Verification and Control of Safety-Factor Profile in Tokamaks

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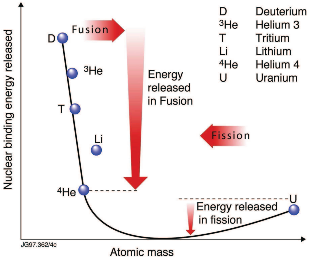
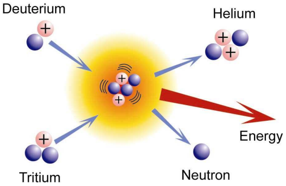
Nuclear Fusion

A Renewable Energy Source

Fusion energy is the potential energy difference between particles in free state and particles bound together by the strong nuclear force.

- ΔE from the strong nuclear force for the ^{2}H + ^{3}H to ^{4}He + ^{1}n reaction is -3.5 MeV/nucleon.
- ΔE from the Coulomb Barrier for the ^{2}H + ^{3}H to ^{4}He + ^{1}n reaction is $+.01$MeV/nucleon
- Nuclear Fission of U^{235} only releases $-.85$ MeV/nucleon
- Unfortunately $.01$MeV/nucleon means a temperature of $120 \cdot 10^6$K.
  - Temperature at center of sun is $15.7 \cdot 10^6$K.
  - From Maxwell-Boltzmann distribution, we only need $\cong 10^6$K for a statistically significant reaction rate



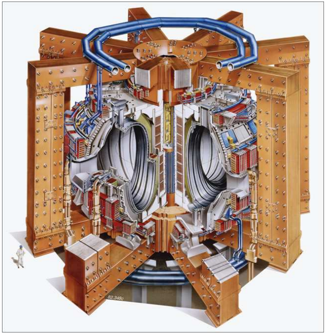


Tokamaks

Magnetic Confinement of Plasma

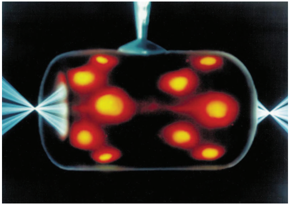
Magnetic Confinement

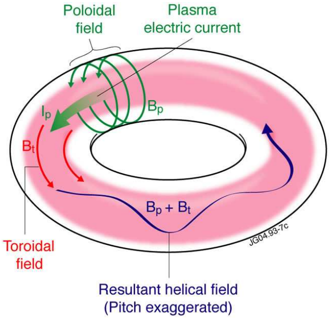
- At high temperature, atoms ionize.
  - Hydrogen $\rightarrow$ ^{2}H ion + electron.
- Charged particles oscillate in a uniform magnetic field.
  - But a uniform field must eventually end.
  - Particles will eventually escape.
- Tokamaks loop the field back on itself.
  - Particles rotate indefinitely.



Inertial Confinement

- Compress the fuel quickly
- Plasma does not have time to expand spatially before creating additional reactions.
  - Similar to a hydrogen bomb.





The plasma is contained through the combined action of toroidal and solenoid field coils.

- The toroidal coils produce a magnetic field, B_ϕ.
  - Field lines are orthogonal to the Z-axis.
- The solenoid produces a plasma current which produces a poloidal magnetic field, B_θ.
  - Field lines in the $R-Z$ plane.

The Safety-Factor and Safety-Factor Profile

A Useful **Heuristic**

The **Safety Factor**, q is the number toroidal field rotations for every poloidal rotation.

- Triggers internal transport barriers which increase energy confinement
- The higher the safety-factor, the better the plasma is contained.

The **Safety-Factor Profile** is the distribution of the safety-factor along an idealized radius.

$$q(x,t) = \frac{\partial \phi(x,t)/\partial x}{\partial \psi(x,t)/\partial x} = \frac{-B_{\phi_0} a^2 x}{\partial \psi(x,t)/\partial x},$$

where

$$\begin{aligned} x &= \text{normalized radius} \\ B_{\phi_0} &= \text{toroidal magnetic field at the plasma center} \\ a &= \text{radius of the last closed magnetic surface (LCMS)} \\ \phi &= \text{magnetic flux of the toroidal field} \\ \psi &= \text{magnetic field of the poloidal field} \end{aligned}$$

To control $q(x,t)$, we control $\psi_x(x,t) = \partial \psi(x,t)/\partial x$.

