction of system behavior.

Lemma 24 (Gronwall-Bellman).

Let λ be continuous and μ be continuous and nonnegative. Let y be continuous and satisfy for $t \le b$,

$$y(t) \le \lambda(t) + \int_a^t \mu(s) y(s) ds.$$

Then

$$y(t) \le \lambda(t) + \int_a^t \lambda(s)\mu(s) \exp\left[\int_s^t \mu(\tau) d\tau\right] ds$$

If λ and μ are constants, then

$$y(t) \le \lambda e^{\mu t}.$$

For $\lambda(t) = y(0)$, the condition can be differentiated to obtain

$$\dot{y}(t) \le \mu(t) y(t).$$

Proof.

We begin by noting that we already satisfy the conditions for existence, uniqueness and asymptotic stability and that $x(t) \in B(r)$.
For simplicity, we take $p = 2$.
Now, observe that

$$\dot{V}(x(t)) \le -c_3\|x(t)\|^2 \le -\frac{c_3}{c_2}V(x(t))$$

Which implies by the **Gronwall-Bellman** inequality ($\mu = \frac{-c_3}{c_2}$, $\lambda = V(x(0))$) that

$$V(x(t)) \le V(x(0))e^{-\frac{c_3}{c_2}t}.$$

Hence

$$\|x(t)\|^2 \le \frac{1}{c_1}V(x(t)) \le \frac{1}{c_1}e^{-\frac{c_3}{c_2}t}V(x(0)) \le \frac{c_2}{c_1}e^{-\frac{c_3}{c_2}t}\|x(0)\|^2.$$

□

Sometimes, we want to prove convergence to a set. Recall

$$V_\gamma = \{x\,,\, V(x) \le \gamma\}$$

Definition 25.

A set, X, is **Positively Invariant** if $x(0) \in X$ implies $x(t) \in X$ for all $t \ge 0$.

Theorem 26.

Suppose that there exists some continuously differentiable function V such that

$$\begin{aligned} V(x) &> 0 \quad \text{for } x \in D, x \neq 0 \\ \nabla V(x)^T f(x) &\le 0 \quad \text{for } x \in D. \end{aligned}$$

for all $x \in D$. Then for any γ such that the level set $X = \{x : V(x) = \gamma\} \subset D$, we have that V_γ is positively invariant.
Furthermore, if $\nabla V(x)^T f(x) \le 0 \quad$ for $x \in D$, then for any δ such that $X \subset V_\delta \subset D$, we have that any trajectory starting in V_δ will approach the sublevel set V_γ.

Converse Lyapunov Theory

In fact, stable systems always have Lyapunov functions.

Suppose that there exists a continuously differentiable function function, called the *solution map*, $g(x, s)$ such that

$$\frac{\partial}{\partial s} g(x, s) = f(g(x, s)) \qquad \text{and} \qquad g(x, 0) = x$$

is satisfied.

Converse Form 1:

$$V(x) = \int_0^{\delta} g(s, x)^T g(s, x) ds$$

Converse Lyapunov Theory

Converse Form 1:

$$V(x) = \int_0^\delta g(s,x)^T g(s,x) ds$$

For a linear system, $g(s,x) = e^{As}x$.

- This recalls the proof of feasibility of the Lyapunov inequality

$$A^T P + PA < 0$$

- The solution was given by

$$x^T P x = \int_0^\infty x^T e^{A^T s} e^{As} x ds = \int_0^\infty g(s,x)^T g(s,x) ds$$

Converse Lyapunov Theory

Theorem 27.

Suppose that there exist K and λ such that g satisfies

$$\|g(x,t)\| \le K\|g(x,0)\|e^{-\lambda t}$$

Then there exists a function V and constants c_1, c_2, and c_3 such that V satisfies

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$
$$\nabla V(x)^T f(x) \le -c_3\|x\|^2$$

Converse Lyapunov Theory

Proof.

There are 3 parts to the proof, of which 2 are relatively minor. But part 3 is tricky.
The main hurdle is to choose $\delta > 0$ sufficiently large
Part 1: Show that $V(x) \leq c_2\|x\|^2$. Then

$$\begin{aligned} V(x) &= \int_0^\delta \|g(s,x)\|^2 ds \\ &\leq K^2\|g(x,0)\|^2 \int_0^\delta e^{-2\lambda s} ds \\ &= \|x\|^2 \frac{K^2}{2\lambda}(1-e^{-2\lambda\delta}) = c_2\|x\|^2 \end{aligned}$$

where $c_2 = \frac{K^2}{2\lambda}(1-e^{-2\lambda\delta})$. This part holds for any $\delta > 0$.

