Partial Differential Equations and Time-Delay Systems

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Lecture 03: PDEs and Delays

Partial Differential Equations: Common Examples

Heat Equation (Newton):

$$u_t = u_{xx}$$

Boundary Conditions:

$u(0) = 0,$	$u(L) = 0$	(Dirichlet)
$u(0) = 0,$	$u_x(0) = 0$	(Neumann)

Wave Equation (d'Alembert):

$$u_{tt} = u_{xx}$$

Boundary Conditions:

$u(0) = 0,$	$u(L) = 0$	(Fixed ends)
$u(0) = 0,$	$u_x(L) = 0$	(Free end)
$au(0) = u_x(0),$	$bu(0) = -u_x(L)$	(S-L)

Partial Differential Equations: Common Examples

Euler-Bernoulli Beam Equation:

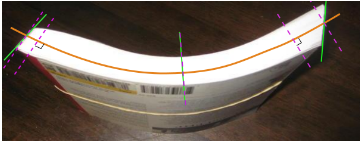
$$u_{tt} = u_{xxxx}$$

Boundary Conditions:

$u(0) = 0,$	$u(L) = 0$	(Fixed ends)
$u(0) = 0,$	$u_x(L) = 0$	(Free end)

Timoschenko Beam Equation:

$$w_{tt} = -\phi_s + w_{ss},$$
$$\phi_{tt} = -\phi + w_s + \phi_{ss}$$



Boundary Conditions:

$\phi(0) = 0,\ w(0) = 0,\ \phi_s(L) = 0,\ w_s(L) - \phi(L) = 0$	(Cantilevered)
$\phi_s(0) = 0,\ w(0) = 0,\ \phi_s(L) = 0,\ w(L) = 0$	(Simply Supported)

Dynamics are usually expressed in the Primal State $\mathbf{x}_p \in X_p$:

$$\mathbf{x}_p \in L^2_{n_1} \times H^1_{n_2} \times H^2_{n_3} := X_p$$

$$\begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}_t = A_0(s) \underbrace{\begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}}_{\mathbf{x}_p} + A_1(s) \begin{bmatrix} x_2(t,s) \\ x_3(t,s) \end{bmatrix}_s + A_2(s) \begin{bmatrix} x_3(t,s) \end{bmatrix}_{ss}$$

Boundary Conditions:

$$B \begin{bmatrix} x_2(0) \\ x_2(L) \\ x_3(0) \\ x_3(L) \\ x_{3s}(0) \\ x_{3s}(L) \end{bmatrix} = 0, \quad \text{rank}(B) = n_2 + 2n_3$$

Euler-Bernoulli Beam:

$$\mathbf{u}_t = \underbrace{\begin{bmatrix} 0 & -c \\ 1 & 0 \end{bmatrix}}_{=A_2 \; (A_0 = A_1 = 0)} \mathbf{u}_{ss}$$

State Space: $u \in H^2_2$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u(0) \\ u(L) \\ u_s(0) \\ u_s(L) \end{bmatrix} = 0$$

Illustration 1: The Euler-Bernoulli Beam

Consider a simple cantilevered E-B beam:

$$u_{tt}(t,s) = -c u_{ssss}(t,s), \qquad \text{where} \quad u(0) = u_s(0) = u_{ss}(L) = u_{sss}(L) = 0$$

Step 1: Eliminate the u_{tt} term (let $u_1 = u_t$)
Step 2: Eliminate u_{ssss} (let $u_2 = u_{ss}$)

$$\dot{u}_1 = u_{tt} = -c u_{ssss} = -c u_{2ss}, \qquad \dot{u}_2 = u_{tss} = u_{1ss}.$$

Universal Formulation:

$$\mathbf{x}_t = \underbrace{\begin{bmatrix} 0 & -c \\ 1 & 0 \end{bmatrix}}_{A_2} \mathbf{x}_{ss}$$

where $A_0 = A_1 = 0$, $n_3 = 2$, and $n_1 = n_2 = 0$.

Boundary Conditions:

$$u_{ss}(L) = u_2(L) = 0 \quad \text{and} \quad u_{sss}(L) = u_{2s}(L) = 0.$$

Insufficient BCs! - $rank(B) = 2$. Differentiate BCs in time to get:

$$u_t(0) = u_1(0) = 0 \quad \text{and} \quad u_{ts}(0) = u_{1s}(0) = 0.$$

This yields $rank(B) = 4$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_1(L) \\ u_2(L) \\ u_{1s}(0) \\ u_{2s}(0) \\ u_{1s}(L) \\ u_{2s}(L) \end{bmatrix} = 0.$$

Illustration 2: The Timoschenko Beam

Consider a simple Timoschenko beam model:

$$\ddot{w} = \partial_s(w_s - \phi) \qquad = -\phi_s + w_{ss}$$
$$\ddot{\phi} = \phi_{ss} + (w_s - \phi) \qquad = -\phi + w_s + \phi_{ss}$$

with boundary conditions

$$\phi(0) = 0, \quad w(0) = 0, \quad \phi_s(L) = 0, \quad w_s(L) - \phi(L) = 0$$

Step 1: Eliminate w_{tt} and ϕ_{tt} - $u_1 = w_t$ and $u_3 = \phi_t$.
Step 2: Use BCs to pick the state - $u_2 = w_s - \phi$ and $u_4 = \phi_s$.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_0} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}}_{\mathbf{x}_2} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_s$$

where $A_2 = []$ and $n_1 = n_3 = 0$ and $n_2 = 4$ - a purely "hyperbolic" form. We only need 4 BCs:

$$u_1(0) = 0, \quad u_3(0) = 0, \quad u_4(L) = 0, \quad u_2(L) = 0$$

This gives a B has row rank $n_2 = 4$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \\ u_4(0) \\ u_1(L) \\ u_2(L) \\ u_3(L) \\ u_4(L) \end{bmatrix} = 0$$

Illustration 2b: The Timoschenko Beam revisited

Consider a modification - naively choose $u_2 = w_s$ and $u_4 = \phi$. This leads to

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A_0} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_s + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{A_2} u_{4ss}$$

where $n_1 = 0$, $n_2 = 3$, and $n_3 = 1$ and with 5 boundary conditions

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \\ u_1(L) \\ u_2(L) \\ u_3(L) \\ u_4(0) \\ u_4(L) \\ u_{4s}(0) \\ u_{4s}(L) \end{bmatrix} = 0.$$

NOT Stable in the given states!

However: If we add a damping term $-cu_{4t} = -cu_3$ to $\dot{u}_3$, then the only change is

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -c & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now Stable for any $c > 0$! Stability is sensitive to definition of states!

Illustration 3: The Tip-Damped Wave Equation

The simplest tip-damped wave equation is

$$u_{tt}(t,s) = u_{ss}(t,s) \qquad u(t,0) = 0 \qquad u_s(t,L) = -k u_t(t,L).$$

Guided by the boundary conditions, we choose

$$u_1(t,s) = u_s(t,s)$$
$$u_2(t,s) = u_t(t,s)$$

This yields

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_s}_{x_2}$$

where $A_0 = 0$, $A_2 = []$ $n_1 = n_3 = 0$ and $n_2 = 2$. The BCs are now

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_1(L) \\ u_2(L) \end{bmatrix} = 0.$$

Illustration 4: Non-"Hyperbolic" Damped Wave Equation

Add u to the dynamics (stable for $a, k \neq 0$)

$$\begin{aligned} & u_{tt}(t,s) = u_{ss}(t,s) - 2au_t(t,s) - a^2 u(t,s) \qquad s \in [0,1] \\ \text{BCs:} \quad & u(t,0) = 0, \qquad\qquad u_s(t,1) = -ku_t(t,1) \end{aligned}$$

Must choose the variables $u_1 = u_t$ and $u_2 = u$. Yields the diffusive form:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \underbrace{\begin{bmatrix} -2a & -a^2 \\ 1 & 0 \end{bmatrix}}_{A_0} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{A_2} \underbrace{u_{2ss}}_{x_3}$$

where $A_1 = 0$, $n_1 = 0$, $n_2 = 1$, and $n_3 = 1$. The BCs on u_1 make us consider this a hyperbolic state!

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_1(L) \\ u_2(0) \\ u_2(L) \\ u_{2s}(0) \\ u_{2s}(L) \end{bmatrix} = 0.$$

Stable!, but not exponentially stable in the given state.

Why are PDEs so hard?

Answer: Dynamics Governed by 3 Separate Equations???

Laplace Equation:

$$(\Delta u)(s) = 0$$

Heat Equation:

$$\dot{u}(t,s) = (\Delta u)(t,s)$$

Boundary Conditions:

$$u(t,s) = 0 \qquad \forall s \in \Gamma$$

Question: Why do we have BCs?

Answer: To make the solution unique.

Q: Are BCs part of the state?

A: No!

Q: Why do we need them?

A: Otherwise solution not unique.

Q: Are all PDE solns sort of the same?

A: No!

Q: Can BCs change the dynamics?

A: Yes!

Who Came up with BCs, anyway?

Semigroup Correction:

$$u \in D(\mathcal{A}) := \{u \in H^2 \;:\; u(0) = 0, u(1) = 0\}$$

Euler-Bernoulli Beam:

$$\mathbf{u}_t = \underbrace{\begin{bmatrix} 0 & -c \\ 1 & 0 \end{bmatrix}}_{=A_2 \; (A_0 = A_1 = 0)} \mathbf{u}_{ss}$$

State Space: $u \in H_2^2$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u(0) \\ u(L) \\ u_s(0) \\ u_s(L) \end{bmatrix} = 0$$

Looking For A Universal Formulation

Dynamics are usually expressed in the Primal State $\mathbf{x}_p \in X_p$:

$$\mathbf{x}_p \in L^2_{n_1} \times H^1_{n_2} \times H^2_{n_3} := X_p$$

$$\begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}_t = A_0(s) \underbrace{\begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}}_{\mathbf{x}_p} + A_1(s) \begin{bmatrix} x_2(t,s) \\ x_3(t,s) \end{bmatrix}_s + A_2(s) \begin{bmatrix} x_3(t,s) \end{bmatrix}_{ss}$$

Boundary Conditions:

$$B \begin{bmatrix} x_2(0) \\ x_2(L) \\ x_3(0) \\ x_3(L) \\ x_{3s}(0) \\ x_{3s}(L) \end{bmatrix} = 0, \quad \text{rank}(B) = n_2 + 2n_3$$

Euler-Bernoulli Beam:

$$\mathbf{u}_t = \underbrace{\begin{bmatrix} 0 & -c \\ 1 & 0 \end{bmatrix}}_{=A_2\ (A_0 = A_1 = 0)} \mathbf{u}_{ss}$$

State Space: $u \in H^2_2$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u(0) \\ u(L) \\ u_s(0) \\ u_s(L) \end{bmatrix} = 0$$

The BCs strongly influence the dynamics!

Extreme Example: $D(\mathcal{A}) = \{\mathbf{u} \in H^2 \ : \ \mathbf{u}(0) = w_1(t), \ \mathbf{u}_s(0) = w_2(t)\}$

$$\dot{\mathbf{u}}(t,s) = \mathbf{u}(t,s), \qquad \mathbf{u}(t,0) = w_1(t), \quad \mathbf{u}_s(t,0) = w_2(t)$$

By the Fundamental Theorem of Calculus:

$$\begin{aligned}\mathbf{u}(s) &= s\mathbf{u}(0) + \mathbf{u}_s(0) + \int_0^s (s-\eta)\mathbf{u}_{ss}(\eta)d\eta \\ &= sw_1(t) + w_2(t) + \int_0^s (s-\eta)\mathbf{u}_{ss}(\eta)d\eta\end{aligned}$$

Now rewrite the dynamics in terms of $\mathbf{u}_{ss}$:

$$\dot{\mathbf{u}}(t,s) = sw_1(t) + w_2(t) + \int_0^s (s-\eta)\mathbf{u}_{ss}(t,\eta)d\eta$$

Time-Delay System:

$$\begin{aligned}\dot{x}(t) &= -x(t) + u(t,-\tau) \\ \mathbf{u}_t(t,s) &= \mathbf{u}_s(t,s), \quad u(t,0) = x(t)\end{aligned}$$

or completely eliminate BCs:

$$\int_0^s \dot{\mathbf{u}}_s(t,\eta)d\eta = \mathbf{u}_s(t,s) + \int_0^\tau \mathbf{u}_s(t,\eta)d\eta$$

Conclusion: The BCs *fundamentally alter* the structure of the dynamics!

What is the Fundamental State? (BCs force us to choose $\mathbf{x}_f = \mathbf{u}_{ss}$)

Problems with the Primal State

Simplify the dynamics

$$\dot{\mathbf{x}}(t,s) = A_0(s)\mathbf{x} + A_1(s)\mathbf{x}_s + A_2(s)\mathbf{x}_{ss}$$

Define a Lyapunov Function:

$$V(\mathbf{x}) = \int_0^L \mathbf{x}(s)^T M(s)\mathbf{x}(s)ds$$

Then $V(x) > 0$ if $M(s) \geq 0$ for all s. However,

$$\dot{V}(\mathbf{x}) = \int_0^L \begin{bmatrix}\mathbf{x}\\ \mathbf{x}_s\\ \mathbf{x}_{ss}\end{bmatrix}(s)^T \underbrace{\begin{bmatrix} A_0(s)^T M(s) + M(s)A_0(s) & M(s)A_1(s) & M(s)A_2(s)\\ A_1(s)^T M(s) & 0 & 0\\ A_2(s)^T M(s) & 0 & 0\end{bmatrix}}_{D(s)} \begin{bmatrix}\mathbf{x}\\ \mathbf{x}_s\\ \mathbf{x}_{ss}\end{bmatrix}(s)ds$$

Problem: $D(s) \not< 0$ for ANY choice of A_i! Why?

Answer: $\mathbf{x}, \mathbf{x}_s, \mathbf{x}_{ss}$ are not independent states!

Old Solution: IBP, Poincaré, Bessel, Jensen, Wirtinger, Agmon, Young, et c.

New Solution: Express the dynamics using the **Fundamental State**

The **Fundamental State:** is the *minimal* part of $\mathbf{x}$ which is needed to define the dynamics

The Traditional PDE Toolbox

Poincare Inequality: For any $u \subset W_0^{1,p}(\Omega)$ there is a $C(\Omega, p)$ such that

$$\|u\|_{L^p} \le C\|\nabla u\|_{L^p}$$

Wirtinger Inequality The 1D Poincare inequality with constant c=1.
If $f : \mathbb{R} \to \mathbb{R} \in \mathbb{C}^1$ satisfies,

$$\int_0^{2\pi} f(x)dx = 0$$

Then

$$\|f\|_{L^2} \le C\|f_s\|_{L^2}$$

Integration By Parts

$$\int_a^b u(s)v_s(s)ds = u(b)v(b) - u(a)v(a) - \int_a^b u_s(s)v(s)ds$$

Leibnitz Rule for Differentiation of Integrals

$$\frac{d}{dt}\int_{a(t)}^{b(t)} u(s,t)ds = \dot{b}(t)u(b(t),t) - \dot{a}(t)u(a(t),t) + \int_{a(t)}^{b(t)} \dot{u}(s,t)ds$$

The Traditional Approach: Stability of Heat Equation

$$u_t = u + u_{ss} \qquad u(0) = u_s(0) = u(1) = u_s(1) = 0$$

$$V(t) = \int_0^1 u(t,s)^2 ds$$

$$\dot{V}(t) = \int_0^1 u(t,s)^T u_t(t,s) ds = \int_0^1 u(t,s)^T u_{ss}(t,s) ds + \int_0^1 u(t,s)^2 ds$$

By IBP,

$$\int_0^1 u(s)^T u_{ss}(s) ds = u(1)u_s(1) - u(0)u_s(0) - \int_0^1 u_s(s)^2 ds = -\int_0^1 u_s(s)^2 ds$$

By Poincare,

$$\int_0^1 u(s)^2 ds \le 1/\pi^2 \int_0^1 u_s(s)^2 ds$$

Hence

$$\begin{aligned}\dot{V}(t) &= \int_0^1 u(t,s)^T u_{ss}(t,s) ds + \int_0^1 u(t,s)^2 ds \\ &\le -\int_0^1 u_s(s)^2 ds + 1/\pi^2 \int_0^1 u_s(s)^2 ds = -\left(1 - \frac{1}{\pi^2}\right) \int_0^1 u_s(s)^2 ds \le 0\end{aligned}$$

The Fundamental Theorem of Calculus - Extended Edition

Goal: An Algorithmic Approach?
Question: How can Computers understand Integration by Parts???