The Dynamics of the Poloidal Flux Gradient

To Control the Safety-Factor Profile, we regulate $\psi_x(x,t) = \frac{\partial}{\partial x}\psi(x,t)$.

$$\frac{\partial \psi_x(x,t)}{\partial t} = \frac{1}{\mu_0 a^2}\frac{\partial}{\partial x}\left(\frac{\eta_{\parallel}(x,t)}{x}\frac{\partial}{\partial x}\left(x\psi_x(x,t)\right)\right) + R_0\frac{\partial}{\partial x}\left(\eta_{\parallel}(x,t)j_{ni}(x,t)\right).$$

where

$$\begin{aligned} R_0 &= \text{magnetic center location} \\ \mu_0 &= \text{permeability of free space} \\ \eta_{\parallel}(x,t) &= \text{ plasma resistivity} \\ j_{ni}(x,t) &= \text{ non-inductive current density} \end{aligned}$$

with the boundary conditions

$$\psi_x(0,t) = 0 \text{ and } \psi_x(1,t) = 0. \tag{1}$$

The dynamics are coupled to electron temperature via **Plasma Resistivity**, $\eta_{\parallel}$.

- Depends on dynamics of temperature, density, etc.
- Nonlinear coupling
- Assume a separation of time-scales

Dynamical System Representation

Lets represent this PDE as an abstract differential system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where A and B are the operators

$$(A\psi)(x) := \frac{1}{\mu_0 a^2} \frac{\partial}{\partial x} \left(\eta_{\parallel}(x) \frac{\partial}{\partial x} (x\psi(x)) \right)$$
$$(Bj_{ni})(x) := \frac{\partial}{\partial x} \left(\eta_{\parallel}(x) j_{eni}(x) \right)$$

We ignore the bootstrap current.

This System generates a strongly continuous semigroup on $X = L_2[0, 1]$ with domain

$$x \in D_A = \{y \in L_2[0, 1] : y, y_x, y_{xx} \in L_2[0, 1], y(0) = y(1) = 0\}.$$

For linear dynamical systems, we have the following characterization of stability. Suppose the operator A generates a strongly continuous semigroup on Hilbert space X with domain D_A.

Theorem 1.

The system

$$\dot{x}(t) = Ax(t)$$

is stable if and only if there exist a positive operator $P \in \mathcal{L}(D_A \to D_A)$ *such that*

$$\langle x, (A^*P + PA)x \rangle_X < \|x\|_X^2$$

for all $x \in D_A$.

Stability is equivalent to a feasibility problem with

- operator-valued variables
- linear inequality constraints

The Variable Substitution Trick:

Theorem 2.

The system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is stabilizable via full-state feedback if and only if there exist operators $P > 0$ and Z such that

$$PA^* + AP + BZ + Z^*B < 0$$

Then $K = ZP^{-1}$.

Here the inequality $PA^* + AP + BZ + Z^*B < 0$ means

$$\langle x, (PA^* + AP + BZ + Z^*B)x \rangle_X$$

for all $x = P^{-1}y$, $y \in D_A$.

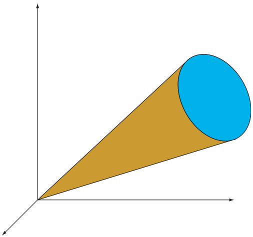
- If $P : D_A \to D_A$, then this is no harder than the Lyapunov inequality.

Tractable or Intractable?

Convex Optimization

Problem:

$$\max \; bx$$
$$\text{subject to} \quad Ax \in C$$



The problem is *convex optimization* if

- C is a convex cone.
- b and A are affine.

Computational Tractability: Convex Optimization over C is, in general, tractable if

- The set membership test for $y \in C$ is in P.
- x is finite dimensional.