□

Converse Lyapunov Theory

Proof.

Part 2: Show that $V(x) \geq c_1\|x\|^2$.
Lipschitz continuity of f implies $\|f(x)\| \leq L\|x\|$. By the fundamental identity

$$\|x(t)\| \leq \|x(0)\| + \int_0^t \|f(x(s))\|ds \leq \|x(0)\| + \int_0^t L\|x(s)\|ds$$

Hence by the **Gronwall-Bellman** inequality

$$\|x(0)\|e^{-Lt} \leq \|x(t)\| \leq \|x(0)\|e^{Lt}.$$

Thus we have that $\|g(x,t)\|^2 \geq \|x\|^2 e^{-Lt}$. This implies

$$\begin{aligned} V(x) &= \int_0^\delta \|g(s,x)\|^2 ds \geq \|x\|^2 \int_0^\delta e^{-2Ls} ds \\ &= \|x\|^2 \frac{1}{2L}(1 - e^{-2L\delta}) = c_1\|x\|^2 \end{aligned}$$

where $c_1 = \frac{1}{2L}(1 - e^{-2L\delta})$. This part also holds for any $\delta > 0$.

□

Converse Lyapunov Theory

Proof, Part 3.

Part 3: Show that $\nabla V(x)^T f(x) \le -c_3\|x\|^2$.
This requires differentiating the solution map with respect to initial conditions.
We first prove the identity

$$g_t(t,x) = -g_x(t,x)f(x)$$

We start with a modified version of the fundamental identity

$$g(t,x) = g(0,x) + \int_0^t f(g(s,x))ds = g(0,x) + \int_{-t}^0 f(g(s+t,x))ds$$

By the Leibniz rule for the differentiation of integrals, we find

$$\begin{aligned} g_t(t,x) &= f(g(0,x)) + \int_{-t}^0 \nabla f(g(s+t,x))^T g_s(s+t,x)ds \\ &= f(x) + \int_0^t \nabla f(g(s,x))^T g_s(s,x)ds \end{aligned}$$

Also, we have $$g_x(t,x) = I + \int_0^t \nabla f(g(s,x))^T g_x(s,x)ds$$

□

Proof, Part 3.

Now

$$\begin{aligned}
&g_t(t,x) - g_x(t,x)f(x) \\
&= x + \int_0^t \nabla f(g(s,x))^T g_s(s,x)ds + f(x) + \int_0^t \nabla f(g(s,x))^T g_x(s,x)f(x)ds \\
&= \int_0^t \nabla f(g(s,x))^T \left(g_s(s,x) - g_x(s,x)f(x)\right) ds
\end{aligned}$$

By, e.g., **Gronwall-Bellman**, this implies

$$g_t(t,x) - g_x(t,x)f(x) = 0.$$

We conclude that

$$g_t(t,x) = g_x(t,x)f(x)$$

Which is interesting.

□

Proof, Part 3.

With this identity in hand, we proceed:

$$\begin{aligned}
\nabla V(x)^T f(x) &= \left(\nabla_x \int_0^\delta g(s,x)^T g(s,x) ds\right)^T f(x) \\
&= 2\int_0^\delta g(s,x)^T g_x(s,x) f(x) ds \\
&= 2\int_0^\delta g(s,x)^T g_s(s,x) ds = \int_0^\delta \frac{d}{ds}\|g(s,x)\|^2 ds \\
&= \|g(\delta,x)\|^2 - \|g(0,x)\|^2 \\
&\le K^2\|x\|^2 e^{-2\lambda\delta} - \|x\|^2 \\
&= -\left(1 - K^2 e^{-2\lambda\delta}\right)\|x\|^2
\end{aligned}$$

Thus the third inequality is satisfied for $c_3 = 1 - K^2 e^{-2\lambda\delta}$. However, this constant is only positive if

$$\delta > \frac{\log K}{\lambda}.$$

Massera's Converse Lyapunov Theory

The Lyapunov function inherits many properties of the solution map and hence the vector field.

$$V(x) = \int_0^\delta g(s,x)^T g(s,x) ds \qquad g(t,x) = g(0,x) + \int_0^t f(g(s,x))ds$$

Massera: Let $D^\alpha = \prod_i \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}$.