Combination of FTC and IBP!

$$\mathbf{x}(s) = x(0) + \int_0^s \mathbf{x}_s(\eta)d\eta$$

$$\mathbf{x}(s) = x(0) + sx_s(0) + \int_0^s (s-\eta)\mathbf{x}_{ss}(\eta)d\eta$$

$$\mathbf{x}(s) = x(0) + sx_s(0) + \frac{s^2}{2}x_{ss}(0) + \int_0^s \frac{(s-\eta)^2}{2}\mathbf{x}_{sss}(\eta)d\eta$$

$$\mathbf{x}(s) = x(0) + sx_s(0) + \frac{s^2}{2}x_{ss}(0) + \frac{s^3}{6}x_{sss}(0) + \int_0^s \frac{(s-\eta)^3}{6}\mathbf{x}_{ssss}(\eta)d\eta$$

Introducing Partial Integral Operators

How to Represent the relationship:

$$\mathbf{x}(s) = x(0) + s x_s(0) + \int_0^s (s-\eta)\mathbf{x}_{ss}(\eta)d\eta$$

$$\mathbf{x}(s) = x(0) + s x_s(0) + \frac{s^2}{2}x_{ss}(0) + \int_0^s \frac{(s-\eta)^2}{2}\mathbf{x}_{sss}(\eta)d\eta?$$

Partial Integral Operators:

$$\left(\mathcal{P}_{\{N_0,N_1,N_2\}}\mathbf{x}\right)(s) := N_0(s)\mathbf{x}(s)ds + \int_a^s N_1(s,\theta)\mathbf{x}(\theta)d\theta + \int_s^b N_2(s,\theta)\mathbf{x}(\theta)d\theta$$

$$\begin{aligned}\mathbf{x}(s) &= x(0) + \mathcal{P}_{\{0,I,0\}}\mathbf{x}_s \\ &= \begin{bmatrix} I & s \end{bmatrix}\begin{bmatrix} x(0) \\ x_s(0)\end{bmatrix} + \mathcal{P}_{\{0,s-\eta,0\}}\mathbf{x}_{ss} \\ &= \begin{bmatrix} I & s & \frac{s^2}{s} \end{bmatrix}\begin{bmatrix} x(0) \\ x_s(0) \\ x_{ss}(0)\end{bmatrix} + \mathcal{P}_{\left\{0,\frac{(s-\eta)^2}{2},0\right\}}\mathbf{x}_{sss}\end{aligned}$$

Partial Integral Equations (PIEs)

An **ALGEBRAIC** Representation of PDEs

$$\dot{\mathbf{x}}_p(t) = \begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}_t = A_0(s) \underbrace{\begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}}_{\mathbf{x}_p} + A_1(s) \begin{bmatrix} x_2(t,s) \\ x_3(t,s) \end{bmatrix}_s + A_2(s) \begin{bmatrix} x_3(t,s) \end{bmatrix}_{ss}$$

Original Form:

$$\mathbf{x} = \mathcal{A}_d \mathbf{x}, \qquad x(0) = Bw(t)$$

where $\mathcal{A}_d$ is a differential (unbounded) operator.

Define the Fundamental (PIE) State:

$$\mathbf{x}_f(t,s) := \begin{bmatrix} x_1(t,s) \\ x_{2s}(t,s) \\ x_{3ss}(t,s) \end{bmatrix}$$

PIE Format: Write the PDE with Partial Integral Operators!

$$\mathcal{E}\dot{\mathbf{x}}_f(t) = \mathcal{A}\mathbf{x}_f(t) + \mathcal{B}w(t)$$

where $\mathcal{E}, \mathcal{A}, \mathcal{B}$ are PIE Operators (bounded).

Examples of PIE Format (no BCs)

Heat Equation: $\dot{\mathbf{u}}(t,s) = \mathbf{u}_{ss}(t,s),\ \mathbf{u}(t,0) = \mathbf{u}_s(t,0) = 0$

$$\int_0^s (s-\eta)\dot{\mathbf{u}}_{ss}(t,\eta)d\eta = \mathbf{u}_{ss}(t,s)$$

$$\mathcal{P}_{\{0,s-\eta,0\}}\dot{\mathbf{u}}(t) = \mathcal{P}_{\{I,0,0\}}\mathbf{u}(t)$$

Previous Example: $\dot{\mathbf{u}}(t,s) = \mathbf{u}(t,s),\ \mathbf{u}(t,0) = w_1(t),\ \mathbf{u}_s(t,0) = w_2(t)$

$$\int_0^s (s-\eta)\dot{\mathbf{u}}_{ss}(t,\eta)d\eta = \int_0^s (s-\eta)\mathbf{u}_{ss}(t,\eta)d\eta + s(w_1(t)-\dot{w}_1(t)) + (w_2(t)-\dot{w}_2(t))$$

$$\mathcal{P}_{\{0,s-\eta,0\}}\dot{\mathbf{u}}(t) = \mathcal{P}_{\{0,s-\eta,0\}}\mathbf{u}(t) + \mathcal{P}_{\{[s \quad 1],0,0\}}\begin{bmatrix} w_1(t)-\dot{w}_1(t) \\ w_2(t)-\dot{w}_2(t)\end{bmatrix}$$

Composition in the PIE $\mathcal{P}_{\{N_0,N_1,N_2\}}$ Operator Algebra

Property 1: Composition

$$\mathcal{P}_{\{R_0,R_1,R_2\}} = \mathcal{P}_{\{B_0,B_1,B_2\}}\mathcal{P}_{\{N_0,N_1,N_2\}}$$

where

$$\begin{aligned}
R_0(s) &= B_0(s)N_0(s) \\
R_1(s,\theta) &= B_0(s)N_1(s,\theta) + B_1(s,\theta)N_0(\theta) + \int_a^\theta B_1(s,\xi)N_2(\xi,\theta)d\xi \\
&\quad + \int_\theta^s B_1(s,\xi)N_1(\xi,\theta)d\xi + \int_s^b B_2(s,\xi)N_1(\xi,\theta)d\xi \\
R_2(s,\theta) &= B_0(s)N_2(s,\theta) + B_2(s,\theta)N_0(\theta) + \int_a^s B_1(s,\xi)N_2(\xi,\theta)d\xi \\
&\quad + \int_s^\theta B_2(s,\xi)N_2(\xi,\theta)d\xi + \int_\theta^b B_2(s,\xi)N_1(\xi,\theta)d\xi
\end{aligned}$$

Triple Notation:

$$\{R_0, R_1, R_2\} = \{B_0, B_1, B_2\} \times \{N_0, N_1, N_2\}$$

Matlab Implementation:

$$\{N_0, N_1, N_2\} = \{T_0, T_1, T_2\} \times \{R_0, R_1, R_2\} \quad \rightarrow \quad \mathcal{P}_{\{N_0,N_1,N_2\}} = \mathcal{P}_{\{T_0,T_1,T_2\}}\mathcal{P}_{\{R_0,R_1,R_2\}}$$

```
[N0, N1, N2] = PL2L_compose(T0,T1,T2,R0,R1,R2,s,th,[a,b])
```

Composition in the PIE $\mathcal{P}_{\{N_0,N_1,N_2\}}$ Operator Algebra

Property 1: Composition

$$\mathcal{P}_{\{R_0,R_1,R_2\}} = \mathcal{P}_{\{B_0,B_1,B_2\}}\mathcal{P}_{\{N_0,N_1,N_2\}}$$

where

$$R_0(s) = B_0(s)N_0(s)$$

$$R_1(s,\theta) = B_0(s)N_1(s,\theta) + B_1(s,\theta)N_0(\theta) + \int_a^\theta B_1(s,\xi)N_2(\xi,\theta)d\xi + \int_\theta^s B_1(s,\xi)N_1(\xi,\theta)d\xi + \int_s^b B_2(s,\xi)N_1(\xi,\theta)d\xi$$

$$R_2(s,\theta) = B_0(s)N_2(s,\theta) + B_2(s,\theta)N_0(\theta) + \int_a^s B_1(s,\xi)N_2(\xi,\theta)d\xi + \int_s^\theta B_2(s,\xi)N_2(\xi,\theta)d\xi + \int_\theta^b B_2(s,\xi)N_1(\xi,\theta)d\xi$$

Triple Notation:

$$\{R_0, R_1, R_2\} = \{B_0, B_1, B_2\} \times \{N_0, N_1, N_2\}$$

Matlab Implementation:

$$\{N_0, N_1, N_2\} = \{T_0, T_1, T_2\} \times \{R_0, R_1, R_2\} \quad \rightarrow \quad \mathcal{P}_{\{N_0,N_1,N_2\}} = \mathcal{P}_{\{T_0,T_1,T_2\}}\mathcal{P}_{\{R_0,R_1,R_2\}}$$

```
[N0, N1, N2] = PL2L_compose(T0,T1,T2,R0,R1,R2,s,th,[a,b])
```

Actually, new parsers work directly with operator objects (As opposed to Triples)

- However, we are not ready to define these operators quite yet.
- These operators act on $\mathbb{R}^m \times L_2^n$.

Matlab Implementation:

```
T.R.R0=T0; T.R.R1=T1; T.R.R2=T2; R.R.R0=R0; R.R.R1=R1; R.R.R2=R2
```

$$\{N_0, N_1, N_2\} = \{T_0, T_1, T_2\}\times\{R_0, R_1, R_2\} \quad \rightarrow \quad \mathcal{P}_{\{N_0,N_1,N_2\}} = \mathcal{P}_{\{T_0,T_1,T_2\}}\mathcal{P}_{\{R_0,R_1,R_2\}}$$

```
N = compose_p(T,R,s,theta,[a,b])
N0=N.R.R0; N1=N.R.R1; N2=N.R.R2
```

Notation:

- Define $P_{\{N_i\}} := P_{\{N_0,N_1,N_2\}}$
- When the N_i are clear from context...

Transpose/Adjoint in the PIE $\mathcal{P}_{\{N_i\}}$ Operator Algebra

Property 2: Transpose/Adjoint

$$\langle \mathbf{x}, \mathcal{P}_{\{\hat{N}_0,\hat{N}_1,\hat{N}_2\}}\mathbf{y}\rangle_{L_2} = \langle \mathcal{P}_{\{N_0,N_1,N_2\}}\mathbf{x},\mathbf{y}\rangle_{L_2}$$

where

$$\hat{N}_0(s) = N_0(s)^T, \quad \hat{N}_1(s,\eta) = N_2(\eta,s)^T, \quad \hat{N}_2(s,\eta) = N_1(\eta,s)^T$$

Triple Notation:

$$\{\hat{R}_0,\hat{R}_1,\hat{R}_2\} = \{R_0,R_1,R_2\}^*$$

Matlab Implementation:

$$\{N_0,N_1,N_2\} = \{T_0,T_1,T_2\}^* \quad \rightarrow \quad \mathcal{P}_{\{N_0,N_1,N_2\}} = \mathcal{P}^*_{\{T_0,T_1,T_2\}}$$

```
[N0, N1, N2] = PL2L_transpose(T0,T1,T2,s,th)
```

Transpose/Adjoint in the PIE $\mathcal{P}_{\{N_i\}}$ Operator Algebra

Property 2: Transpose/Adjoint

$$\langle \mathbf{x}, \mathcal{P}_{\{\hat{N}_0,\hat{N}_1,\hat{N}_2\}}\mathbf{y}\rangle_{L_2} = \langle \mathcal{P}_{\{N_0,N_1,N_2\}}\mathbf{x}, \mathbf{y}\rangle_{L_2}$$

where

$$\hat{N}_0(s) = N_0(s)^T, \quad \hat{N}_1(s,\eta) = N_2(\eta,s)^T, \quad \hat{N}_2(s,\eta) = N_1(\eta,s)^T$$

Triple Notation:

$$\{\hat{R}_0, \hat{R}_1, \hat{R}_2\} = \{R_0, R_1, R_2\}^*$$

Matlab Implementation:

$$\{N_0, N_1, N_2\} = \{T_0, T_1, T_2\}^* \quad \rightarrow \quad \mathcal{P}_{\{N_0,N_1,N_2\}} = \mathcal{P}^*_{\{T_0,T_1,T_2\}}$$

```
[N0, N1, N2] = PL2L_transpose(T0,T1,T2,s,th)
```

- The composition property is surprising and non-trivial.
- Two integrations can be expressed using a single integral.
- Two derivatives can NOT be expressed using a single derivative.

New Parser Format:

Matlab Implementation:

```
T.R.R0=T0; T.R.R1=T1; T.R.R2=T2;
```

$$\{N_0, N_1, N_2\} = \{T_0, T_1, T_2\}^* \quad \rightarrow \quad \mathcal{P}_{\{N_0,N_1,N_2\}} = \mathcal{P}^*_{\{T_0,T_1,T_2\}}$$

```
N = transpose_p(T,s,theta,[a,b])
N0=N.R.R0; N1=N.R.R1; N2=N.R.R2
```

Other Operations in the PIE $\mathcal{P}_{\{N_i\}}$ Operator Algebra

Property 3: Scalar Multiplication

$$\alpha\mathcal{P}_{\{N_0,N_1,N_2\}} = \mathcal{P}_{\{\alpha N_0,\alpha N_1,\alpha N_2\}}$$

Triple Notation:

$$\alpha\{\hat{R}_0,\hat{R}_1,\hat{R}_2\} = \{\alpha R_0,\alpha R_1,\alpha R_2\}^*$$

Property 4: Addition

$$\mathcal{P}_{\{M_0+N_0,M_1+N_1,M_2+N_2\}} = \mathcal{P}_{\{M_0,M_1,M_2\}} + \mathcal{P}_{\{N_0,N_1,N_2\}}$$

Triple Notation:

$$\{M_0+N_0,M_1+N_1,M_2+N_2\} = \{M_0,M_1,M_2\} + \{N_0,N_1,N_2\}$$

Theorem 1.

For any functions $Z(s)$ and $Z(s,\theta)$, and $g(s) \ge 0$ for all $s \in [a,b]$

$$N_0(s) = g(s)Z(s)^T P_{11} Z(s)$$

$$\begin{aligned}N_1(s,\theta) &= g(s)Z(s)^T P_{12} Z(s,\theta) + g(\theta)Z(\theta,s)^T P_{31} Z(\theta) + \int_a^\theta g(\nu)Z(\nu,s)^T P_{33} Z(\nu,\theta)d\nu \\ &\quad + \int_\theta^s g(\nu)Z(\nu,s)^T P_{32} Z(\nu,\theta)d\nu + \int_s^L g(\nu)Z(\nu,s)^T P_{22} Z(\nu,\theta)d\nu\end{aligned}$$

$$\begin{aligned}N_2(s,\theta) &= g(s)Z(s)^T P_{13} Z(s,\theta) + g(\theta)Z(\theta,s)^T P_{21} Z(\theta) + \int_a^s g(\nu)Z(\nu,s)^T P_{33} Z(\nu,\theta)d\nu \\ &\quad + \int_s^\theta g(\nu)Z(\nu,s)^T P_{23} Z(\nu,\theta)d\nu + \int_\theta^L g(\nu)Z(\nu,s)^T P_{22} Z(\nu,\theta)d\nu,\end{aligned}$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \ge 0,$$

then $\mathcal{P}^*_{\{N_0,N_1,N_2\}} = \mathcal{P}_{\{N_0,N_1,N_2\}}$ and $\langle \mathbf{x}, \mathcal{P}_{\{N_0,N_1,N_2\}}\mathbf{x}\rangle_{L_2} \ge 0$ for all $\mathbf{x} \in L_2[a,b]$.