The Stabilization Problem is Convex

Optimization Problem: Find $P \in \mathcal{L}(Z)$ and $Z \in \mathcal{L}(D_A)$ such that

$$PA^* + AP + BZ + Z^*B < 0$$
$$P > 0$$

Inequality represents the convex cone of positive operators on D_A with inner product X.

- Composition and adjoint are linear operations.
- Convex combinations of positive operators are positive.

Problems

- The space of operators is infinite-dimensional.
- Verifying positivity of an operator is hard.

Solving Linear Operator Inequalities

A Finite-Dimensional Subspace

Question: How to parameterize the set of operators?

- Later, we will enforce positivity.

Classes of Operators: $x \in \mathbb{R} \times \mathcal{C}[-\tau, 0]$

$$(Ax)(s) = M(s)x(s) + \int_0^1 N(s,t)x(t)dt$$

- $M(s)$ is the multiplier of a **Multiplier Operator**.
- $N(s,t)$ is the kernel of an **Integral Operator**.

Question: How to parameterize multiplier and integral operators

- We consider *polynomial* multipliers and kernels

$$M(s) = c^T D(s)$$

  - $D(s)$ is a monomial basis
  - c is a vector of decision variables.
  - For a finite basis, the set of operators is finite-dimensional

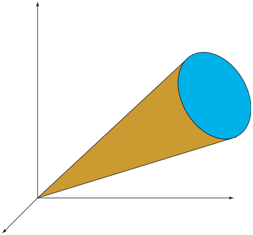
Now, how do we enforce positivity on D_A?

Optimization of Polynomials

Problem:

$$\max\ b^T x$$

subject to $A_0(y) + \sum_{i=1}^{n} x_i A_i(y) \succeq 0 \quad \forall y$



The A_i are matrices of polynomials in y. e.g. Using multi-index notation,

$$A_i(y) = \sum_{\alpha} A_{i,\alpha}\ y^{\alpha}$$

Computationally Intractable

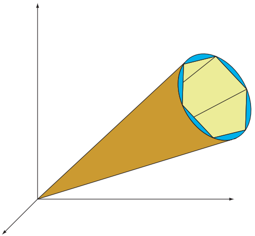
The problem: "Is $p(x) \geq 0$ for all $x \in \mathbb{R}^n$?" (i.e. "$p \in \mathbb{R}^+[x]$?") is NP-hard.

Sum-of-Squares (SOS) Programming

Problem:

$$\max \; b^T x$$

$$\text{subject to} \quad A_0(y) + \sum_{i=1}^{n} x_i A_i(y) \in \Sigma_s$$



Definition 3.

$\Sigma_s \subset \mathbb{R}^+[x]$ is the cone of *sum-of-squares* matrices. If $S \in \Sigma_s$, then for some $G_i \in \mathbb{R}[x]$,

$$S(y) = \sum_{i=1}^{r} G_i(y)^T G_i(y)$$

Computationally Tractable: $S \in \Sigma_s$ is an SDP constraint.

SOS Programming:

Why is $M \in \Sigma_s$ an SDP?

Let $Z_d^n(x)$ be the vector of monomial bases in dimension n of degree d or less. e.g., if $x \in \mathbb{R}^2$, then

$$Z_2^1(x)^T = \begin{bmatrix} 1 & x_1 & x_2 & x_1 x_2 & x_1^2 & x_2^2 \end{bmatrix}$$

and

$$Z_1^2(x)^T = \begin{bmatrix} 1 & x_1 & x_2 & & & \\ & & & 1 & x_1 & x_2 \end{bmatrix} = \begin{bmatrix} Z_1^1(x) & \\ & Z_1^1(x) \end{bmatrix}$$

Feasibility Test:

Lemma 4.

Suppose M is polynomial of degree $2d$. $M \in \Sigma_s$ iff there exists some $Q \succeq 0$ such that

$$M(x) = Z_d(x)^T Q Z_d(x).$$

Let

$$X := \left\{ x : \begin{matrix} p_i(x) \geq 0 & i = 1, \ldots, k \\ q_j(x) = 0 & j = 1, \ldots, m \end{matrix} \right\}$$

Theorem 5 (Putinar).