- $D^\alpha V(x)$ is continuous if $D^\alpha f(x)$ is continuous.

Massera's Converse Lyapunov Theory

Formally, this means

Theorem 28 (Massera).

Consider the system defined by $\dot{x} = f(x)$ *where* $D^\alpha f \in \mathcal{C}(\mathbb{R}^n)$ *for any* $\|\alpha\|_1 \leq s$. *Suppose that there exist constants* $\mu, \delta, r > 0$ *such that*

$$\|(Ax_0)(t)\|_2 \leq \mu \|x_0\|_2 e^{-\delta t}$$

for all $t \geq 0$ *and* $\|x_0\|_2 \leq r$. *Then there exists a function* $V : \mathbb{R}^n \to \mathbb{R}$ *and constants* $\alpha, \beta, \gamma > 0$ *such that*

$$\begin{aligned} \alpha\|x\|_2^2 \leq & V(x) \leq \beta\|x\|_2^2 \\ \nabla V(x)^T f(x) \leq & -\gamma\|x\|_2^2 \end{aligned}$$

for all $\|x\|_2 \leq r$. *Furthermore,* $D^\alpha V \in \mathcal{C}(\mathbb{R}^n)$ *for any* α *with* $\|\alpha\|_1 \leq s$.

Finding $L = \sup_x \|D^\alpha V\|$

Given a Lipschitz bound for f, lets find a Lipschitz constant for V?

$$V(x) = \int_0^\delta g(s,x)^T g(s,x) ds \qquad g(t,x) = x + \int_0^t f(g(s,x)) ds$$

We first need a Lipschitz bound for the solution map:

$$L_g = \sup_x \|\nabla_x g(s,x)\|$$

From the identity

$$g_x(t,x) = I + \int_0^t \nabla f(g(s,x)) g_x(s,x) ds$$

we get

$$\|g_x(t,x)\| = \leq 1 + \int_0^t L \|g_x(s,x)\| ds$$

which implies by **Gronwall-Bellman** that $\|g_x(t,x)\| \leq e^{Lt}$

Finding $L = \sup_x \|D^\alpha V\|$

Faà di Bruno's formula

What about a bound for $\|D^\alpha V(x)\|$?

$$D^\alpha g(t,x) = \int_0^t D^\alpha f(g(s,x)) g_x(s,x) ds$$

Faà di Bruno's formula: For scalar functions

$$\frac{d^n}{dx^n} f(g(y)) = \sum_{\pi \in \Pi} f^{(|\pi|)}(g(y)) \cdot \prod_{B \in \pi} g^{(|B|)}(x).$$

where Π is the set of partitions of $\{1, \ldots, n\}$, and $|\cdot|$ denotes cardinality.

We can generalize Faà di Bruno's formula to functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$.

The combinatorial notation allows us to keep track of terms.

A Generalized Chain Rule

Definition 29.

Let Ω_r^i denote the set of partitions of $(1,\ldots,r)$ into i non-empty subsets.

Lemma 30 (Generalized Chain Rule).

Suppose $f:\mathbb{R}^n \to \mathbb{R}$ *and* $z:\mathbb{R}^n \to \mathbb{R}^n$ *are* r*-times continuously differentiable. Let* $\alpha \in \mathbb{N}^n$ *with* $|\alpha|_1 = r$*. Let* $\{a_i\} \in Z^r$ *be any decomposition of* α *so that* $\alpha = \sum_{i=1}^r a_i$*.*

$$D_x^\alpha f(z(x)) =$$

$$\sum_{i=1}^{r} \sum_{j_1=1}^{n} \cdots \sum_{j_i=1}^{n} \frac{\partial^i}{\partial x_{j_1} \cdots \partial x_{j_i}} f(z(x)) \times$$

$$\sum_{\beta \in \Omega_r^i} \prod_{k=1}^{i} D^{\sum_{l \in \beta_k} a_l} z_{j_k}(x)$$

A Quantitative Massera-style Converse Lyapunov Result

We can use the generalized chain rule to get the following.

Theorem 31.