Positivity in the PIE N_0, N_1, N_2 Algebra

Proof: Let

$$\{Z_0, Z_1, Z_2\} := \left\{ \begin{bmatrix} \sqrt{g(s)} Z_{d1}(s) & & \\ & & \end{bmatrix}, \begin{bmatrix} & \sqrt{g(s)} Z_{d2}(s,\theta) & \\ & & \end{bmatrix}, \begin{bmatrix} & & \\ & & \sqrt{g(s)} Z_{d2}(s,\theta) \end{bmatrix} \right\}$$

Then

$$\{N_0, N_1, N_2\} = \{Z_0, Z_1, Z_2\}^* \times \{P, 0, 0\} \times \{Z_0, Z_1, Z_2\}$$

Matlab Implementation:

$$\{N_0, N_1, N_2\} \in \Phi_d \qquad \to \qquad \mathcal{P}_{\{N_0,N_1,N_2\}} \geq 0$$

```
[prog, N0, N1, N2]= sospos_PL2L(prog,n,d,d,s,th,[a,b])
```

Positivity in the PIE N_0, N_1, N_2 Algebra

Proof: Let

$$\{Z_0, Z_1, Z_2\} := \left\{ \begin{bmatrix} \sqrt{g(s)} Z_{d1}(s) \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \sqrt{g(s)} Z_{d2}(s,\theta) \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \sqrt{g(s)} Z_{d2}(s,\theta) \end{bmatrix} \right\}$$

Then

$$\{N_0, N_1, N_2\} = \{Z_0, Z_1, Z_2\}^* \times \{P, 0, 0\} \times \{Z_0, Z_1, Z_2\}$$

Matlab Implementation:

$$\{N_0, N_1, N_2\} \in \Phi_d \qquad \to \qquad \mathcal{P}_{\{N_0, N_1, N_2\}} \geq 0$$

```
[prog, N0, N1, N2]= sospos_PL2L(prog,n,d,d,s,th,[a,b])
```

New Parser Implementation

```
[prog, Pv] = sospos_L2L_matker(prog,np,n1,n2,s,th,X);
[prog, Pv] = sospos_L2L_ker(prog,np,n1,n2,s,th,X);
[prog, Pv] = sospos_L2L_ker_psatz(prog,np,n1,n2,s,th,X);
```

Conversion Between PIE and PDE States

$$\dot{\mathbf{x}}_p(t) = \begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}_t = A_0(s) \underbrace{\begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}}_{\mathbf{x}_p} + A_1(s) \begin{bmatrix} x_2(t,s) \\ x_3(t,s) \end{bmatrix}_s + A_2(s) \begin{bmatrix} x_3(t,s) \end{bmatrix}_{ss}$$

Write

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \mathcal{P}_{\{G_0,G_1,G_2\}} \begin{bmatrix} x_1(t) \\ x_{2s}(t) \\ x_{3ss}(t) \end{bmatrix}$$

$$\begin{bmatrix} x_2(t) \\ x_3(t) \end{bmatrix}_s = \mathcal{P}_{\{H_0,H_1,H_2\}} \begin{bmatrix} x_1(t) \\ x_{2s}(t) \\ x_{3ss}(t) \end{bmatrix}$$

$$\begin{bmatrix} x_3(t) \end{bmatrix}_{ss} = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_{2s}(t) \\ x_{3ss}(t) \end{bmatrix}$$

Boundary Conditions:

$$B \begin{bmatrix} x_2(0) \\ x_2(L) \\ x_3(0) \\ x_3(L) \\ x_{3s}(0) \\ x_{3s}(L) \end{bmatrix} = 0, \quad \operatorname{rank}(B) = n_2 + 2n_3$$

$$\dot{\mathbf{x}}_p(t) = \left(\mathcal{P}_{\{G_0,G_1,G_2\}} + \mathcal{P}_{\{H_0,H_1,H_2\}} + \begin{bmatrix} 0 & 0 & I \end{bmatrix}\right) \begin{bmatrix} x_1(t,s) \\ x_{2s}(t,s) \\ x_{3s}(t,s) \end{bmatrix}$$

Converting from PDE state to PIE state

$$\mathbf{x}_p(t,s) := \begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}, \qquad \mathbf{x}_f(t,s) := \begin{bmatrix} x_1(t,s) \\ x_{2s}(t,s) \\ x_{3ss}(t,s) \end{bmatrix}, \qquad \begin{bmatrix} x_2(a) \\ x_2(b) \\ x_3(a) \\ x_3(b) \\ x_{3s}(a) \\ x_{3s}(b) \end{bmatrix} = 0$$

Part 1: Fundamental Theorem of Calculus in Selected BCs

$$\mathbf{x}_p(s) = K(s) \begin{bmatrix} x_2(a) \\ x_3(a) \\ x_{3s}(a) \end{bmatrix} + (\mathcal{P}_{\{L_0,L_1,0\}}\mathbf{x}_f)(s).$$

Part 2: Convert Given BCs to Selected BCs

$$B \begin{bmatrix} x_2(a) \\ x_2(b) \\ x_3(a) \\ x_3(b) \\ x_{3s}(a) \\ x_{3s}(b) \end{bmatrix} = BT \begin{bmatrix} x_2(a) \\ x_3(a) \\ x_{3s}(a) \end{bmatrix} + B\mathcal{P}_{\{0,Q,Q\}}\mathbf{x}_f = 0 \quad \text{or} \quad \begin{bmatrix} x_2(a) \\ x_3(a) \\ x_{3s}(a) \end{bmatrix} = -(BT)^{-1}B\mathcal{P}_{\{0,Q,Q\}}\mathbf{x}_f.$$

Part 3: Substitute

$$\mathbf{x}_p = \mathcal{P}_{\{G_0,G_1,G_2\}}\mathbf{x}_f$$

where

$$G_0(s) = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_1(s,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (s-\theta)I \end{bmatrix} + G_2(s,\theta), \quad G_2(s,\theta) = -K(s)(BT)^{-1}BQ(s,\theta)$$

$$T = \begin{bmatrix} I & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & (b-a)I \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix}, \quad Q(s,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (b-\theta)I \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad K(s) = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & (s-a) \end{bmatrix}$$

Converting a PDE to a PIE

$$\dot{\mathbf{x}}_p = A_0(s)\mathbf{x}_p + A_1(s)\begin{bmatrix} x_2(t,s) \\ x_3(t,s) \end{bmatrix}_s + A_2(s)\begin{bmatrix} x_3(t,s) \end{bmatrix}_{ss}$$
$$= \Big(\{A_0,0,0\} \times \{G_0,G_1,G_2\} + \{A_1,0,0\} \times \{H_0,H_1,H_2\} + \begin{bmatrix} 0 & 0 & A_2 \end{bmatrix}\Big)\mathbf{x}_f$$
$$= \mathcal{P}_{\{J_0,J_1,J_2\}}\mathbf{x}_f(t)$$

with the more fundamental version:

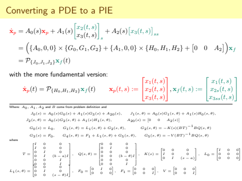
$$\dot{\mathbf{x}}_p(t) = \mathcal{P}_{\{H_0,H_1,H_2\}}\mathbf{x}_f(t) \qquad \mathbf{x}_p(t,s) := \begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}, \mathbf{x}_f(t,s) := \begin{bmatrix} x_1(t,s) \\ x_{2s}(t,s) \\ x_{3ss}(t,s) \end{bmatrix}$$

Where: A_0, A_1, A_2 and B come from problem definition and

$$J_0(s) = A_0(s)G_0(s) + A_1(s)G_3(s) + A_{20}(s), \qquad J_1(s,\theta) = A_0(s)G_1(s,\theta) + A_1(s)H_0(s,\theta),$$
$$J_2(s,\theta) = A_0(s)G_2(s,\theta) + A_1(s)H_1(s,\theta), \qquad A_{20}(s) = \begin{bmatrix} 0 & 0 & A_2(s) \end{bmatrix}$$
$$G_0(s) = L_0, \qquad G_1(s,\theta) = L_1(s,\theta) + G_2(s,\theta), \qquad G_2(s,\theta) = -K(s)(BT)^{-1}BQ(s,\theta)$$
$$G_3(s) = F_0, \qquad G_4(s,\theta) = F_1 + L_1(s,\theta) + G_5(s,\theta), \qquad G_5(s,\theta) = -V(BT)^{-1}BQ(s,\theta)$$

where

$$T = \begin{bmatrix} I & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & (b-a)I \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix}, \quad Q(s,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (b-\theta)I \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad K(s) = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & (s-a) \end{bmatrix}, \quad L_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$L_1(s,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (s-\theta)I \end{bmatrix}, \quad F_0 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$



Matlab Implementation:

`assemble_operators_stab_odepde.m`

Illustration 1: Heat Equation

Primal Formulation:

$$\mathbf{x}_t = \underbrace{\lambda}_{A_0}\mathbf{x} + \underbrace{1}_{A_2}\mathbf{x}_{ss}$$

where $A_1 = 0$, $n_3 = 1$, and $n_1 = n_2 = 0$.

Boundary Conditions:

$$\underbrace{\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}}_{B}\begin{bmatrix}x(0)\\ x(1)\\ x_s(0)\\ x_s(1)\end{bmatrix} = 0.$$

PIE Formulation:

$$\dot{\mathbf{x}}_p = \mathcal{P}_{\{H_0,H_1,H_2\}}\mathbf{x}_f \qquad \mathbf{x}_p = \mathcal{P}_{\{G_0,G_1,G_2\}}\mathbf{x}_f$$

where

$$G_0 = 0 \qquad G_1 = \frac{1}{2}(s\theta + s - \theta - 1) \qquad G_2 = \frac{1}{2}(s\theta - s + \theta - 1)$$

$$H_0 = 1 \qquad H_1 = \frac{\lambda}{2}(s\theta + s - \theta - 1) \qquad H_2 = \frac{\lambda}{2}(s\theta - s + \theta - 1);$$

Illustration 4: Non-"Hyperbolic" Damped Wave Equation

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \underbrace{\begin{bmatrix} -2a & -a^2 \\ 1 & 0 \end{bmatrix}}_{A_0} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{A_2} \underbrace{u_{2ss}}_{x_3}$$

where $A_1 = 0$, $n_1 = 0$, $n_2 = 1$, and $n_3 = 1$. The BCs on u_1 make us consider this a hyperbolic state!

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_1(L) \\ u_2(0) \\ u_2(L) \\ u_{2s}(0) \\ u_{2s}(L) \end{bmatrix} = 0.$$

PIE Formulation:

$$\dot{\mathbf{x}}_p = \mathcal{P}_{\{H_0,H_1,H_2\}}\mathbf{x}_f \qquad \mathbf{x}_p = \mathcal{P}_{\{G_0,G_1,G_2\}}\mathbf{x}_f$$

where

$$G_0 = 0 \qquad G_1 = \begin{bmatrix} 1 & 0 \\ -s & \theta \end{bmatrix} \qquad G_2 = \begin{bmatrix} 0 & 0 \\ -s & -s \end{bmatrix}$$

$$H_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad H_1 = \begin{bmatrix} a^2 s - 2a & a^2\theta \\ 1 & 0 \end{bmatrix} \qquad H_2 = \begin{bmatrix} a^2 s & a^2 s \\ 0 & 0 \end{bmatrix};$$

Illustration 2: The Euler-Bernoulli Beam

Recall the cantilevered E-B beam: **Primal Formulation:**

$$\mathbf{x}_t = \underbrace{\begin{bmatrix} 0 & -c \\ 1 & 0 \end{bmatrix}}_{A_2} \mathbf{x}_{ss}$$

where $A_0 = A_1 = 0$, $n_3 = 2$, and $n_1 = n_2 = 0$.

Boundary Conditions:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_1(L) \\ u_2(L) \\ u_{1s}(0) \\ u_{2s}(0) \\ u_{1s}(L) \\ u_{2s}(L) \end{bmatrix} = 0.$$

PIE Formulation:

$$\dot{\mathbf{x}}_p = \mathcal{P}_{\{H_0,H_1,H_2\}} \mathbf{x}_f \qquad \mathbf{x}_p = \mathcal{P}_{\{G_0,G_1,G_2\}} \mathbf{x}_f$$

where

$$G_0 = 0 \qquad G_1 = \begin{bmatrix} s-\theta & 0 \\ 0 & 0 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 0 & 0 \\ 0 & \theta - s \end{bmatrix}$$

$$H_0 = \begin{bmatrix} 0 & -c \\ 1 & 0 \end{bmatrix} \qquad H_1 = 0 \qquad H_2 = 0;$$

Lyapunov (Energy) Stability - Converting an LMI to a LOI

LOI Stability Condition: $\mathbf{x}_p = \mathcal{P}_{\{G_i\}}\mathbf{x}_f$

$$\dot{\mathbf{x}}_p(t) = \mathcal{P}_{\{H_i\}}\mathbf{x}_f(t)$$

We now propose a Lyapunov function of the form

$$V(\mathbf{x}_p) = \langle \mathbf{x}_p, \mathcal{P}_{\{N_i\}}\mathbf{x}_p \rangle$$

The time-derivative of the Lyapunov function is

$$\begin{aligned}\dot{V}(\mathbf{x}_p(t)) &= 2\langle \mathbf{x}_p, \mathcal{P}_{\{N_i\}}\dot{\mathbf{x}}_p \rangle \\ &= 2\langle \mathbf{x}_p, \mathcal{P}_{\{N_i\}}\mathcal{P}_{\{H_i\}}\mathbf{x}_f \rangle \\ &= 2\langle \mathcal{P}_{\{G_i\}}\mathbf{x}_f, \mathcal{P}_{\{N_i\}}\mathcal{P}_{\{H_i\}}\mathbf{x}_f \rangle \\ &= 2\langle \mathbf{x}_f, \mathcal{P}^*_{\{G_i\}}\mathcal{P}_{\{N_i\}}\mathcal{P}_{\{H_i\}}\mathbf{x}_f \rangle \\ &= \langle \mathbf{x}_f, \mathcal{P}_{\{K_i\}}\mathbf{x}_f \rangle + \langle \mathbf{x}_f, \mathcal{P}^*_{\{K_i\}}\mathbf{x}_f \rangle\end{aligned}$$

Stability Condition: $\mathcal{P}_{\{N_0,N_1,N_2\}} > 0$ and

$$\mathcal{P}_{\{K_0,K_1,K_2\}} + \mathcal{P}^*_{\{K_0,K_1,K_2\}} \le 0$$

LMI Equivalent: $x_p = Ex$

$$\dot{x}_p(t) = Ax(t)$$

$$V(x) = {x_p}^T P E x_p$$

$$\begin{aligned}\dot{V}(x_p) &= 2{x_p}^T P \dot{x}_p \\ &= 2x^T(E^T P A)x \\ &= x^T(E^T PA + A^T PE)x\end{aligned}$$

$$E^T PA + A^T PE < 0$$

Theorem 2.

For any functions $Z(s)$ and $Z(s,\theta)$, and $g(s) \geq 0$ for all $s \in [a,b]$

$$N_0(s) = g(s)Z(s)^T P_{11} Z(s)$$

$$N_1(s,\theta) = g(s)Z(s)^T P_{12} Z(s,\theta) + g(\theta)Z(\theta,s)^T P_{31} Z(\theta) + \int_a^\theta g(\nu)Z(\nu,s)^T P_{33} Z(\nu,\theta)d\nu + \int_\theta^s g(\nu)Z(\nu,s)^T P_{32} Z(\nu,\theta)d\nu + \int_s^L g(\nu)Z(\nu,s)^T P_{22} Z(\nu,\theta)d\nu$$

$$N_2(s,\theta) = g(s)Z(s)^T P_{13} Z(s,\theta) + g(\theta)Z(\theta,s)^T P_{21} Z(\theta) + \int_a^s g(\nu)Z(\nu,s)^T P_{33} Z(\nu,\theta)d\nu + \int_s^\theta g(\nu)Z(\nu,s)^T P_{23} Z(\nu,\theta)d\nu + \int_\theta^L g(\nu)Z(\nu,s)^T P_{22} Z(\nu,\theta)d\nu,$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \geq 0,$$

then $\mathcal{P}^*_{\{N_i\}} = \mathcal{P}_{\{N_i\}} \geq 0$.