Suppose X is "compact+" and $v(x) \geq 1$ for $x \in X$. Then there exist $s_i \in \Sigma_s$ and $t_i \in \mathbb{R}[x]$ such that

$$v(x) - \sum_{i=1}^{k} s_i(x) p_i(x) + \sum_{i=1}^{m} t_i(x) q_i(x) - s_0 = 0$$

Recall that for the Tokamak problem, we have

$$(A\psi)(x) := \frac{1}{\mu_0 a^2}\frac{\partial}{\partial x}\left(\eta_{\parallel}(x)\frac{\partial}{\partial x}(x\psi(x))\right)$$

$$(Bj_{ni})(x) := \frac{\partial}{\partial x}\left(\eta_{\parallel}(x)j_{eni}(x)\right)$$

and $X = L_2[0,1]$ with domain

$$D_A = \{y \in L_2[0,1] : y, y_x, y_{xx} \in L_2[0,1], y(0) = y(1) = 0\}.$$

We parameterize our operator P simply using a multiplier as

$$(Px)(s) = M(s)x(s)$$

An important choice is that of the controller: $K : D_A \to X$

$$(K\psi)(x) = K_1(x)\psi(x) + \frac{d}{dx}(Z_2(x)\psi(x))$$

The structure of K and P imposes a structure on $Z = KP$:

$$(Z\psi)(x) = (KP\psi)(x) = Z_1(x)\psi(x) + \frac{d}{dx}(Z_2(x)\psi(x))$$

First Constraint: Enforcing positivity of P is easy.

$$\langle x, Px\rangle = \int_0^1 x(s)M(s)x(s)ds \geq 0$$

if and only if

$$M(s) \geq 0 \qquad \text{for all} s \in [0,1]$$

Second Constraint: Enforcing negativity of $PA^* + AP + BZ + Z^*B$ can be reformulated as

$$\langle x, (PA^*+AP+BZ+Z^*B)x\rangle = \int_0^1 x(s)R_1(s)x(s)ds + \int_0^1 \dot{x}(s)R_2(s)\dot{x}(s)ds \leq 0$$

where R_1 and R_2 are linear in variables Z_1, Z_2, and M (Next Slide). We require both

$$R_1(s) \leq 0 \qquad \text{and} \qquad R_2(s) \leq 0 \qquad \text{for all } s \in [0,1].$$

Enforcing Positivity

As promised:

$$R_1(s) := \frac{1}{\mu_0 a^2} b_1\left(x, \frac{d}{dx}\right) M(x) + b_2\left(x, \frac{d}{dx}\right) Z_1(x) + b_3\left(x, \frac{d}{dx}\right) Z_2(x)$$

$$R_2(s) := \frac{1}{\mu_0 a^2} c_1(x) M(x) + c_2(x) Z_2(x).$$

where

$$\begin{aligned} b_1\left(x, \frac{d}{dx}\right) &= f(x)\left(\frac{\eta_{\|,x}}{x} - \frac{\eta_{\|}}{x^2}\right) + f'(x)\left(-\frac{\eta_{\|}}{x} + \eta_{\|,x}\right) \\ &\quad + f''(x)\eta_{\|} + \frac{f(x)\eta_{\|}}{x}\frac{d}{dx} + \left(f(x)\eta_{\|} + f(x)\eta_{\|,x}\right)\frac{d^2}{dx^2}, \\ b_2\left(x, \frac{d}{dx}\right) &= -f'(x) + f(x)\frac{d}{dx}, \\ b_3\left(x, \frac{d}{dx}\right) &= \eta_{\|,x} f'(x) + \eta_{\|} f''(x) + \eta_{\|,x} f(x)\frac{d}{dx} + \eta_{\|} f(x)\frac{d^2}{dx^2}, \end{aligned}$$

$c_1(x) = -\eta_{\|} f(x), c_2(x) = -2\eta_{\|} f(x)$ and $f(x) = x^2(1-x)$.