- *Suppose that* $\|D^\beta f\|_\infty \le L$ *for* $\|\beta\|_\infty \le r$
- *Suppose* $\dot{x} = f(x)$ *satisfies* $\|x(t)\| \le k\|x(0)\|e^{-\lambda t}$ *for* $\|x(0)\| \le 1$.

Then there exists a function V *such that*

- V *is exponentially decreasing on* $\|x\| \le 1$.
- $D^\alpha V$ *is continuous on* $\|x\| \le 1$ *for* $\|\alpha\|_\infty \le r$ *with upper bound*

$$\max_{|\alpha|_1 < r} \|D^\alpha V(x)\| \le c_1 2^r \left(B(r) \frac{L}{\lambda} e^{c_2 \frac{L}{\lambda}} \right)^{er!}$$

for some $c_1(k,n)$ *and* $c_2(k,n)$.

- Also a bound on the continuity of the solution map.
- $B(r)$ is the Ball number.

Approximation Theory for Lyapunov Functions

Theorem 32 (Approximation).

- *Suppose f is bounded on compact X.*
- *Suppose that $D^\alpha V$ is continuous for $\|\alpha\|_\infty \le 3$.*

Then for any $\delta > 0$, there exists a polynomial, p, such that for $x \in X$,

$$\|V(x) - p(x)\| \le \delta \|x\|^2 \qquad \textit{and} \qquad \left\|\nabla (V(x) - p(x))^T f(x)\right\| \le \delta \|x\|^2$$

- Polynomials can approximate differentiable functions arbitrarily well in Sobolev norms with a quadratic upper bound on the error.

Consider the system

$$\dot{x}(t) = f(x(t))$$

Theorem 33.

- *Suppose $\dot{x}(t) = f(x(t))$ is exponentially stable for $\|x(0)\| \le r$.*
- *Suppose $D^\alpha f$ is continuous for $\|\alpha\|_\infty \le 3$.*

Then there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ such that

- *V is exponentially decreasing on $\|x\| \le r$.*
- *V is a* **polynomial**.

Implications:

- Using polynomials is not conservative.

Question:

- What is the degree of the Lyapunov function
  - How many coefficients do we need to optimize?

Degree Bounds in Approximation Theory

This result uses the Bernstein polynomials to give a degree bound as a function of the error bound.

Theorem 34.

- *Suppose* $V : R^n \to \mathbb{R}^n$ *has Lipschitz constant* L *on the unit ball.*

$$\|V(x) - V(y)\|_2 < L\|x - y\|_2$$

Then for any $\epsilon > 0$*, there exists a* **polynomial**, p*, which satisfies*

$$\sup_{\|x\| \le 1} \|p(x) - V(x)\|_2 < \epsilon$$

where

$$\textit{degree}(p) \le \frac{n}{4^2} \left(\frac{L}{\epsilon}\right)^2$$

To find a bound on L, we can use a bound on $D^\alpha V$.

A Bound on the Complexity of Lyapunov Functions

Theorem 35.

- *Suppose* $\|x(t)\| \leq K\|x(0)\|e^{-\lambda t}$ *for* $\|x(0)\| \leq r$.
- *Suppose* f **is polynomial** *and* $\|\nabla f(x)\| \leq L$ *on* $\|x\| \leq r$.

Then there exists a polynomial $V \in \Sigma_s$ *such that*

- V *is exponentially decreasing on* $\|x\| \leq r$.
- *The degree of* V *is less than*

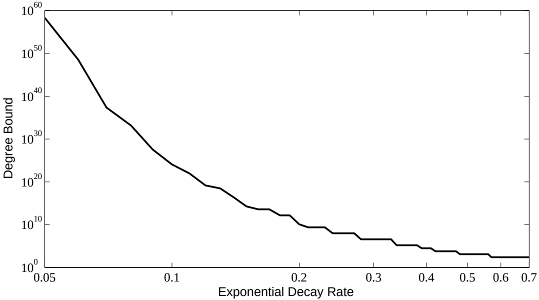
$$\text{degree}(V) \leq 2q^{2(Nk-1)} \cong 2q^{2c_1\frac{L}{\lambda}}$$

where q *is the degree of the vector field,* f.

$$V(x) = \int_0^{\delta} G_k(x,s)^T G_k(x,s)$$

- G_k is an extended Picard iteration.
- k is the number of Picard iterations and N is the number of extensions.