Proof: Let

$$\{Z_0, Z_1, Z_2\} := \left\{ \begin{bmatrix} \sqrt{g(s)}Z_{d1}(s) \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \sqrt{g(s)}Z_{d2}(s,\theta) \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \sqrt{g(s)}Z_{d2}(s,\theta) \end{bmatrix} \right\}$$

Then

$$\{N_0, N_1, N_2\} = \{Z_0, Z_1, Z_2\}^* \times \{P, 0, 0\} \times \{Z_0, Z_1, Z_2\}$$

Matlab Toolbox Implementation (Stability Analysis)

$$\{N_0, N_1, N_2\} \in \Phi_d \qquad \to \qquad \mathcal{P}_{\{N_i\}} \geq 0$$

```
[prog, N0, N1, N2]= sospos_PL2L(prog,n,d,d,s,th,[a,b])
```

$$\{N_0, N_1, N_2\} = \{T_0, T_1, T_2\} \times \{R_0, R_1, R_2\} \quad \to \quad \mathcal{P}_{\{N_i\}} = \mathcal{P}_{\{T_i\}}\mathcal{P}_{\{R_i\}}$$

```
[N0, N1, N2] = PL2L_compose(T0,T1,T2,R0,R1,R2,s,th,[a,b])
```

$$\{N_0, N_1, N_2\} = \{T_0, T_1, T_2\}^* \quad \to \quad \mathcal{P}_{\{N_i\}} = \mathcal{P}^*_{\{T_i\}}$$

```
[N0, N1, N2] = PL2L_transpose(T0,T1,T2,s,th)
```

Almost Complete Matlab Code:

```
pvar s th
[prog, G0, G1, G2]=...
[prog, H0, H1, H2]=...
prog = sosprogram([s th])
[prog, M, N1, N2]= sospos_PL2L(prog,n,d,d,s,th,II)
[J0, J1, J2] = PL2L_compose(M+ep*I,N1,N2,G0,G1,G2,s,th,II)
[H0s, H1s, H2s] = PL2L_transpose(H0,H1,H2,s,th)
[K0, K1, K2] = PL2L_compose(H0s,H1s,H2s,J0,J1,J2,s,th,II)
[K0s, K1s, K2s] = PL2L_transpose(K0,K1,K2,s,th)
[prog, [],N1e, N2e] = sospos_PL2L(prog,n,d+2,d+2,s,th,II)
[prog, [],gN1e, gN2e] = sospos_PL2L_psatz(prog,n,d+2,d+2,s,th,II)
[prog] = sosmateq(prog,K1+K1s+N1eq+gN1eq)
prog = sossolve(prog,pars)
```

Stability Conditions:

$\{N_i\} - \{\epsilon I, 0, 0\} \in \Phi_d$

$\{K_i\} = \{G_i\}^* \times \{N_i\} \times \{H_i\}$

$-\{K_i\} - \{K_i\}^* \in \Phi_{d+2}$

Accuracy:

Example 1: Adapted from Valmorbida, 2014:

$$\dot{x}(t,s) = \lambda x(t,s) + x_{ss}(t,s) \qquad x(0) = x(1) = 0$$

Stable iff $\lambda < \pi^2 \cong 9.8696$. We prove stability for $\lambda = 9.8696$.

Example 2: From Valmorbida, 2016,

$$\dot{x}(t,s) = \lambda x(t,s) + x_{ss}(t,s) \qquad x(0) = 0, \quad x_s(1) = 0$$

Unstable for $\lambda > 2.467$. We prove stability for $\lambda = 2.467$.

Example 3: From Gahlawat, 2017:

$$\dot{x}(t,s) = (-.5s^3 + 1.3s^2 - 1.5s + .7 + \lambda)x(t,s) + (3s^2 - 2s)x_s(t,s) + (s^3 - s^2 + 2)x_{ss}(t,s)$$

with $x(0) = 0$ and $x_s(1) = 0$. Unstable for $\lambda > 4.65$. For $d = 1$, we prove stability for $\lambda = 4.65$.

Example 4: From Valmorbida,2014,

$$\dot{x}(t,s) = \begin{bmatrix} 1 & 1.5 \\ 5 & .2 \end{bmatrix} x(t,s) + R^{-1}x_{ss}(t,s), \qquad x(0) = x_s(1) = 0$$

With $d = 1$, we prove stability for $R = 2.93$ (improvement over $R = 2.45$).

Example 5: From Valmorbida,2016,

$$\dot{x}(t,s) = \begin{bmatrix} 0 & 0 & 0 \\ s & 0 & 0 \\ s^2 & -s^3 & 0 \end{bmatrix} x(t,s) + R^{-1}x_{ss}(t,s), \qquad x(0) = x_s(1) = 0$$

Using $d = 1$, we prove stability for $R = 21$ (and greater) with a computation time of $4.06s$.

Computational Complexity vs. Number of PDEs

Consider a simple n-dimensional diffusion equation

$$\dot{x}(t,s) = x(t,s) + x_{ss}(t,s)$$

where $x(t,s) \in \mathbb{R}^n$.

Computation Time:

n (# of states)	1	5	10	20
CPU sec	.54	37.4	745	31620

Illustration 2: The Timoschenko Beam

Consider a simple Timoschenko beam model:

$$\ddot{w} = \partial_s(w_s - \phi) \qquad\qquad = -\phi_s + w_{ss}$$

$$\ddot{\phi} = \phi_{ss} + (w_s - \phi) \qquad\qquad = -\phi + w_s + \phi_{ss}$$

with boundary conditions

$$\phi(0) = 0, \quad w(0) = 0, \quad \phi_s(L) = 0, \quad w_s(L) - \phi(L) = 0$$

Step 1: Eliminate w_{tt} and ϕ_{tt} - $u_1 = w_t$ and $u_3 = \phi_t$.
Step 2: Use BCs to pick the state - $u_2 = w_s - \phi$ and $u_4 = \phi_s$.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_0} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}}_{\mathbf{x}_2} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_s$$

where $A_2 = []$ and $n_1 = n_3 = 0$ and $n_2 = 4$ - a purely "hyperbolic" form. We only need 4 BCs:

$$u_1(0) = 0, \quad u_3(0) = 0, \quad u_4(L) = 0, \quad u_2(L) = 0$$

This gives a B has row rank $n_2 = 4$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \\ u_4(0) \\ u_1(L) \\ u_2(L) \\ u_3(L) \\ u_4(L) \end{bmatrix} = 0$$

Stable! However, not exponentially stable ($\dot{V} \not< 0$) in all the given states.

Illustration 2b: The Timoschenko Beam revisited

Consider a modification - naively choose $u_2 = w_s$ and $u_4 = \phi$. This leads to

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A_0} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_s + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{A_2} u_{4ss}$$

where $n_1 = 0$, $n_2 = 3$, and $n_3 = 1$ and with 5 boundary conditions

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \\ u_1(L) \\ u_2(L) \\ u_3(L) \\ u_4(0) \\ u_4(L) \\ u_{4s}(0) \\ u_{4s}(L) \end{bmatrix} = 0.$$

NOT Stable in the given states!

However: If we add a damping term $-cu_{4t} = -cu_3$ to $\dot{u}_3$, then the only change is

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -c & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now Stable for any $c > 0$! Stability is sensitive to definition of states!

Illustration 3: The Tip-Damped Wave Equation

The simplest tip-damped wave equation is

$$u_{tt}(t,s) = u_{ss}(t,s) \qquad u(t,0) = 0 \qquad u_s(t,L) = -k u_t(t,L).$$

Guided by the boundary conditions, we choose

$$u_1(t,s) = u_s(t,s)$$
$$u_2(t,s) = u_t(t,s)$$

This yields

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_s}_{x_2}$$

where $A_0 = 0$, $A_2 = []$ $n_1 = n_3 = 0$ and $n_2 = 2$. The BCs are now

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_1(L) \\ u_2(L) \end{bmatrix} = 0.$$

We prove exp. stability in the given states u_t, u_s for $k > 0$.

Illustration 4: Non-“Hyperbolic” Damped Wave Equation

Add u to the dynamics (stable for $a, k \neq 0$)

$$u_{tt}(t,s) = u_{ss}(t,s) - 2au_t(t,s) - a^2u(t,s) \qquad s \in [0,1]$$

$$\text{BCs:} \qquad u(t,0) = 0, \qquad\qquad u_s(t,1) = -ku_t(t,1)$$

Must choose the variables $u_1 = u_t$ and $u_2 = u$. Yields the diffusive form:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \underbrace{\begin{bmatrix} -2a & -a^2 \\ 1 & 0 \end{bmatrix}}_{A_0} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{A_2} \underbrace{u_{2ss}}_{x_3}$$

where $A_1 = 0$, $n_1 = 0$, $n_2 = 1$, and $n_3 = 1$. The BCs on u_1 make us consider this a hyperbolic state!

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_1(L) \\ u_2(0) \\ u_2(L) \\ u_{2s}(0) \\ u_{2s}(L) \end{bmatrix} = 0.$$

Stable!, but not exponentially stable in the given state (confirmed analytically).

Converting an LMI to an LOI:

The LMI to LOI conversion process:

Step 1: Write the dynamics

$$\dot{\mathbf{x}}_p(t) = \mathcal{A}\mathbf{x}_f(t) + \mathcal{B}w(t), \qquad y(t) = \mathcal{C}\mathbf{x}_f(t) + Dw(t), \qquad \mathbf{x}_p(t) = \mathcal{H}\mathbf{x}_f$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are in the $\{N_0, N_1, N_2\}$ algebra.

Step 2: Replace Matrices with Operators (e.g. KYP Lemma)

$$\begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \le 0 \;\rightarrow\; \begin{bmatrix} u \\ v \\ \mathbf{x}_f \end{bmatrix}^T \begin{bmatrix} -\gamma I & D^T & \mathcal{B}^*\mathcal{P}\mathcal{H} \\ D & -\gamma I & \mathcal{C} \\ \mathcal{H}^*\mathcal{P}\mathcal{B} & \mathcal{C}^* & \mathcal{A}^*\mathcal{P}\mathcal{H} + \mathcal{H}^*\mathcal{P}\mathcal{A} \end{bmatrix} \begin{bmatrix} u \\ v \\ \mathbf{x}_f \end{bmatrix} \prec 0$$

Why Does This Work?:

- The conversion between primal and fundamental state is a $\{N_0, N_1, N_2\}$ operator.
- We express the dynamics as a $\{N_0, N_1, N_2\}$ operator.
- We express the Lyapunov Functions using a $\{N_0, N_1, N_2\}$ operator.
- $\{N_0, N_1, N_2\}$ operators are closed under composition, adjoint, and addition.
- We can parameterize $\{N_0, N_1, N_2\}$ operators using real numbers
- We can enforce positivity of $\{N_0, N_1, N_2\}$ operators.

Algebras on $\mathbb{R}^n \times L_2$

How to Enforce:

$$\begin{bmatrix} u \\ v \\ \mathbf{x}_f \end{bmatrix}^T \begin{bmatrix} -\gamma I & D^T & \mathcal{B}^*\mathcal{P}\mathcal{H} \\ D & -\gamma I & \mathcal{C} \\ \mathcal{H}^*\mathcal{P}\mathcal{B} & \mathcal{C}^* & \mathcal{A}^*\mathcal{P}\mathcal{H}+\mathcal{H}^*\mathcal{P}\mathcal{A} \end{bmatrix} \begin{bmatrix} u \\ v \\ \mathbf{x}_f \end{bmatrix} \prec 0 \qquad \forall \begin{bmatrix} u \\ v \\ \mathbf{x}_f \end{bmatrix} \in \mathbb{R}^{m+p} \times L_2^n ?$$

ODEs Coupled with PDEs:

Algebra of Operators on $\mathbb{R}^m \times L_2^n[a,b]$

$$\left(\mathcal{P}\left\{\begin{smallmatrix} P, & Q_1 \\ Q_2, & \{R_i\} \end{smallmatrix}\right\} \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_a^b Q_1(s)\mathbf{x}(s)ds \\ Q_2(s)x + \left(\mathcal{P}_{\{R_i\}}\mathbf{x}\right)(s) \end{bmatrix}.$$

$$\begin{bmatrix} P & \mathcal{Q}_1 \\ \mathcal{Q}_2 & \mathcal{P} \end{bmatrix} \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix} = \mathcal{P}\left\{\begin{smallmatrix} P, & Q_1 \\ Q_2, & \{R_i\} \end{smallmatrix}\right\} \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}$$

where

$$P : \mathbb{R}^m \to \mathbb{R}^m \qquad y = Px$$

$$\mathcal{Q}_1 : \mathcal{L}_2^n \to \mathbb{R}^m \qquad y = \int Q_1(s)\mathbf{x}(s)$$

$$\mathcal{Q}_2 : \mathbb{R}^m \to L_2^n \qquad \mathbf{y}(s) = Q_2(s)x$$

$$\mathcal{P} : L_2^n \to L_2^n \qquad \mathbf{y} = \mathcal{P}_{\{R_i\}}\mathbf{x}$$

Operators on $\mathbb{R} \times L_2$ in a Matlab structure

A general operator on $\mathcal{P}\left\{\begin{smallmatrix} P, & Q_1 \\ Q_2, & \{R_i\}\end{smallmatrix}\right\} : \mathbb{R}^p \times L_2^q[a,b] \to \mathbb{R}^m \times L_2^n[a,b]$

$$\left(\mathcal{P}\left\{\begin{smallmatrix} P, & Q_1 \\ Q_2, & \{R_i\}\end{smallmatrix}\right\} \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_a^b Q_1(s)\mathbf{x}(s)ds \\ Q_2(s)x + \left(\mathcal{P}_{\{R_i\}}\mathbf{x}\right)(s) \end{bmatrix}.$$

MATLAB structure has following elements.

1. P.P: a $m \times p$ matrix
2. P.Q1, P.Q2: $m \times q$ and $n \times p$ matrix valued polynomials in s, respectively
3. P.R: a structure with entities R_0, R_1, and R_2
4. P.R.R0 : $n \times q$ matrix valued polynomial in s
5. P.R.R1, P.R.R2 : $n \times q$ matrix valued polynomials in s and θ
6. P.dim: $\begin{bmatrix} m & p \\ n & q \end{bmatrix}$.

Composition

$$\mathcal{P}_{\left[\begin{smallmatrix} L, & M_1 \\ M_2, & \{N_i\}\end{smallmatrix}\right]} \mathcal{P}_{\left[\begin{smallmatrix} P, & Q_1 \\ Q_2, & \{R_i\}\end{smallmatrix}\right]} \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}$$

$$= \begin{bmatrix} L\left(Px + \int_a^b Q_1(s)\mathbf{x}(s)ds\right) + \int_a^b M_1(s)\left(Q_2(s)x + \left(\mathcal{P}_{\{R_i\}}\mathbf{x}\right)(s)\right)ds \\ M_2(s)\left(Px + \int_a^b Q_1(s)\mathbf{x}(s)ds\right) + \left(\mathcal{P}_{\{N_i\}}\mathbf{Q}_2(s)x + \left(\mathcal{P}_{\{R_i\}}\mathbf{x}\right)(s)\right)(s) \end{bmatrix}.$$

$$= \begin{bmatrix} \left(LP + \int_a^b M_1(s)Q_2(s)\right)x + \int_a^b LQ_1(s)\mathbf{x}(s)ds + \int_a^b M_1(s)\left(\mathcal{P}_{\{R_i\}}\mathbf{x}\right)(s)ds \\ \left(M_2(s)P + \mathcal{P}_{\{N_i\}}Q_2 ds\right)x + M_2(s)\int_a^b Q_1(\theta)\mathbf{x}(\theta)d\theta + \left(P_{\{N_i\}}\mathcal{P}_{\{R_i\}}\mathbf{x}\right)(s)(s) \end{bmatrix}$$

Triple-Triple Notation:

$$\begin{bmatrix} P, & Q_1\} \\ Q_2, & \{R_i\} \end{bmatrix} = \begin{bmatrix} L, & M_1 \\ M_2, & \{N_i\} \end{bmatrix} \times \begin{bmatrix} \{F, & G_1 \\ G_2, & \{H_i\} \end{bmatrix}$$

Matlab Implementation:

$$\mathcal{P}\left\{\begin{smallmatrix} P, & Q_1 \\ Q_2, & \{R_i\} \end{smallmatrix}\right\} = \mathcal{P}\left\{\begin{smallmatrix} L, & M_1 \\ M_2, & \{N_i\} \end{smallmatrix}\right\} \mathcal{P}\left\{\begin{smallmatrix} F, & G_1 \\ G_2, & \{H_i\} \end{smallmatrix}\right\}$$

```
P_comp = compose_p(P1,P2,s,theta,[a,b])
```

Positivity using

$$P \geq 0 \quad \rightarrow \quad \left\{\begin{smallmatrix} I, & Z_1, & Z_2 \\ Z_3, & Z_4, & Z_5 \end{smallmatrix}\right\}^* \times \left\{\begin{smallmatrix} P_1, & P_2, & P_3 \\ P_4, & 0, & 0 \end{smallmatrix}\right\} \times \left\{\begin{smallmatrix} I, & Z_1, & Z_2 \\ Z_3, & Z_4, & Z_5 \end{smallmatrix}\right\} \succ 0$$

Illustration 1: The Tip-Damped Wave Equation with Disturbance

Adding a uniform disturbance to the tip-damped wave equation

$$u_{tt}(t,s) = u_{ss}(t,s) + w(t) \qquad u(t,0) = 0 \qquad u_s(t,L) = -ku_t(t,L).$$

How does the disturbance affect tip displacement? (i.e. $u(t,L)$)
Change the states to

$$u_1(t,s) = u_s(t,s)$$
$$u_2(t,s) = u_t(t,s)$$

Then

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_s}_{x_2} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_1} w(t), \qquad y(t) = u(t,L) = \underbrace{\int_0^L u_1(t,s)ds}_{\mathcal{C}x_2}$$

and $A_0 = 0$, $A_2 = []$, $C.Q1 = 1$, $D = 0$, $n_1 = n_3 = 0$ and $n_2 = 2$. The BCs are

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_1(L) \\ u_2(L) \end{bmatrix} = 0.$$

Matlab code to find a bound on L_2-gain for PDEs

Define system:

```
pvar s, th, gamma;
A0=..;A1=..;A2=..;B1=..;C = ..;D=..;B=..;a=..; b=..;
prog = sosprogram([s;th],gamma);
```

$$\mathcal{P} \succ 0$$

```
[prog, Pv] = sospos_L2L_matker(prog, np, n1, n2, s, th, X);
```

$$\mathcal{P}_{eq} = \begin{bmatrix} -\gamma I & D^T & \mathcal{B}^*\mathcal{P}\mathcal{H} \\ D & -\gamma I & \mathcal{C} \\ \mathcal{H}^*\mathcal{P}\mathcal{B} & \mathcal{C}^* & \mathcal{A}^*\mathcal{P}\mathcal{H} + \mathcal{H}^*\mathcal{P}\mathcal{A} \end{bmatrix}$$

```
HPA = compose_p (transpose_p(H,s,th),compose_p(Pv,Af,s,th,X),s,th,X);
BPH = compose_p (transpose_p(B,s,th),compose_p(Pv,H,s,th,X),s,th,X);
Peq.P = [-gamma*eye(nw) D'; D -gamma*eye(ny)];
Peq.Q1 = [ BPH.Q1 Cf.Q1];
Peq.R.R0 = HPA.R.R0+HPA.R.R0';
Peq.R.R1 = HPA.R.R1+var_swap(HPA.R.R2',s,theta);
```

Matlab code to find a bound on L_2-gain for PDEs- cont.

