Optimal Thermostat Programming for Peak and Time-of-Use Pricing Plans with Thermal Energy Storage and Implications for Regulated Utilities

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***Abstract*—In this paper, we solve the optimal thermostat programming problem for consumers with combined demand ($/kW) and time-of-use ($/kWh) pricing plans. We account for energy storage in interior floors and surfaces using a partial-differential model of diffusion. We consider 2 types of thermostats: the first can be programmed to vary continuously in time and the second is limited to 4 constant set-points. Thermostat settings were constrained to lie within a desired interval. Numerical testing shows that the resulting algorithm can reduce monthly electricity bills by up to 25% in the summer with average savings of 9.2% over a variety of tested building models and using pricing from Arizona utility Salt River Project (SRP). Furthermore, we examined how optimal thermostat programming can influence electricity pricing by using a highly simplified model of utility generation costs to determine the optimal ratio of demand to time-of-use prices. Our results show that pricing electricity at the marginal cost of generation in this scenario is sub-optimal.**

I. Introduction

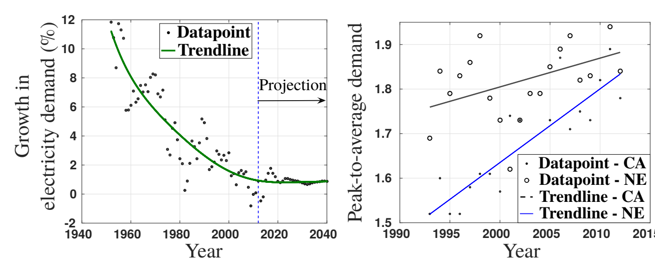
Growth in the US demand for electricity has plateaued [1] and is expected to remain flat (less than 1% growth as shown in Fig. 1.1) for the indefinite future. Flat demand growth coupled with increasing use of solar/wind generation is expected to reduce the amount of carbon-producing fossil fuels used by electrical utility companies in coming years. Specifically, according to the US Climate Action Report [2], the amount of carbon-dioxide emissions from the energy sector in 2020 is expected to drop by 8-12 percent below the 2005 levels.

While flat demand growth and increasing integration of renewables will lead to a reduction in greenhouse gases, these structural changes may also have a negative economic impact on electrical utility companies. For example, as solar generation by users increases, the total energy provided by the utility will decrease - implying a reduction in revenue for utility companies which charge users based on their total energy consumption (Time-Of-Use (TOU) charges). However, because solar generation peaks at ≅12:00 and electricity consumption peaks at around 17:00, the use of solar does not typically change the maximum power provided by the utility over a typical 24 hr period. Because utilities must build and maintain generation and distribution capacity as determined by peak demand, the increasing use of solar will result in a decrease in revenue, but no decrease in expenses tied to generation and distribution capacity. These structural changes in consumption can be seen as an increase in the ratio of demand peak to average demand (see Fig. 1.2) in recent years.

To maintain economic viability, Arizona utilities have recently begun to recoup capacity and distribution costs from commercial and residential consumers [4], [5] through the use of a form of pricing which charges consumers not only based on their total energy consumption ($/kWh) (TOU charges), but also based on the maximum *rate* of consumption ($/kW) (a demand charge). With demand charges, for example, a single transient peak in consumption will significantly increase the demand charge, while not substantially altering the TOU charges. Demand charges are particularly significant for consumers with rooftop solar since solar reduces total electricity consumption (TOU charges), but typically has minimal effect on late-day peaks (approximately at 19:00) which determine the demand charge. Moreover, while demand charges more accurately align the costs of the utility and those of the consumer, there is significant uncertainty in how a consumer can modify usage in order to minimize such charges.

In this paper, we focus on the use of energy storage to minimize the cost of electricity for consumers with both TOU and demand charges. For example, if we imagine a consumer with perfect knowledge of load and a sufficiently large battery with no restrictions on charging rate, then by charging when load is above average and discharging when load is below average, one could achieve a demand charge precisely equal to the average demand - an ideal scenario. Indeed, several papers have recently studied the problem of optimal use of battery storage in a TOU scenario, often including a portfolio of appliances to dispatch [6], [7]. The disadvantage of this approach, however, is that it requires a significant capital investment from the consumer. While Li-Ion battery prices have decreased in recent years, the cost of purchase and installation of these devices significantly reduces the savings gained from optimal scheduling.