Note that the Lyapunov function is a square of polynomials.



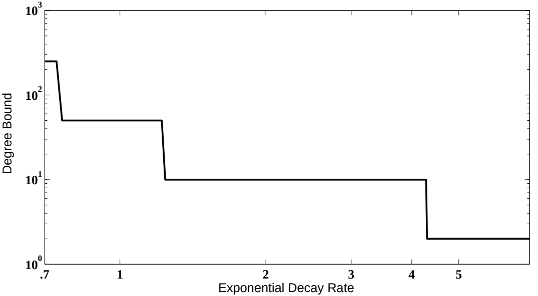


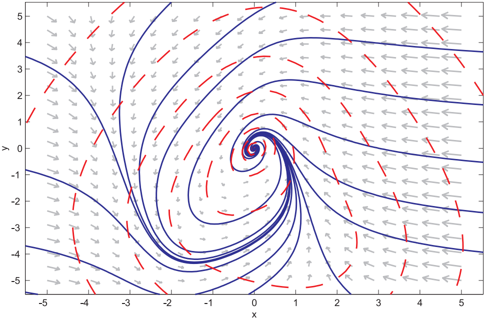
Figure : Degree bound vs. Convergence Rate for $K = 1.2$, $r = L = 1$, and $q = 5$

Returning to the Lyapunov Stability Conditions

Consider

$$\dot{x}(t) = f(x(t))$$

with $x(0) \in \mathbb{R}^n$.



Theorem 36 (Lyapunov Stability).

Suppose there exists a continuous V and $\alpha, \beta, \gamma > 0$ where

$$\beta\|x\|^2 \le V(x) \le \alpha\|x\|^2$$
$$-\nabla V(x)^T f(x) \ge \gamma\|x\|^2$$

for all $x \in X$. Then any sub-level set of V in X is a **Domain of Attraction**.

The Stability Problem is Convex

Convex Optimization of Functions: Variables $V \in \mathcal{C}[\mathbb{R}^n]$ and $\gamma \in \mathbb{R}$

$$\max_{V,\gamma} \; \gamma$$

subject to

$$V(x) - x^T x \geq 0 \qquad \forall x$$
$$\nabla V(x)^T f(x) + \gamma x^T x \leq 0 \qquad \forall x$$

Moreover, since we can assume V is polynomial with bounded degree, the problem is *finite-dimensional*.

Convex Optimization of Polynomials: Variables $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$

$$\max_{c,\gamma} \; \gamma$$

subject to

$$c^T Z(x) - x^T x \geq 0 \qquad \forall x$$
$$c^T \nabla Z(x) f(x) + \gamma x^T x \leq 0 \qquad \forall x$$

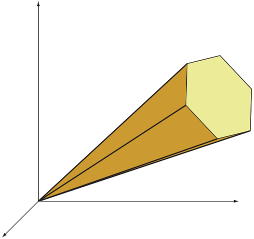
- $Z(x)$ is a fixed vector of monomial bases.

Can we solve optimization of polynomials?

Problem:

$$\max\ b^T x$$

$$\text{subject to } A_0(y) + \sum_i^n x_i A_i(y) \succeq 0 \quad \forall y$$



The A_i are matrices of polynomials in y. e.g. Using multi-index notation,

$$A_i(y) = \sum_\alpha A_{i,\alpha}\ y^\alpha$$

Computationally Intractable

The problem: "Is $p(x) \geq 0$ for all $x \in \mathbb{R}^n$?" (i.e. "$p \in \mathbb{R}^+[x]$?") is NP-hard.

Conclusions

Nonlinear Systems are relatively well-understood.

Well-Posed

- Existence and Uniqueness guaranteed if vector field and its gradient are bounded.
  - Contraction Mapping Principle
- The dependence of the solution map on the initial conditions
  - Properties are inherited from the vector field via Gronwall-Bellman

Lyapunov Stability

- Lyapunov's conditions are necessary and sufficient for stability.
  - Problem is to find a Lyapunov function.
- Converse forms provide insight.
  - Capture the inherent energy stored in an initial condition
- We can assume the Lyapunov function is polynomial of bounded degree.
  - Degree may be very large.
  - We need to be able to optimize the cone of positive polynomial functions.