Question: How to Enforce:

$$\mathcal{P}_{eq} \preccurlyeq 0$$

```
[prog, Pe] = sospos_RL2RL_ker(prog, no, np, d1, d2, s, th, X);
[prog, Pf] = sospos_RL2RL_ker_psatz(prog,no, np, d1, d2, s, th, X);
prog = sosmateq(prog, Pe.P+Pf.P+Pheq.P);
prog = sosmateq(prog, Pe.Q1+Pf.Q1+Pheq.Q1);
prog = sosmateq(prog, Pe.R.R1+Pf.R.R1+Pheq.R.R1);
```

How does the disturbance affect tip displacement in tip-damped wave equation?

Answer: We get $\|y\|_{L_2} \leq 0.5\|w\|_{L_2}$ for $k = 2$.

Some more examples

Example 1: Adapted from Valmorbida, 2014:

$$\dot{x}(t,s) = \lambda x(t,s) + x_{ss}(t,s) + w(t) \qquad x(0) = x(1) = 0 \qquad y(t) = \int_0^1 x(t,s)ds$$

We get $\gamma = 8.214$ vs 8.253 from discretization for $\lambda = 9.86$.

Example 2: From Valmorbida, 2016,

$$\dot{x}(t,s) = \lambda x(t,s) + x_{ss}(t,s) + w(t) \qquad x(0) = 0, \quad x_s(1) = 0 \qquad y(t) = \int_0^1 x(t,s)ds$$

$\gamma = 12.03$ vs 12.3 from discretization for $\lambda = 2.4$.

Example 3: From Valmorbida,2014,

$$\dot{x}(t,s) = \begin{bmatrix} 1 & 1.5 \\ 5 & .2 \end{bmatrix} x(t,s) + R^{-1} x_{ss}(t,s) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t), \qquad x(0) = x_s(1) = 0 \qquad y(t) = \int_0^1 x_1(t,s)ds$$

We get $\gamma = 1.67$ vs 1.66 from discretization for $R = 2.7$.

Example 4: From Valmorbida,2016,

$$\dot{x}(t,s) = \begin{bmatrix} 0 & 0 & 0 \\ s & 0 & 0 \\ s^2 & -s^3 & 0 \end{bmatrix} x(t,s) + R^{-1} x_{ss}(t,s) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} w(t), \quad x(0) = x_s(1) = 0, \quad y(t) = \int_0^1 x_2(t,s)ds$$

We get $\gamma = 3.58$ vs 3.97 from discretization for $R = 21$.

Stable for $\lambda < 4.65$.

$$u_t(s,t) = A_0(s)u(s,t) + A_1(s)u_s(s,t) + A_2(s)u_{ss}(s,t) + w(t)$$
$$u(0,t) = 0 \qquad u_s(1,t) = 0$$
$$y(t) = \int_0^1 u(s,t)ds$$

$A_0(s) = (-0.5s^3+1.3s^2-1.5s+0.7+\lambda)$, $A_1(s) = (3s^2-2s)$, $A_2(s) = (s^3-s^2+2)$

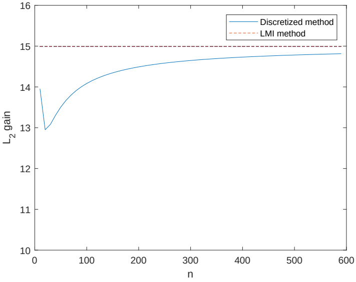


Figure: Compare with Discretization ($d=1$)

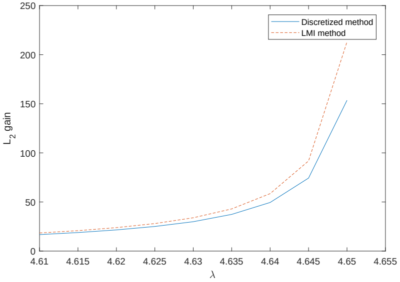
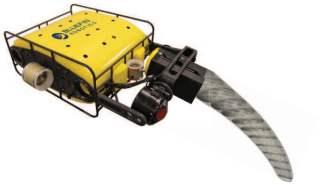


Figure: H_∞ Gain as a function of λ ($d=1$)

Wing attached to a Fuselage
Position of Body: $z(t)$ **Deflection and Curvature:** $w(s,t), w_{ss}(s,t)$
Disturbance: $d(t), u(t)$ **Output:** $w_{ss}(s,t)$

$$\ddot{z}(t) = -F w_{sss}(0,t) + d(t),$$
$$\ddot{w}(s,t) = -\frac{EI}{\mu} w_{ssss}(s,t) + u(t),$$
$$w(0,t) = z(t), w_s(0,t) = 0,$$
$$w_{ss}(L,t) = 0, w_{sss}(L,t) = 0$$



Things to Note:

- The ODE state is affected by the boundary of the PDE
- The BCs of the PDE are affected by the ODE State

ODE/PDE Models (Fluid-Structure)

General Form: A PDE -

$$\begin{bmatrix}\dot{\mathbf{z}}_1\\ \dot{\mathbf{z}}_2\\ \dot{\mathbf{z}}_3\end{bmatrix}(s,t) = A_0(s)\begin{bmatrix}\mathbf{z}_1\\ \mathbf{z}_2\\ \mathbf{z}_3\end{bmatrix}(s,t) + A_1(s)\partial_s\begin{bmatrix}\mathbf{z}_2\\ \mathbf{z}_3\end{bmatrix}(s,t) + A_2(s)\partial_s^2\mathbf{z}_3(s,t),;$$

coupled with a linear ODE

$$\dot{x}(t) = Ax(t) + B_2 z_b(t),$$

coupled at the boundary using

$$\begin{aligned} Bz_b(t) &= B_1 x(t) \\ z_b(t) &= \mathrm{col}\big(\mathbf{z}_2(a,t), \mathbf{z}_2(b,t), \mathbf{z}_3(a,t), \mathbf{z}_3(b,t), \partial_s\mathbf{z}_3(a,t), \partial_s\mathbf{z}_3(b,t)\big) \end{aligned}$$

ODE/PDE Models (Fluid-Structure)

Illustrative Example A string coupled with an ODE [Barreau].

$$\begin{aligned}
\ddot{w}(s,t) &= c w_{ss}(s,t),\\
\dot{x}(t) &= Ax(t) + Bw(1,t),\\
w(0,t) &= Kx(t),\\
w_s(1,t) &= -c_0 \dot{w}(1,t),
\end{aligned}$$

- $w(s,t)$ is transverse displacement of the string

These equations may be rewritten in the proposed form as

$$\begin{aligned}
&\begin{bmatrix}\dot{\mathbf{z}}_1\\ \dot{\mathbf{z}}_3\end{bmatrix}(s,t) = \overbrace{\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}}^{A_0(s)}\begin{bmatrix}\mathbf{z}_1\\ \mathbf{z}_3\end{bmatrix}(s,t) + \overbrace{\begin{bmatrix}c\\ 0\end{bmatrix}}^{A_2(s)}\frac{\partial^2}{\partial s^2}\mathbf{z}_3(s,t)\\
&\dot{x}(t) = Ax(t) + B\mathbf{z}_3(1,t),\\
&\mathbf{z}_3(0,t) = Kx(t),\\
&\mathbf{z}_{3s}(1,t) = -c_0\mathbf{z}_1(1,t)
\end{aligned}$$

where $\mathbf{z}_1 = \dot{w}$ and $\mathbf{z}_3 = w$.

In this case, the BCs become

$$B\begin{bmatrix} x_2(a) \\ x_2(b) \\ x_3(a) \\ x_3(b) \\ x_{3s}(a) \\ x_{3s}(b) \end{bmatrix} = BT\begin{bmatrix} x_2(a) \\ x_3(a) \\ x_{3s}(a) \end{bmatrix} + B\mathcal{P}_{\{0,Q,Q\}}\mathbf{x}_f = B_z z(t) \qquad \text{or}$$

$$\begin{bmatrix} x_2(a) \\ x_3(a) \\ x_{3s}(a) \end{bmatrix} = (BT)^{-1}B_z z(t) - (BT)^{-1}B\mathcal{P}_{\{0,Q,Q\}}\mathbf{x}_f$$

which yields a new identity of the form

$$\mathbf{x}_p = \mathcal{H}\mathbf{x}_f + K(BT)^{-1}B_z z(t)$$

We now write a set of BC-free dynamics of the form

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\mathbf{x}}_p \end{bmatrix} = \underbrace{\begin{bmatrix} A_0 & \mathcal{A}_1 \\ \mathcal{A}_2 & \mathcal{A}_3 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} z(t) \\ \mathbf{x}_f \end{bmatrix} + \underbrace{\begin{bmatrix} B_0 \\ \mathcal{B}_1 \end{bmatrix}}_{\mathcal{B}} \begin{bmatrix} u(t) \\ d(t) \end{bmatrix}.$$

Here

$$\mathcal{A} = \mathcal{P}\left\{\begin{smallmatrix} A_0, & A_1 \\ A_2, & \{A_i\} \end{smallmatrix}\right\}, \qquad \mathcal{B} = \mathcal{P}\left\{\begin{smallmatrix} B_0, & \emptyset \\ B_1, & \{\emptyset\} \end{smallmatrix}\right\}$$

Field Estimation and ODE/PDE Models (Fluid-Structure)

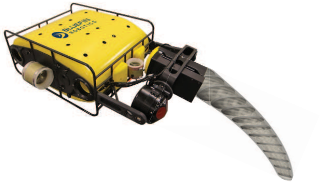
1D Flexible Arm attached to Rigid Body
Position of Body: $z(t)$ **Deflection and Curvature:** $w(s,t), w_{ss}(s,t)$
Disturbance: $d(t), u(t)$ **Output:** $w_{ss}(s,t)$

$$\ddot{z}(t) = -F w_{sss}(0,t) + d(t),$$

$$\ddot{w}(s,t) = -\frac{EI}{\mu} w_{ssss}(s,t) + u(t),$$

$$w(0,t) = z(t), w_s(0,t) = 0,$$

$$w_{ss}(L,t) = 0, w_{sss}(L,t) = 0$$



Result: For $\frac{EI}{\mu} = 10$, we get $\|z\|_{L_2}^2 \leq .8936 \|u\|_{L_2}^2$.

Distributed State Estimation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \lambda \begin{bmatrix}1 & 0.3\\ 0.1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_{1ss}\\ x_{2ss}\end{bmatrix} + \begin{bmatrix}s-s^2\\ 0\end{bmatrix} w(t),$$

$$y = \int_a^b \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(s)\\ x_2(s)\end{bmatrix} \mathsf{d}s,\; z(t) = \int_a^b \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(s)\\ x_2(s)\end{bmatrix} \mathsf{d}s.$$

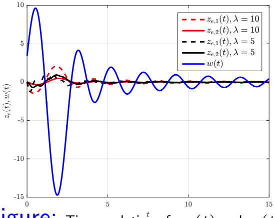


Figure: Time evolution of $z_e(t)$ and $w(t)$ for $\lambda = 5, 10$ where $w(t)$ is generated by damped sinusoidal functions.

PDEs in 2 Spatial Dimensions:

Algebra of Operators on $L_2[[a,b] \times [c,d]]$

$$(\mathcal{P}\mathbf{u})(s) :=$$

$$N_0(x,y)\mathbf{u}(x,y) + \int_a^x \int_c^y N_1(x,y,s,\theta)\mathbf{u}(s,\theta) ds d\theta + \int_x^b \int_y^d N_1(x,y,s,\theta)\mathbf{u}(s,\theta) ds d\theta.$$

State Estimation and Multiple Spatial Dimensions

Distributed State Estimation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \lambda \begin{bmatrix}1 & 0.3\\ 0.1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_{1ss}\\ x_{2ss}\end{bmatrix} + \begin{bmatrix}s-s^2\\ 0\end{bmatrix} w(t),$$

$$y = \int_a^b \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(s)\\ x_2(s)\end{bmatrix} \mathsf{d}s, \; z(t) = \int_a^b \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(s)\\ x_2(s)\end{bmatrix} \mathsf{d}s.$$

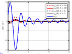


Figure: Time evolution of $z_E(t)$ and $w(t)$ for $\lambda = 5, 10$ where $w(t)$ is generated by damped sinusoidal functions.

PDEs in 2 Spatial Dimensions:
Algebra of Operators on $L_2[[a,b]\times[c,d]]$

$(\mathcal{P}\mathbf{u})(s) :=$

$$N_0(x,y)\mathbf{u}(x,y) + \int_a^x \int_c^y N_1(x,y,s,\theta)\mathbf{u}(s,\theta) ds d\theta + \int_x^b \int_y^d N_1(x,y,s,\theta)\mathbf{u}(s,\theta) ds d\theta.$$

Code for these problems is not yet available. Sorry :(

Time-Delay Systems

What is a Time-Delay System

$$
\begin{aligned}
\dot{x}(t) &= A_0 x(t) + \sum_i A_i x(t-\tau_i) + B_1 w(t) + B_2 u(t), \\
y(t) &= C_1 x(t) + D_1 w(t) + D_2 u(t) \\
z(t) &= C_2 x(t) + D_3 w(t) + D_4 u(t)
\end{aligned}
$$

To Simplify, we

- Assume the Delay is known
- Ignore time-varying delay
- Ignore Distributed Delay

Modeling the Athenaeum Showers

Tracking Control with integral feedback

- T_i is the water temperature
- x_i is the tap position
- τ_i is the time for water to move from tap to showerhead
- w_i is the desired water temperature (Not available to controller!)
- Opening the tap by user i decreases the water temperature of users $j \neq i$
- $u_i(t)$ is the controlled input

$$\dot{x}_i(t) = T_i(t) - w_i(t)$$

$$\dot{T}_i(t) = -\alpha_i \left(T_i(t-\tau_i) - w_i(t)\right) + \sum_{j \neq i} \gamma_{ij} \alpha_j \left(T_j(t-\tau_j) - w_j(t)\right) + u_i(t)$$

$$y_i(t) = \begin{bmatrix} x_i(t) \\ .1 u_i(t) \end{bmatrix} \qquad \text{Sensed Output}$$

Fixing the Athenaeum Showers

$$\dot{x}(t) = A_0 x(t) + \sum_i A_i x(t-\tau_i) + B_1 w(t) + B_2 u(t), \quad y(t) = Cx(t) + D_1 w(t) + D_2 u(t)$$

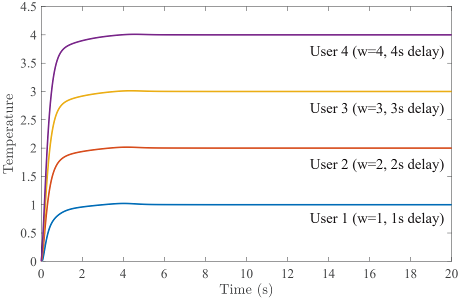
where

$$A_0 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & 0 \\ 0 & \hat{A}_i \end{bmatrix}, \quad B_1 = \begin{bmatrix} -I \\ -\hat{\Gamma} + \text{diag}(\alpha_1 \ldots \alpha_K) \end{bmatrix}$$

$$\hat{A}_i(:,i) = \alpha_i \begin{bmatrix} \gamma_{i,1} & \ldots & \gamma_{i,i-1} & -1 & \gamma_{i,i-1} & \ldots & \gamma_{i,K} \end{bmatrix}^T$$

$$\hat{\Gamma}_{ij} = \alpha_j \gamma_{ij} = \begin{bmatrix} q_1 & \ldots & q_K \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$C_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \; C_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; D_2 = \begin{bmatrix} 0 \\ .1I \end{bmatrix}$$



Complexity: 8 states, 4 delays, 4 inputs, 4 disturbances, 8 regulated outputs

Results: A Matlab simulation of the step response of the closed-loop temperature dynamics ($T_{2i}(t)$) with 4 users (w_i and τ_i as indicated) coupled with the controller with closed-loop gain of $.48$

PDE Representation of Delay System

A linear time-delay system is the interconnection of an ODE and a simple transport PDE with point actuation and point observation.