Simulation

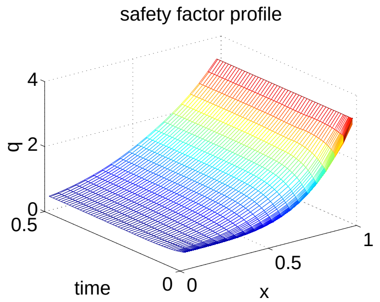


Figure: Time evolution of the safety factor profile or the q-profile.

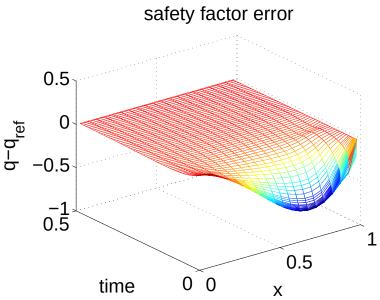


Figure: Time evolution of the q-profile Error, $q(x,t) - q_{ref}(x)$. Here x is the normalized spatial variable.

Simulation

Note: Although not discussed, we also constraint $j_{ni} \leq 3$MA

- We use finite difference methods to obtain the numerical solution of the closed loop system.
- To simulate the controller under realistic scenarios we use the plasma resistivity $\eta_{\|}(x,t)$ data from the **Tore Supra** Tokamak.
- The other data used from the **Tore Supra** Tokamak are:

$$\begin{aligned} I_p(\text{plasma current}) &= 0.6MA \\ B_{\phi_0}(\text{toroidal magnetic field at the plasma center}) &= 1.9T \\ a(\text{Radius of the last closed magnetic surface}) &= .72m \\ R_0(\text{magnetic center location}) &= 2.38m. \end{aligned}$$

- From this data the boundary conditions for $\psi_x(x,t)$ are calculated to be

$$\psi_x(0,t) = 0 \text{ and } \psi_x(1,t) = -0.2851.$$

Simulation

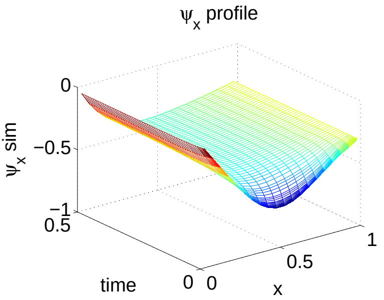


Figure: Time evolution of ψ_x-profile.

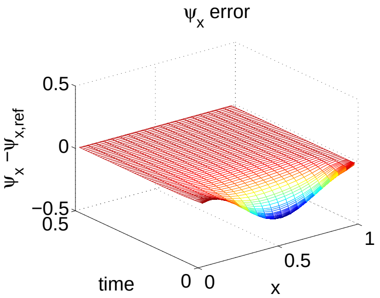


Figure: ψ_x-profile error, $\psi_x - \psi_{x,ref}$. Here $\psi_{x,ref}$ is obtained from the reference q-profile, q_{ref}.

Simulation

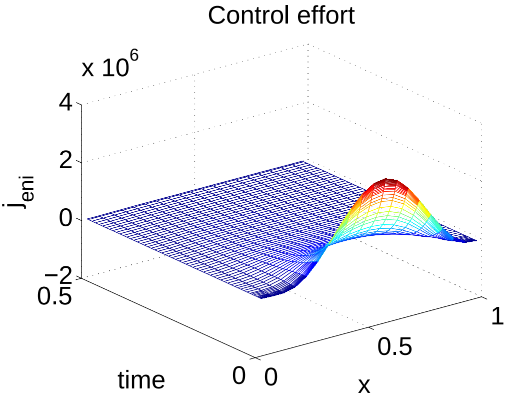


Figure: External non-inductive current deposit, $j_{eni}(x,t)$.

Ongoing Work

The Non-Inductive Source Term

We can improve our model of actuator control

$$R_0 \frac{\partial}{\partial x} \left(\eta_{\|}(x,t) j_{ni}(x,t)\right)$$

The control is via the **Non-Inductive Source Term**, j_{ni}.