An energy storage alternative to batteries which requires no capital expense is the use of thermostat programming to store thermal energy in interior surfaces such as floors and walls - a strategy which has been validated both experimentally [8], [9] and in-silico [10], [11],[12]. In this approach interior mass acts as a *passive* thermal energy storage system where charging occurs by lowering interior temperatures when loads are light



1.1: Percent of growth of electricity demand and its trend-line in the US [1]

1.2: Peak-to-average demand in California and New-England [3]

Fig. 1. Demand growth and peak-to-average demand of electricity in the US

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or electricity is cheap - thereby gradually lowering the internal temperature of the floors and walls through diffusion. Then, later in the day, when demand peaks, one allows the interior temperature to rise. However, because of the latency of thermal diffusion, the thermal mass will continue to absorb heat for some time - thereby reducing load on the HVAC system. Thermal storage strategies such as this have been studied for some time, the most well-known of which is *pre-cooling* [13], [14]. Note, however, that ad-hoc strategies such as pre-cooling may not significantly affect demand charges. This is because if the building has a small thermal mass to size ratio, under a pre-cooling strategy, the thermal energy stored in the structure will have depleted by the end of day when demand peaks typically occur.

In this paper, we use a partial-differential equation (PDE) model of thermal diffusion to create an algorithm which determines the thermostat settings which minimize the electricity bill for a consumer with both TOU and demand charges. These settings are based on: a range of acceptable interior temperatures (comfort zone); prediction data for exterior temperature; and estimates of building's thermal properties. We pose the optimal thermostat programming for HVAC as a constrained dynamic optimization problem and present a Dynamic Programming (DP) algorithm which is guaranteed to converge to the solution. This yields temperature set-points which minimize the monthly electricity bill consisting of on-peak, off-peak and demand costs to the residential customer.

Note that this result is unique in three ways. First, it uses a PDE model to accurately capture the latency of the thermal diffusion process - as opposed to simplified battery models (linear ordinary differential equations) [12], [15]. Modeling the latency of thermal diffusion is significant in that energy which is stored deep inside the mass must diffuse to the surface before it is available for use in reducing electricity consumption. Several efforts have been made in recent years to capture this effect, including the use of electric circuit models and the concept of *deep* and *shallow* mass in [13] and the use of a PDE model in [16]. Note, however, that the latter work did not consider the problem of optimizing demand charges.

The second unique contribution of the paper is that it combines both demand charges and TOU pricing - as opposed to thermostat programming with only demand charges [13] or only TOU prices [17], [18] or only *real-time* pricing (prices which are constantly changing) [19], [16], [15]. Indeed, it seems that the combination of TOU and demand charges does not appear in the literature, while practiced by Arizona's utility companies. The third unique contribution of the paper is to consider the problem of programming a 4 set-point thermostat[1]. That is, for the benefit of the consumers who do not have access to continuously adjustable thermostats, we consider the case of optimization of four thermostat programming periods, wherein the algorithm determines the 4 start/stop times and 4 constant settings corresponding to the intervals between those times. This problem does not appear in existing literature.

[1]This paper is an extension to the conference paper [20]. The new content includes:1) Designing four-setpoint optimal thermostat programs; 2) Evaluating the benefits of our optimal thermostat across a wide range (147 scenarios) of building types; 3) Quantifying the effects of solar integration on electricity bills; 4) Discussing the implications of optimal behavior of consumers for regulated utilities; 5) An experimental validation for our thermodynamics model for the building's thermal mass.

To better understand the potential impact of optimal thermostat programming, we have applied our results to a number of cases and scenarios. First, in Subsection IV-A, we applied our algorithm to 143 combinations of building parameters and loads to find an average of 9.2% reduction in monthly electricity bills. Next, in Subsection IV-B, we used the optimal thermostat program as a model for the behavior of a rational consumer in response to changes in on-peak, off-peak, and demand prices. This allowed us to study the question of whether basing the demand charge on the marginal cost of adding generation and distribution capacity is suboptimal. For this problem, we used a simplified model of generation, distribution, and capacity costs for a vertically integrated regulated utility based on estimates provided by Arizona utility SRP (note that conclusions drawn from this model of costs do not apply to utilities in the ISO framework). Using a Nelder-Mead simplex algorithm, we were able to demonstrate that such pricing proportional to marginal costs can be suboptimal and we determined prices which can reduce overall generation and distribution costs while maintaining the acceptable range of comfort for all consumers. Finally, in Subsection IV-C, we studied the question of whether high levels of solar penetration significantly affects the price of electricity for non-solar consumers in a scenario where all consumers pay demand charges and all thermostats are optimized. By using a model of a regulated utility, wherein prices are scaled so that revenue equals a fixed fraction of the generation costs - and considering a mixed consumer base of 50% solar consumers and 50% non-solar consumers (all using optimal thermostat programs) - we found that the presence of solar consumers increases the bills of non-solar consumers, but that this increase is less than 2%.

II. PROBLEM STATEMENT

In this section, we first define a model of the thermodynamics which govern heating and cooling of the interior structures of a building. We then use this model to pose our optimal thermostat programming (*consumer-level*) problem in Subsections II-C and II-D as minimization of a monthly electricity bill (with on/peak, off-peak and demand charges) subject to constraints on the interior temperature of the building. In Subsection II-E, we use our optimal thermostat program as a model of a rational consumer to explore the consumer's response to the prices. In particular, we pose our *utility-level* problem as optimization over on/peak, off-peak and demand prices for minimizing a simplified model of generation, distribution and capacity costs - assuming that the consumers respond optimally to the prices.