ODE: The system G_1

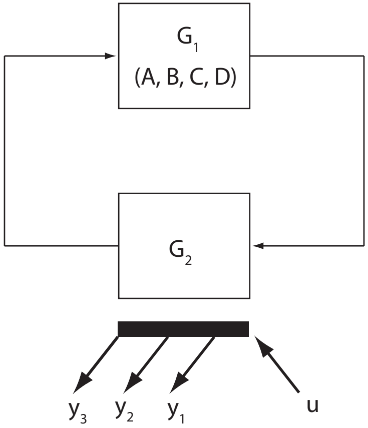
$$\dot{x}_1(t) = Ax_1(t) + Bu_1(t)$$
$$u_2(t) = Cx_1(t) + Du_1(t)$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} A_0 & \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix} \\ \hline I & 0 \end{array}\right]$$

PDE: The system G_2

$$\frac{\partial}{\partial t} x_2(t,s) = \frac{\partial}{\partial s} x_2(t,s) \quad x_2(t,0) = u_2(t),$$

$$u_1(t) = \begin{bmatrix} x_2(-\tau_1) \\ \vdots \\ x_2(-\tau_K) \end{bmatrix}$$



Of course, the solution is just $x_2(t,s) = u_2(t-s)$.

PIE Representation of a time-delay System

ODE: The system G_1

$$\dot{x}_1(t) = \left(\sum_{i=0}^{K} A_i\right) x_1(t) - \sum_{i=1}^{K} \int_{-\tau_i}^{0} A_i \mathbf{x}_{2s}(\theta)$$

$$\dot{\mathbf{x}}_2(t,s) = \mathbf{x}_{2s}(t,s)$$

$$u_2(t) = Cx_1(t) + Du_1(t)$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{\mathbf{x}}_2(t)\end{bmatrix} = \mathcal{P}_{\left[\substack{\{P,\, Q_1,\, Q_2\}\\ \{R_0,\, R_1,\, R_2\}}\right]} \begin{bmatrix}x_1(t)\\ \mathbf{x}_2(t)\end{bmatrix}$$

where

$$P = \sum_{i=0}^{K} A_i, \quad Q_{1i} = A_i, \quad Q_{2i} = 0, \quad \{R_0, R_1, R_2\} = \{I, 0, 0\}$$

The H_∞-Optimal Full-State Feedback Controller Synthesis Problem

Consider solutions of

$$\dot{x}(t) = A_0 x(t) + \sum_i A_i x(t-\tau_i) + B_1 w(t) + B_2 u(t),$$
$$z(t) = Cx(t) + D_1 w(t) + D_2 u(t).$$

Problem Definition:
Minimize γ such that there exist K_0, K_{1i} and $K_{2i}(s)$ such that if

$$u(t) = K_0 x(t) + \sum_i K_{1i} x(t-\tau_i) + \sum_i \int_{-\tau_i}^{0} K_{2i}(s) x(t+s) ds$$

then for any $w \in L_2$, $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$.

Recall the LMI for Optimal Control of **ODE**s

Get rid of the delays and we have

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t), \qquad y(t) = Cx(t) + D_1 w(t) + D_2 u(t).$$

Lemma 3 (Full-State Feedback Controller Synthesis).

Define:

$$\hat{G}(s) = \left[\begin{array}{c|c} A + B_2 K & B_1 \\ \hline C + D_2 K & D_1 \end{array}\right].$$

The following are equivalent.

- *There exists a* K *such that* $\|\hat{G}\|_{H_\infty} \le \gamma$.
- *There exists a* $P > 0$ *and* Z *such that*

$$\begin{bmatrix} PA^T + AP + Z^T B_2^T + B_2 Z & B_1 & PC_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 P + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0$$

The Controller is recovered as $K = ZP^{-1}$.

- $P > 0$ ensures P is invertible.

Operator Formulation of the System

Write the DDE as

$$\dot{x}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t), \quad z(t) = \mathcal{C}\mathbf{x}(t) + \mathcal{D}_1 w(t) + \mathcal{D}_2 u(t), \quad u(t) = \mathcal{K}\mathbf{x}(t).$$

where

$$\mathcal{A}\begin{bmatrix} x \\ \phi_i \end{bmatrix}(s) := \begin{bmatrix} A_0 x + \sum_{i=1}^K A_i \phi_i(-\tau_i) \\ \dot{\phi}_i(s) \end{bmatrix}, \quad \left(\mathcal{C}\begin{bmatrix} x \\ \phi_i \end{bmatrix}\right) := \left[C_0 x + \sum_i C_i \phi_i(-\tau_i)\right]$$

$$(\mathcal{B}_1 w)(s) := \begin{bmatrix} B_1 w \\ 0 \end{bmatrix}, \; (\mathcal{B}_2 u)(s) := \begin{bmatrix} B_2 u \\ 0 \end{bmatrix}, \; (\mathcal{D}_1 w) := D_1 w, \; (\mathcal{D}_2 u) := D_2 u$$

$$\mathcal{K}\begin{bmatrix} x \\ \phi_i \end{bmatrix} := K_0 x(t) + \sum_i K_{1i}\phi_i(t-\tau_i) + \sum_i \int_{-\tau_i}^0 K_{2i}(s) x(t+s) ds$$

Details: $\mathcal{A} : X \to Z_{n,K}$, $\mathcal{B}_1 : \mathbb{R}^m \to Z_{n,K}$, $\mathcal{B}_2 : \mathbb{R}^p \to Z_{n,n,K}$, $\mathcal{D}_1 : \mathbb{R}^m \to \mathbb{R}^q$, $\mathcal{D}_2 : \mathbb{R}^p \to \mathbb{R}^q$, and $\mathcal{C} : Z_{n,n,K} \to \mathbb{R}^p$ where

$$Z_{m,n,K} := \{\mathbb{R}^m \times L_2^n[-\tau_1, 0] \times \cdots \times L_2^n[-\tau_K, 0]\}$$

$$X := \left\{ \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{n,K} \; : \; \begin{matrix} \phi_i \in W_2^n[-\tau_i, 0] \text{ and} \\ \phi_i(0) = x \text{ for all } i \in [K] \end{matrix} \right\}.$$

Duality Theorem for Controller Synthesis: Suppose $\mathcal{Y} \geq \epsilon I$, $\mathcal{Y} : X \to X$ and

$$\begin{bmatrix} w \\ v \\ \mathbf{x}_f \end{bmatrix}^T \begin{bmatrix} -\gamma I & D_1^T & \mathcal{B}_1^* \\ D_1 & -\gamma I & \mathcal{C}\mathcal{Y} + \mathcal{D}_2\mathcal{Z} \\ \mathcal{B}_1 & (\mathcal{C}\mathcal{Y} + \mathcal{D}_2\mathcal{Z})^* & \mathcal{A}\mathcal{Y} + \mathcal{B}_2\mathcal{Z} + (\star)^* \end{bmatrix} \begin{bmatrix} w \\ v \\ \mathbf{x}_f \end{bmatrix} \prec 0$$

then if $\mathcal{K} = \mathcal{Z}\mathcal{Y}^{-1}$, we have $\|z\|_{L_2} \leq \gamma \|w\|_{L_2}$

DPS Version of Controller Synthesis: Minimize γ such that $\exists \mathcal{P} : X \to X$ (coercive, $\mathcal{P} = \mathcal{P}^*$, $\mathcal{P}(X) = X$) and $\mathcal{Z}$ such that

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z} \rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w \rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z} \rangle_Z - \gamma w^T w$$
$$+ v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v + v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v \leq -\epsilon \|z\|_Z^2$$

for all $\mathbf{z} \in Z$, $w \in \mathbb{R}^m$, $v \in \mathbb{R}^q$

Put Each Term in the PQRS Framework

Define $\mathbf{z} = \begin{bmatrix} x \\ \phi_i \end{bmatrix}$ and $h = \begin{bmatrix} v^T & w^T & x^T & \phi_1(-\tau_1)^T & \cdots & \phi_K(-\tau_K)^T \end{bmatrix}^T$.

H_∞-optimal Controller Synthesis Condition: Let $\mathcal{P} = \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z}\rangle_Z - \gamma w^T w$$
$$+ v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v + v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v \le -\epsilon\|z\|$$

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z}\rangle_{Z_{n,K}} + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z}\rangle_{Z_{n,K}} = \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1, E_{1i}, \dot{S}_i, G_{ij}\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}}$$

where

$$D_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & C_0 + C_0^T & C_1 & \dots & C_K \\ 0 & 0 & C_1^T & -S_1(-\tau_1) & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & C_K^T & 0 & 0 & -S_K(-\tau_K) \end{bmatrix}, \quad C_0 := A_0 P + \tau_K \sum_{i=1}^K \left(A_i Q_i(-\tau_i)^T + \tfrac{1}{2} S_i(0)\right), \quad C_i := \tau_K A_i S_i(-\tau_i),$$

$$E_{1i}(s) := \begin{bmatrix} 0 & 0 & B_i(s)^T & 0 & \cdots & 0 \end{bmatrix}^T, \quad B_i(s) := A_0 Q_i(s) + \dot{Q}_i(s) + \sum_{j=1}^K A_j R_{ji}(-\tau_j, s),$$

$$G_{ij}(s,\theta) := \frac{\partial}{\partial s} R_{ij}(s,\theta) + \frac{\partial}{\partial \theta} R_{ji}(s,\theta)^T.$$

Put Each Term in the PQRS Framework

$$\left(\mathcal{Z}\begin{bmatrix}\psi\\ \phi_i\end{bmatrix}\right) := \left[Z_0\psi + \sum_i Z_{1i}\phi_i(-\tau_i) + \sum_i \int_{-\tau_i}^0 Z_{2i}(s)\phi_i(s)ds\right]$$

$$\langle \mathcal{A}\mathcal{P}\mathbf{z},\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z},\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w,\mathbf{z}\rangle_Z - \gamma w^T w$$

$$+ v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v + v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v \le -\epsilon\|z\|$$

$$\langle \mathcal{B}_2\mathcal{Z}\mathbf{z},\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z = 2\tau_K x^T\left[B_2 Z_0 x + \sum_i B_2 Z_{1i}\phi_i(-\tau_i) + \sum_i \int_{-\tau_i}^0 B_2 Z_{2i}(s)\phi_i(s)ds\right]$$

$$= \tau_K \underbrace{\begin{bmatrix} v\\ w\\ x\\ \phi_1(-\tau_1)\\ \vdots\\ \phi_K(-\tau_K)\end{bmatrix}^T}_{h^T} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0\\ *^T & 0 & 0 & 0 & \dots & 0\\ *^T & *^T & B_2 Z_0 + Z_0^T B_2^T & B_2 Z_{11} & \dots & B_2 Z_{1K}\\ *^T & *^T & *^T & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ *^T & *^T & *^T & *^T & \dots & 0\end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} v\\ w\\ x\\ \phi_1(-\tau_1)\\ \vdots\\ \phi_K(-\tau_K)\end{bmatrix}}_{h}$$

$$+ 2\tau_K \sum_{i=1}^K \int_{-\tau_i}^0 \begin{bmatrix} v\\ w\\ x\\ \phi_1(-\tau_1)\\ \vdots\\ \phi_K(-\tau_K)\end{bmatrix}^T \underbrace{\begin{bmatrix} 0\\ 0\\ B_2 Z_{2i}(s)\\ 0\\ \vdots\\ 0\end{bmatrix}}_{E_{2i}(s)} \phi_i(s)ds = \left\langle \begin{bmatrix} h\\ \phi_i\end{bmatrix}, \mathcal{P}_{\{D_2, E_{2i}, 0, 0\}}\begin{bmatrix} h\\ \phi_i\end{bmatrix}\right\rangle_{Z_{r,n,K}}.$$

Put Each Term in the PQRS Framework

$$\langle \mathcal{AP}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{AP}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z}\rangle_Z - \gamma w^T w$$
$$+ v^T(\mathcal{CP}\mathbf{z}) + (\mathcal{CP}\mathbf{z})^T v + v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v \le -\epsilon\|z\|$$

$$\langle \mathbf{z}, \mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z}\rangle_Z - \gamma w^T w + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v$$
$$= \tau_K x^T B_1 w + \tau_K (B_1 w)^T x - \gamma w^T w + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v$$
$$= \tau_K \underbrace{\begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K)\end{bmatrix}^T}_{h^T} \underbrace{\frac{1}{\tau_K}\begin{bmatrix} -\gamma I & D_1 & 0 & 0 & \dots & 0 \\ D_1^T & -\gamma I & \tau_K B_1^T & 0 & \dots & 0 \\ 0 & \tau_K B_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}}_{D_3} \underbrace{\begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K)\end{bmatrix}}_{h}$$
$$= \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_3,0,0,0\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}}$$

Put Each Term in the PQRS Framework

$$\langle \mathcal{A}\mathcal{P}\mathbf{z},\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z},\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w,\mathbf{z}\rangle_Z - \gamma w^T w$$
$$+ v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v + v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v \le -\epsilon\|z\|$$

$$v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v = 2v^T\Big[\Big(C_0P + \sum_i \tau_K C_i Q_i(-\tau_i)^T\Big)x + \tau_K \sum_i C_i S_i(-\tau_i)\phi_i(-\tau_i) + \sum_{i=1}^K \int_{-\tau_i}^0 \Big(C_0 Q_i(s) + \sum_j C_j R_{ji}(-\tau_j,s)\Big)\phi_i(s)ds\Big]$$

$$= \tau_K \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K)\end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & 0 & \frac{C_0P}{\tau_K} + \sum_i C_i Q_i(-\tau_i)^T & C_1 S_1(-\tau_1) & \dots & C_K S_K(-\tau_K) \\ *^T & 0 & 0 & 0 & \dots & 0 \\ *^T & *^T & 0 & 0 & \dots & 0 \\ *^T & *^T & *^T & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ *^T & *^T & *^T & *^T & \dots & 0 \end{bmatrix}}_{D_4} \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K)\end{bmatrix}$$

$$+ 2\tau_K \sum_{i=1}^K \int_{-\tau_i}^0 \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K)\end{bmatrix}^T \underbrace{\frac{1}{\tau_K}\begin{bmatrix} C_0 Q_i(s) + \sum_j C_j R_{ji}(-\tau_j,s) \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{E_{4i}(s)} \phi_i(s)ds = \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_4, E_{4i}, 0, 0\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}}.$$

Put Each Term in the PQRS Framework

$$\langle \mathcal{A}\mathcal{P}\mathbf{z},\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z},\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w,\mathbf{z}\rangle_Z - \gamma - \gamma w^T w$$
$$+ v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v + v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v \le -\epsilon \|z\|$$

$$v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v = 2v^T \left[D_2 Z_0 x + \sum_i D_2 Z_{1i}\phi_i(-\tau_i) + \sum_i \int_{-\tau_i}^0 D_2 Z_{2i}(s)\phi_i(s)ds\right]$$

$$= \tau_K \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}^T \underbrace{\frac{1}{\tau_K}\begin{bmatrix} 0 & 0 & D_2 Z_0 & D_2 Z_{11} & \dots & D_2 Z_{1K} \\ *^T & 0 & 0 & 0 & \dots & 0 \\ *^T & *^T & 0 & 0 & \dots & 0 \\ *^T & *^T & *^T & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ *^T & *^T & *^T & *^T & \dots & 0 \end{bmatrix}}_{D_5} \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}$$

$$+ 2\tau_K \sum_{i=1}^K \int_{-\tau_i}^0 \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}^T \underbrace{\frac{1}{\tau_K}\begin{bmatrix} D_2 Z_{2i}(s) \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{E_{5i}(s)} \phi_i(s)ds = \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_5, E_{5i}, 0, 0\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}}.$$

Combine Terms and enforce Constraint

And, finally,

$$\epsilon\|z\|_Z^2 = \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{\hat{I},0,I,0\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}} \qquad \text{where} \quad \hat{I} = \text{diag}(0_{q+m}, I_n, 0_{nK})$$

Find P, Q_i, S_i, R_{ij}, Z_0, Z_{1i}, and Z_{2i} such that

$$\left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D+\hat{I},E_i,\dot{S}_i+I,G_{ij}\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}} \le 0,$$

where $D = \sum_{i=1}^5 D_i$,and $E_i(s) = \sum_{j=1}^5 E_{ij}(s)$. Then there exists a feedback controller $u(t) = \mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t)$ which achieves CL H_∞ norm γ.