- Spatially-distributed
- A sum of Gaussians

$$j_{ni}(x,t) = a_1 e^{\frac{(x-b_1)^2}{c_1}} + a_2 e^{\frac{(x-b_2)^2}{c_2}} + a_3 e^{\frac{(x-b_3)^2}{c_3}}$$

We can parameterize a Gaussian as

$$a_1 e^{\frac{(x-b_1)^2}{c_1}} = a p_a(x) + b p_b(x) + c p_c(x) = \begin{bmatrix} p_a(x) & p_b(x) & p_c(x) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = P(x)^T u$$

We can look for a controller as

$$u = \int_0^1 K(x)\psi_x(x) + \frac{d}{dx}\left(K_2(x)\psi(x)\right) dx$$

Ongoing Work: Observing PDE systems

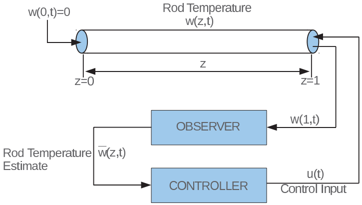
Heat Equation Example

Problem: Feedback requires a knowledge of the heat distribution.

- Sensors can only measure heat at a single point.

Consider the dynamics of heat flux.

$$w_t(x,t) = w_{xx}(x,t)$$



Suppose we only observe at a single point,

$$y(t) = w(1,t)$$

and control the gradient at the same point:

$$w_z(1,t) = u(t)$$

To know the state, we want Luenberger Observer, L:

$$\dot{\hat{z}}(t) = (A + LC + BF)\hat{z}(t) - Ly(y)$$

with both $A + LC$ and $A + BF$ stable. Then $\hat{z}(s,t) \to z(s,t)$.

Ongoing Work: Observer-Based Controller

The Heat Equation

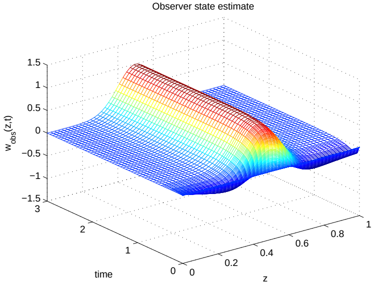


Figure: Estimate of the State

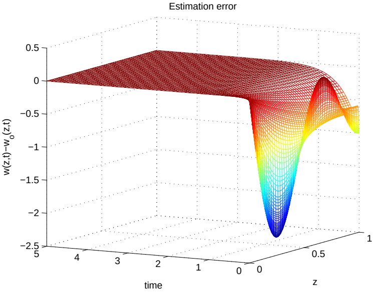


Figure: Error in the Observed State

Ongoing Work: Observer-Based Controller

The Heat Equation

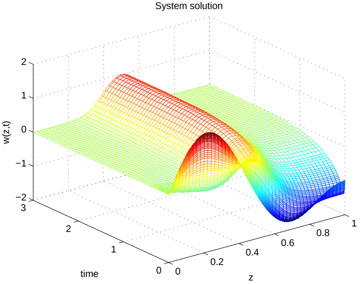


Figure: Effect of Observer-Based Boundary Control

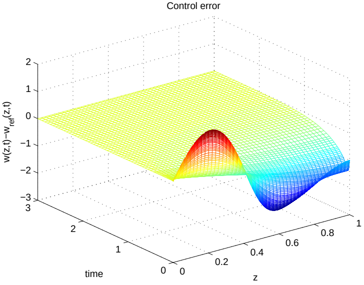


Figure: Error in Observer-Based Boundary Control

Concluding Remarks: Research Directions

Directions:

- Theory of Linear Operator Inequalities
  - Duality
  - Optimal H_∞ Control
- Parallel Algorithms for SOS/Polya
  - GPU Computing
  - Analysis/Synthesis

Other Research:

- Immunology/Cancer
  - Identify Feedback Mechanisms
  - Decentralized Control
- Tokamaks
  - Observers
  - Temperature/Density Coupling
  - RF-Heating

Some algorithms are available for download at:

`http://mmae.iit.edu/~mpeet`

Thanks for Listening