Matlab Code: `solver_ndelay_opt_control.m`

```
[P,Q,R,S] = sosjointpos_mat_ker_ndelay_PQRS_vZ
...
[P2,Q2,R2,S2] = sosjointpos_mat_ker_ndelay_PQRS_vZ
sosmateq(prog,D+P2); sosmateq(prog,Q2{i}+E{i});
sosmateq(prog,S2{i}+F{i}); sosmateq(prog,R2{i,j}+G{i,j});
```

How to ensure $\mathcal{P}(X) = X$

Not Needed for Optimal Estimator Synthesis

Recall PQRS Operators have the form

$$\begin{bmatrix} x' \\ \phi_i' \end{bmatrix}(s) = \left(\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix}\right)(s)$$

$$= \begin{bmatrix} Px + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(s)\phi_i(s)ds \\ \tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(s,\theta)\phi_j(\theta)\, d\theta \end{bmatrix}$$

So to achieve $x' = \phi_i'(0)$, we need

$$P = \tau_K(Q_i(0)^T + S_i(0)), \qquad Q_j(s) = R_{ij}(0,s) \qquad \forall i,j$$

These are linear constraints on P and the coefficients of polynomials Q_i, S_i, R_{ij}.

Matlab Code:

```
for i=1:n_delay
 prog = sosmateq(prog,P-tau*(subs(Q{i}',s,0)+subs(S{i},s,0)));
for j=1:n_delay
 prog = sosmateq(prog,Q{i}-subs(var_swap(R{j,i},s,th),th,0));
 end
end
```

The Inverse of a $PQRS$ Operator is a $PQRS$ Operator!

How to find

$$\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}_{\{P,Q,S,R\}}?$$

Extract Coefficients: $Q(s) = HZ(s)$ and $R(s,\theta) = Z(s)^T \Gamma Z(\theta)$.
Then $\mathcal{P}^{-1}_{\{P,Q,S,R\}} = \mathcal{P}_{\{\hat{P},\hat{Q},\hat{S},\hat{R}\}}$ where

$$\hat{P} = \left(I - \hat{H} V H^T\right) P^{-1}, \qquad \hat{Q}(s) = \frac{1}{\tau}\hat{H} Z(s) S(s)^{-1}$$

$$\hat{S}(s) = \frac{1}{\tau^2} S(s)^{-1} \qquad \hat{R}(s,\theta) = \frac{1}{\tau} S(s)^{-1} Z(s)^T \hat{\Gamma} Z(\theta) S(\theta)^{-1},$$

where

$$\hat{H} = P^{-1} H \left(V H^T P^{-1} H - I - V\Gamma\right)^{-1}$$

$$\hat{\Gamma} = -(\hat{H}^T H + \Gamma)(I + V\Gamma)^{-1},$$

$$V = \int_{-\tau}^{0} Z(s) S(s)^{-1} Z(s)^T ds$$

Analytic Formula for Operator Inversion

Suppose $\mathcal{P} := \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$, $Q_i(s) = H_i Z(s)$ and $R_{ij}(s,\theta) = Z(s)^T \Gamma_{ij} Z(\theta)$. Then $\mathcal{P}^{-1} = \mathcal{P}_{\left\{\hat{P},\hat{Q}_i,\hat{S}_i,\hat{R}_{ij}\right\}}$ where if we define

$$H = \begin{bmatrix} H_1 & \dots & H_K \end{bmatrix} \qquad \text{and} \qquad \Gamma = \begin{bmatrix} \Gamma_{11} & \dots & \Gamma_{1K} \\ \vdots & & \vdots \\ \Gamma_{K,1} & \dots & \Gamma_{K,K} \end{bmatrix},$$

then

$$\hat{P} = \left(I - \hat{H} V H^T\right) P^{-1}, \quad \hat{Q}_i(s) = \frac{1}{\tau_K} \hat{H}_i Z(s) S_i(s)^{-1}$$

$$\hat{S}_i(s) = \frac{1}{\tau_K^2} S_i(s)^{-1} \qquad \hat{R}_{ij}(s,\theta) = \frac{1}{\tau_K} S_i(s)^{-1} Z(s)^T \hat{\Gamma}_{ij} Z(\theta) S_i(\theta)^{-1},$$

where

$$\begin{bmatrix} \hat{H}_1 & \dots & \hat{H}_K \end{bmatrix} = \hat{H} = P^{-1} H \left(V H^T P^{-1} H - I - V\Gamma\right)^{-1}$$

$$\begin{bmatrix} \hat{\Gamma}_{11} & \dots & \hat{\Gamma}_{1K} \\ \vdots & & \vdots \\ \hat{\Gamma}_{K,1} & \dots & \hat{\Gamma}_{K,K} \end{bmatrix} = \hat{\Gamma} = -(\hat{H}^T H + \Gamma)(I + V\Gamma)^{-1}, \quad V = \begin{bmatrix} V_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & V_K \end{bmatrix}$$

$$V_i = \int_{-\tau_i}^{0} Z(s) S_i(s)^{-1} Z(s)^T ds$$

Reconstructing the Full-State Feedback Controller Gains

Finally, we recover the controller as

$$u(t) = K_0 x(t) + \frac{1}{\tau_K}\sum_i K_{1i} x(t-\tau_i) + \frac{1}{\tau_K}\sum_i \int_{-\tau_i}^0 K_{2i}(s) x(t+s) ds$$

where (Z_0, Z_{1i}, Z_{2i} are variables, Z is a vector of monomials)

$$K_0 = Z_0 \hat{P} + \sum_j \left(Z_{1j} S_j(-\tau_j)^{-1} Z(-\tau_j)^T + O_j \right) \hat{H}_j^T$$

$$K_{1i} = Z_{1i} S_i(-\tau_i)^{-1}, \qquad O_i = \int_{-\tau_j}^0 Z_{2j}(s) S_j(s)^{-1} Z(s)^T ds$$

$$K_{2i}(s) = \left(Z_0 \hat{H}_i Z(s) + Z_{2i}(s) + \sum_{j=1}^{K} \left(Z_{1j} S_j(-\tau_j)^{-1} Z(-\tau_j)^T + O_j \right) \hat{\Gamma}_{ji} Z(s) \right) S_i(s)^{-1}$$

Note: This is *Full-State* Feedback.

- Contrast with output feedback: $u(t) = Kx(t)$ or $u(t) = Ky(t-r)$.

H_∞-Optimal Observer Synthesis Problem to be Solved

Consider solutions of

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + Bw(t)$$
$$y(t) = C_2 x(t)$$

With a PDE observer (observed errors)(nominal dynamics)(corrective gains)

$$\dot{\hat{x}}(t) = A_0\hat{x}(t) + A_1\hat{\phi}(t,-\tau) + L_1\left(C_2\hat{x}(t) - y(t)\right) + L_2\left(C_2\hat{\phi}(t,-\tau) - y(t-\tau)\right)$$
$$\qquad + \int_{-\tau}^{0} L_3(\theta)\left(C_2\hat{\phi}(t,\theta) - y(t+\theta)\right)d\theta$$
$$\partial_t\hat{\phi}(t,s) = \partial_s\hat{\phi}(t,s) + L_4(s)\left(C_2\hat{x}(t) - y(t)\right) + L_5(s)\left(C_2\hat{\phi}(t,-\tau) - y(t-\tau)\right)$$
$$\qquad + L_6(s)\left(C_2\hat{\phi}(t,s) - y(t+s)\right) + \int_{-\tau}^{0} L_7(s,\theta)\left(C_2\hat{\phi}(t,\theta) - y(t+\theta)\right)d\theta$$
$$\hat{\phi}(t,0) = \hat{x}(t)$$

Problem Definition:
Minimize γ such that there exist L_i such that if $z_e(t) = C_1(x(t) - \hat{x}(t))$, then for any $w \in L_2$, $\|z_e\|_{L_2} \le \gamma \|w\|_{L_2}$.

Operator Version of the Dynamics

Write the DDE as

$$\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}w(t), \qquad \underbrace{z(t) = \mathcal{C}_1 x(t)}_{\text{Regulated Output}}, \qquad \underbrace{\mathbf{y}(t) = \mathcal{C}_2\mathbf{x}(t)}_{\text{Observed Output}}$$

where

$$\mathcal{A}\begin{bmatrix} x \\ \phi \end{bmatrix}(s) := \begin{bmatrix} A_0 x + A_1\phi(-\tau) \\ \dot{\phi}(s) \end{bmatrix}, (\mathcal{B}w)(s) := \begin{bmatrix} B_1 w \\ 0 \end{bmatrix}$$

$$\left(\mathcal{C}_1\begin{bmatrix} x_1 \\ \phi \end{bmatrix}\right) := \begin{bmatrix} C_1 x_1 \end{bmatrix}, \qquad \left(\mathcal{C}_2\begin{bmatrix} x_1 \\ \phi \end{bmatrix}\right)(s) := \begin{bmatrix} C_2 x_1 \\ C_2\phi(s) \end{bmatrix}$$

Details: $\mathcal{A} : X \to Z_{n,n}$, $\mathcal{B} : \mathbb{R}^m \to Z_{n,n}$, $\mathcal{C}_1 : Z_{n,n} \to \mathbb{R}^p$, and $\mathcal{C}_2 : Z_{n,n} \to Z_{q,q}$ where

$$Z_{m,n} := \{\mathbb{R}^m \times L_2^n[-\tau_1, 0]\}$$

$$X := \left\{ \begin{bmatrix} x \\ \phi \end{bmatrix} \in Z_{n,n} \; : \; \substack{\phi \in W_2^n[-\tau,0] \text{ and} \\ \phi(0)=x} \right\}.$$

Operator Version of the Observer

Write the Observer dynamics as

$$\dot{\hat{\mathbf{x}}}(t) = \underbrace{\mathcal{A}\hat{\mathbf{x}}(t)}_{\text{Nominal Dynamics}} + \underbrace{\mathcal{L}\left(\mathcal{C}_2\hat{\mathbf{x}}(t) - \mathbf{y}(t)\right)}_{\text{Correction Term}}, \qquad \underbrace{z_e(t) = \mathcal{C}_1(\hat{\mathbf{x}}(t) - \mathbf{x}(t))}_{\text{Regulated Error}}$$

where

$$\mathcal{A}\begin{bmatrix} x \\ \phi \end{bmatrix}(s) := \begin{bmatrix} A_0 x + A_1\phi(-\tau) \\ \dot{\phi}(s) \end{bmatrix},$$

$$\left(\mathcal{C}_1\begin{bmatrix} x_1 \\ \phi \end{bmatrix}\right) := \left[C_1 x_1\right], \qquad \left(\mathcal{C}_2\begin{bmatrix} x_1 \\ \phi \end{bmatrix}\right)(s) := \begin{bmatrix} C_2 x_1 \\ C_2\phi(s) \end{bmatrix}$$

$$\mathcal{L}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}(s) = \begin{bmatrix} L_1 y_1 + L_2 y_2(-\tau) + \int_{-\tau}^0 L_3(\theta) y_2(\theta) d\theta \\ L_4(s) y_1 + L_5(s) y_2(-\tau) + L_6(s) y_2(s) + \int_{-\tau}^0 L_7(s,\theta) y_2(\theta) d\theta \end{bmatrix}$$

Details: $\mathcal{A} : X \to Z_{n,n}$, $\mathcal{L} : Z_{q,q} \to Z_{n,n}$, $\mathcal{C}_1 : Z_{n,n} \to \mathbb{R}^p$, and $\mathcal{C}_2 : Z_{n,n} \to Z_{q,q}$ where

$$Z_{m,n} := \{\mathbb{R}^m \times L_2^n[-\tau_1, 0]\}$$

$$X := \left\{ \begin{bmatrix} x \\ \phi \end{bmatrix} \in Z_{n,n} \; : \; \begin{matrix} \phi \in W_2^n[-\tau, 0] \text{ and} \\ \phi(0) = x \end{matrix} \right\}.$$

Operator Version of the Error Dynamics

Write the Error dynamics ($\mathbf{e}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t)$) as

$$\dot{\mathbf{e}}(t) = (\mathcal{A} + \mathcal{L}\mathcal{C}_2)\mathbf{e}(t) - \mathcal{B}w(t), \qquad \underbrace{z_e(t) = \mathcal{C}_1\mathbf{e}(t)}_{\text{Regulated Error}}$$

where

$$\mathcal{A}\begin{bmatrix} x \\ \phi \end{bmatrix}(s) := \begin{bmatrix} A_0 x + A_1\phi(-\tau) \\ \dot{\phi}(s) \end{bmatrix}, \qquad (\mathcal{B}w)(s) := \begin{bmatrix} B_1 w \\ 0 \end{bmatrix}$$

$$\left(\mathcal{C}_1 \begin{bmatrix} x_1 \\ \phi \end{bmatrix}\right) := \begin{bmatrix} C_1 x_1 \end{bmatrix}, \qquad \left(\mathcal{C}_2 \begin{bmatrix} x_1 \\ \phi \end{bmatrix}\right)(s) := \begin{bmatrix} C_2 x_1 \\ C_2\phi(s) \end{bmatrix}$$

$$\mathcal{L}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}(s) = \begin{bmatrix} L_1 y_1 + L_2 y_2(-\tau) + \int_{-\tau}^0 L_3(\theta) y_2(\theta) d\theta \\ L_4(s) y_1 + L_5(s) y_2(-\tau) + L_6(s) y_2(s) + \int_{-\tau}^0 L_7(s,\theta) y_2(\theta) d\theta \end{bmatrix}$$

Details: $\mathcal{A} : X \to Z_{n,n}$, $\mathcal{L} : Z_{q,q} \to Z_{n,n}$, $\mathcal{C}_1 : Z_{n,n} \to \mathbb{R}^p$, and $\mathcal{C}_2 : Z_{n,n} \to Z_{q,q}$ where

$$Z_{m,n} := \{\mathbb{R}^m \times L_2^n[-\tau, 0]\}$$

$$X := \left\{ \begin{bmatrix} x \\ \phi \end{bmatrix} \in Z_{n,n} \; : \; \begin{smallmatrix} \phi \in W_2^n[-\tau,0] \text{ and} \\ \phi(0) = x \end{smallmatrix} \right\}.$$

An LMI for Optimal *Estimation* of **ODE**s

Get rid of the delays and we have

$$\dot{x}(t) = Ax(t) + B_1 w(t), \qquad y(t) = C_2 x(t) + Dw(t)$$

Observer:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + L(C_2\hat{x}(t) - y(t)), \quad z_e(t) = C_1(\hat{x}(t) - x(t))$$

Lemma 4 (H_∞-Optimal Observer Synthesis).

Define the map $w \mapsto z_e$:

$$\hat{G}(s) = \left[\begin{array}{c|c} A + LC_2 & -(B + LD) \\ \hline C_1 & 0 \end{array}\right].$$

The following are equivalent.

- *There exists a L such that $\|\hat{G}\|_{H_\infty} \le \gamma$.*
- *There exists a $P > 0$ and Z such that*

$$\begin{bmatrix} A^T P + C_2^T Z^T + PA + ZC_2 & -(PB + ZD) \\ -(PB + ZD)^T & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C_1^T C_1 & 0 \\ 0 & 0 \end{bmatrix} < 0.$$

The Observer Gain is recovered as $L = P^{-1}Z$.

The DPS/DDE Equivalent of the Observer LMI

LMI Version of Observer Synthesis: Minimize γ such that $\exists P > 0$ and $Z \in \mathbb{R}^{p\times n}$ such that

$$\begin{bmatrix} e \\ w \end{bmatrix}^T \left[\begin{bmatrix} A^T P + C_2^T Z^T + PA + ZC_2 & -(PB + ZD) \\ -(PB+ZD)^T & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C_1^T C_1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} e \\ w \end{bmatrix}$$

$$= (PAe)^T e + (PAe)^T e + (ZCe)^T e + (ZC_2 e)^T e$$

$$-e^T PBw - (PBw)^T e - \gamma w^T w + \frac{1}{\gamma}(C_1 e)^T (C_1 e) < 0$$

for all $e \in \mathbb{R}^n$, $w \in \mathbb{R}^m$

DPS Version of Observer Synthesis: Minimize γ such that $\exists \mathcal{P} > 0$ and $\mathcal{Z}$ such that

$$\langle \mathcal{P}\mathcal{A}\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{P}\mathcal{A}\mathbf{e}\rangle_{L_2} + \langle \mathcal{Z}\mathcal{C}_2\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{L_2}$$

$$-\langle \mathbf{e}, \mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle \mathcal{B}w, \mathcal{P}\mathbf{e}\rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma}(\mathcal{C}_1\mathbf{e})^T(\mathcal{C}_1\mathbf{e}) < -\epsilon\|\mathbf{e}\|^2 \qquad \forall \mathbf{e} \in X,\ w \in \mathbb{R}^m$$

Observer: Put Each Term in the PQRS Framework

Define $\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ and $h = \begin{bmatrix} w^T & e_1^T & e_2(-\tau)^T \end{bmatrix}^T$.

$$\langle \mathcal{P}\mathcal{A}\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{P}\mathcal{A}\mathbf{e}\rangle_{L_2} + \langle \mathcal{Z}\mathcal{C}_2\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{L_2}$$

$$- \langle \mathbf{e}, \mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle \mathcal{B}w, \mathcal{P}\mathbf{e}\rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma}(\mathcal{C}_1\mathbf{e})^T(\mathcal{C}_1\mathbf{e}) < -\epsilon\|\mathbf{e}\|^2 \qquad \forall \mathbf{e} \in X,\ w \in \mathbb{R}^m,$$

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z}\rangle_{Z_n} + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z}\rangle_{Z_{n,K}}$$

$$= \int_{-\tau}^{0} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix}^T \begin{bmatrix} D_1(s) & \tau E_1(s) \\ \tau E_1(s)^T & -\tau \dot{S}(s) \end{bmatrix} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix} ds + \tau \int_{-\tau}^{0}\int_{-\tau}^{0} e_2(s)^T G(s,\theta) e_2(\theta) d\theta$$

$$= \left\langle \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix}, \mathcal{P}_{\{D_1, E_1, -\dot{S}, G\}} \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix} \right\rangle_{L_2}$$

where

$$D_1(s) = \begin{bmatrix} 0 & * & * \\ 0 & PA_0 + A_0^T P + Q(0) + Q(0)^T + S(0) & * \\ 0 & A_1^T P - Q(-\tau)^T & -S(-\tau) \end{bmatrix}$$

$$E(s) = \begin{bmatrix} 0 \\ A_0^T Q(s) + R(s,0)^T - \dot{Q}(s) \\ A_1^T Q(s) - R(s,-\tau)^T \end{bmatrix} \qquad G(s,\theta) = -R_\theta(s,\theta) - R_s(s,\theta).$$

Observer: Put Each Term in the PQRS Framework

$$\langle \mathcal{P}\mathcal{A}\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{P}\mathcal{A}\mathbf{e}\rangle_{L_2} + \langle \mathcal{Z}\mathcal{C}_2\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{L_2}$$

$$- \langle \mathbf{e}, \mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle \mathcal{B}w, \mathcal{P}\mathbf{e}\rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma}(\mathcal{C}_1\mathbf{e})^T(\mathcal{C}_1\mathbf{e}) < -\epsilon\|\mathbf{e}\|^2 \qquad \forall \mathbf{e} \in X,\ w \in \mathbb{R}^m,$$

where

$$\mathcal{Z}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}(s) = \begin{bmatrix} Z_1 y_1 + Z_2 y_2(-\tau) + \int_{-\tau}^0 Z_3(\theta) y_2(\theta) d\theta \\ Z_4(s) y_1 + Z_5(s) y_2(-\tau) + Z_6(s) y_2(s) + \int_{-\tau}^0 Z_7(s,\theta) y_2(\theta) d\theta \end{bmatrix}$$

$$\langle \mathcal{Z}\mathcal{C}_2\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{L_2} = 2\tau e_1^T \left(Z_1 C_2 e_1 + Z_2 C_2 e_2(-\tau) + \int_{-\tau}^0 Z_3(\theta) C_2 e_2(\theta) d\theta \right)$$

$$+ 2\tau \int_{-\tau}^0 e_2(s)^T \left(Z_4(s) C_2 e_1 + Z_5(s) C_2 e_2(-\tau) + Z_6(s) C_2 e_2(s) + \int_{-\tau}^0 Z_7(s,\theta) C_2 e_2(\theta) d\theta \right)$$

$$= \tau \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & Z_1 C_2 & Z_2 C_2 \\ 0 & C_2^T Z_2 & 0 \end{bmatrix}}_{D_2} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix} + 2\tau \int_{-\tau}^0 \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 \\ C_2^T Z_4(s)^T + Z_3(s) C_2 \\ C_2^T Z_5(s)^T \end{bmatrix}}_{E_2} e_2(s) ds$$

$$+ \tau \int_{-\tau}^0 e_2(s)^T \underbrace{\left(Z_6(s) C_2 + C_2^T Z_6(s)^T \right)}_{F_2} e_2(s) ds + \tau \int_{-\tau}^0 \int_{-\tau}^0 e_2(s)^T \underbrace{\left(Z_7(s,\theta) C_2 + C_2^T Z_7(\theta, s)^T \right)}_{G_2} e_2(\theta) d\theta$$

$$= \left\langle \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix}, \mathcal{P}_{\{D_2, E_2, F_2, G_2\}} \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix} \right\rangle_{L_2}$$

Observer: Put Each Term in the PQRS Framework

$$\langle \mathcal{P}\mathcal{A}\mathbf{e},\mathbf{e}\rangle_{L_2} + \langle \mathbf{e},\mathcal{P}\mathcal{A}\mathbf{e}\rangle_{L_2} + \langle \mathcal{Z}\mathcal{C}_2\mathbf{e},\mathbf{e}\rangle_{L_2} + \langle \mathbf{e},\mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{L_2}$$
$$-\langle \mathbf{e},\mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle \mathcal{B}w,\mathcal{P}\mathbf{e}\rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma}(\mathcal{C}_1\mathbf{e})^T(\mathcal{C}_1\mathbf{e}) < -\epsilon\|\mathbf{e}\|^2 \qquad \forall \mathbf{e}\in X,\ w\in\mathbb{R}^m,$$

$$-\langle \mathbf{e},\mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle \mathcal{B}w,\mathcal{P}\mathbf{e}\rangle_{L_2} = 2\int_{-\tau}^{0} e_1^T PBw ds + 2\int_{-\tau}^{0} e_2(s)^T \tau Q(s)^T Bw ds$$
$$= \tau \begin{bmatrix} w \\ e_1 \\ e_2(-\tau)\end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & -B^T P & 0 \\ -PB & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}}_{D_3} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau)\end{bmatrix} + 2\tau\int_{-\tau}^{0} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau)\end{bmatrix}^T \underbrace{\begin{bmatrix} -B^T Q(s) \\ 0 \\ 0\end{bmatrix}}_{E_3} e_2(s) ds$$
$$= \left\langle \begin{bmatrix} h \\ \mathbf{e}_2\end{bmatrix}, \mathcal{P}_{\{D_3,E_3,0,0\}} \begin{bmatrix} h \\ \mathbf{e}_2\end{bmatrix}\right\rangle_{L_2}$$

Observer: Put Each Term in the PQRS Framework

$$\langle \mathcal{P}\mathcal{A}\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{P}\mathcal{A}\mathbf{e}\rangle_{L_2} + \langle \mathcal{Z}\mathcal{C}_2\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{L_2}$$
$$- \langle \mathbf{e}, \mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle \mathcal{B}w, \mathcal{P}\mathbf{e}\rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma}(\mathcal{C}_1\mathbf{e})^T(\mathcal{C}_1\mathbf{e}) + \epsilon\|\mathbf{e}\|^2 \le 0 \qquad \forall \mathbf{e} \in X,\ w \in \mathbb{R}^m$$

$$-\gamma w^T w + \frac{1}{\gamma}(\mathcal{C}_1\mathbf{e})^T(\mathcal{C}_1\mathbf{e}) + \epsilon\|\mathbf{e}\|^2$$
$$= \tau \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix}^T \underbrace{\begin{bmatrix} -\frac{\gamma}{\tau} & 0 & 0 \\ 0 & \frac{1}{\gamma\tau}C_1^T C_1 + \epsilon I & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D_4} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix}$$
$$= \left\langle \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix}, \mathcal{P}_{\{D_4,0,0,0\}} \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix} \right\rangle_{L_2}$$

Combine Terms and enforce Constraint

Suppose there exist P, Q, S, R, Z_i such that $\mathcal{P}_{\{P,Q,S,R\}} > 0$ and

$$\left\langle \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix}, \mathcal{P}_{\{D,E,F,G\}} \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix} \right\rangle_{L_2} \le 0,$$

where $D = \sum_{i=1}^{5} D_i$, $E(s) = \sum_{j=1}^{3} E_i(s)$ and $G(s,\theta) = \sum_{j=1}^{2} G_i(s,\theta)$. Then if

$$\mathcal{L} = \mathcal{P}_{\{P,Q,S,R\}}^{-1} \mathcal{Z}$$

and $\dot{\hat{\mathbf{x}}}(t) = \mathcal{A}\hat{\mathbf{x}}(t) + \mathcal{L}\left(\mathcal{C}_2\hat{\mathbf{x}}(t) - \mathbf{y}(t)\right)$ and $z_e(t) = \hat{z}(t) - z(t)$, we have $\|z_e\|_{L_2} \le \gamma \|w\|_{L_2}$.

Matlab Code: `solver_ndelay_opt_estimator.m`

```
[P,Q,R,S] = sosjointpos_mat_ker_ndelay_PQRS_vZ
...
[P2,Q2,R2,S2] = sosjointpos_mat_ker_ndelay_PQRS_vZ
sosmateq(prog,D+P2); sosmateq(prog,Q2+E); sosmateq(prog,S2+F);
sosmateq(prog,R2+G);
```

Observer Gains Reconstruction

Let $\mathcal{L} = \mathcal{P}_{\{\hat{P},\hat{Q},\hat{S},\hat{R}\}}\mathcal{Z}$. Then the observer dynamics are given by

$$\dot{\hat{\mathbf{x}}}(t) = \mathcal{A}\hat{\mathbf{x}}(t) + \mathcal{L}(\mathcal{C}_2\hat{\mathbf{x}}(t) - \mathbf{y}(t)) \qquad \text{or:}$$

$$\dot{\hat{x}}(t) = A_0\hat{x}(t) + A_1\hat{\phi}(t,-\tau) + L_1\left(C_2\hat{x}(t) - y(t)\right) + L_2\left(C_2\hat{\phi}(t,-\tau) - y(t-\tau)\right)$$

$$+ \int_{-\tau}^{0} L_3(\theta)\left(C_2\hat{\phi}(t,\theta) - y(t+\theta)\right)d\theta, \qquad \hat{\phi}(t,0) = \hat{x}(t)$$

$$\partial_t\hat{\phi}(t,s) = \partial_s\hat{\phi}(t,s) + L_4(s)\left(C_2\hat{x}(t) - y(t)\right) + L_5(s)\left(C_2\hat{\phi}(t,-\tau) - y(t-\tau)\right)$$

$$+ L_6(s)\left(C_2\hat{\phi}(t,s) - y(t+s)\right) + \int_{-\tau}^{0} L_7(s,\theta)\left(C_2\hat{\phi}(t,\theta) - y(t+\theta)\right)d\theta$$

where

$$L_1 = \hat{P}Z_1 + \int_{-\tau}^{0}\hat{Q}(\theta)Z_4(\theta)d\theta, \quad L_2 = \hat{P}Z_2 + \int_{-\tau}^{0}\hat{Q}(\theta)Z_5(\theta)d\theta$$

$$L_3(\theta) = \hat{P}Z_3(\theta) + \hat{Q}(\theta)Z_6(\theta) + \int_{-\tau}^{0}\hat{Q}(s)Z_7(s,\theta)ds$$

$$L_4(s) = \hat{Q}(s)^T Z_1 + \hat{S}(s)Z_4(s) + \int_{-\tau}^{0}\hat{R}(s,\theta)Z_4(\theta)d\theta$$

$$L_5(s) = \hat{Q}(s)^T Z_2 + \hat{S}(s)Z_5(s) + \int_{-\tau}^{0}\hat{R}(s,\theta)Z_5(\theta)d\theta, \qquad L_6(s) = \hat{S}(s)Z_6(s)$$

$$L_7(s,\theta) = \hat{Q}(s)^T Z_3(\theta) + \hat{S}(s)Z_7(s,\theta) + \hat{R}(s,\theta)Z_6(\theta) + \int_{-\tau}^{0}\hat{R}(s,\xi)Z_7(\xi,\theta)d\xi.$$

Boring Numerical Examples

Numerical Example 1 In this example, we consider the unstable system

$$\dot{x}(t) = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t),$$

$$y(t) = \begin{bmatrix} 0 & 7 \end{bmatrix} x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)$$

Applying the Ricatti approach in [Fattouh 1998] with $\epsilon = .001$ we obtain a L_2-gain of $\gamma = .580$. Applying the LOI, we obtain an L_2-gain of .236. Of all the systems we tested, this one showed the least improvement in performance.

Numerical Example 2 A modified form of [Fridman 2001].

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -.9 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t),$$

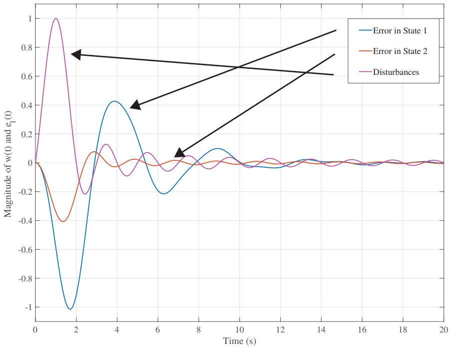
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

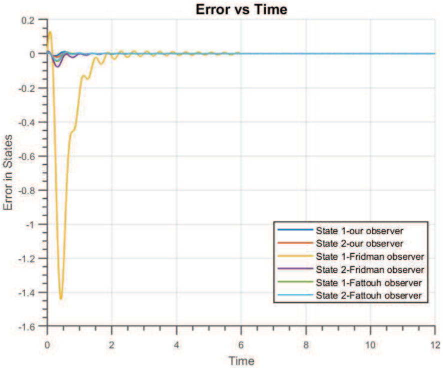
Using the original system with $\tau = 1$, a closed-loop gain of 22.8 was obtained in [Fridman 2001]. For this problem, [Fattouh 1998] was infeasible for any value of gain. Applying the LOI, we obtained a closed-loop gain of 2.33 using polynomials of degree 4.

Boring Numerical Examples

$$\dot{x}(t) = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t),$$

$$y(t) = \begin{bmatrix} 0 & 7 \end{bmatrix} x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)$$





The Last Slide (Thanks to ONR #N000014-17-1-2117)

$\mathcal{P}_{\{N_0,N_1,N_2\}}$ Operators extend LMI techniques to PDEs and Delay Systems.

- $A^T P + PA < 0$ becomes

$$\underbrace{\mathcal{P}^*_{\{H_0,H_1,H_2\}}}_{A^T}\underbrace{\mathcal{P}_{\{N_0,N_1,N_2\}}}_{P}\mathcal{P}_{\{G_0,G_1,G_2\}}+\mathcal{P}^*_{\{G_0,G_1,G_2\}}\underbrace{\mathcal{P}_{\{N_0,N_1,N_2\}}}_{P}\underbrace{\mathcal{P}_{\{H_0,H_1,H_2\}}}_{A} \le 0$$

Conclusions:

PROs:

- Computationally Efficient
- A more rational treatment of boundary conditions.
- No Conservatism (Almost N+S)
- Easily Extended to New Problems
  - e.g. higher order derivatives
  - e.g. distributed dynamics

CONs:

- Operator Theory
- No Very Good Parsers
- PDE Must be Stable in all States

Extensions:

- Input-Output Properties (ACC, 2019)
  - H_∞ Gain
  - passivity
- Optimal Estimator Synthesis
- Optimal Controller Synthesis

Solvable (in order of difficulty)

- Extension to 3D
- Duality (Stability of $\mathcal{A}^*$)
- Inversion of the $\mathcal{P}_{\{N_0,N_1,N_2\}}$ Operator
  - Want an Analytic Formula

The VERY Last Slide

Everything Here is a TOOL!

Good Luck
Be Productive

With Luck, you won't need luck