A. A model for the building thermodynamics

In 1822, J. Fourier proposed a PDE to model the dynamics of temperature and energy in a solid mass. Now known as the classical one-dimensional unsteady heat conduction equation, this PDE can be applied to an interior wall as

$$T_t(t,x) = \alpha T_{xx}(t,x), \tag{1}$$

where $T : \mathbb{R}^+ \times [0, L_{in}] \to \mathbb{R}$ represents the temperature distribution in the interior walls/floor with nominal width L_{in}, and

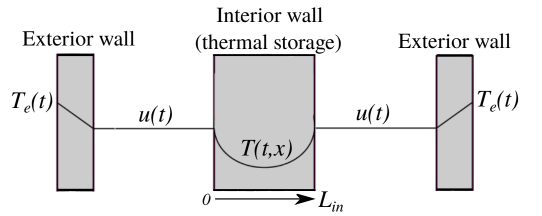


Fig. 2. A schematic view of our thermal mass model

where $\alpha = k_{in}/(\rho C_p)$ is the coefficient of thermal diffusivity. Here k_{in} is the coefficient of thermal conductivity, ρ is the density and C_p is the specific heat capacity. The wall is coupled to the interior air temperature using Dirichlet boundary conditions, i.e., $T(t,0) = T(t,L_{in}) = u(t)$ for all $t \in \mathbb{R}^+$, where $u(t)$ represents the interior temperature which we assume can be controlled instantaneously by the thermostat. In the Fourier model, the heat/energy flux through the surface of the interior walls is modelled as

$$q_{in}(T(t,x)) := 2C_{in}\frac{\partial T}{\partial x}(t,0), \tag{2}$$

where $C_{in} = k_{in}A_{in}$ is the thermal capacitance of the interior walls and A_{in} is the nominal area of the interior walls. We assume that all energy storage occurs in the interior walls and surfaces and that energy transport through exterior walls can be modelled using a steady-state version of the heat equation. This implies that the heat flux q_{loss} through the exterior walls is the linear sink

$$q_{loss}(t,u(t)) := \frac{T_e(t)-u(t)}{R_e}, \tag{3}$$

where $T_e(t)$ is the outside temperature and $R_e = L_e/(k_eA_e)$ is the thermal resistance of the exterior walls, where L_e is the nominal width of exterior walls, k_e is the coefficient of thermal conductivity and A_e is the nominal area of the exterior walls. By conservation of energy, the power required from the HVAC to maintain the interior air temperature is

$$q(t,u(t),T(t,x)) = q_{loss}(u(t),T_e(t)) + q_{in}(T(x,t)). \tag{4}$$

See Fig. 2 for an illustration of the model.

Eqn. (1) is a PDE. For optimization purposes, we discretize (1) in space, using $T(t) \in \mathbb{R}^M$ to replace $T(t,x) \in \mathbb{R}$, where $T_i(t)$ denotes $T(t,i\Delta x)$, where $\Delta x := \frac{L_{in}}{M+1}$. Then

$$\dot{T}(t) = AT(t) + Bu(t), \tag{5}$$

$$\text{where } A = \frac{\alpha}{\Delta x^2}\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}, \; B = \frac{\alpha}{\Delta x^2}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^M.$$

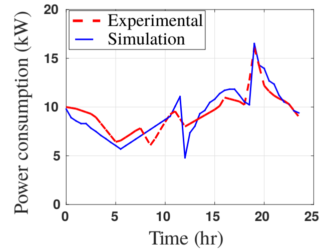
We then discretize in time, using $\dot{T}(t) \approx (T(t+\Delta t) - T(t))/\Delta t$ to rewrite Equation (5) as a difference equation.

$$T^{k+1} = \begin{bmatrix} T_1^{k+1} \\ \vdots \\ T_M^{k+1} \end{bmatrix} = f(T^k,u_k) = \begin{bmatrix} f_1(T^k,u_k) \\ \vdots \\ f_M(T^k,u_k) \end{bmatrix} = (I + A\Delta t)T^k + B\Delta t\, u_k \tag{6}$$

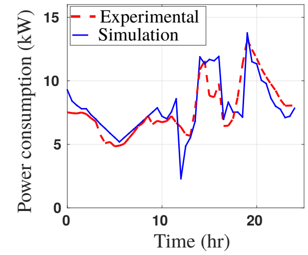
for $k = 0,\cdots,N_f-1$, where $T^k = T(k\Delta t)$ and $u_k = u(k\Delta t)$.

B. Calibrating the thermodynamics model

To find empirical values for the parameters α, C_{in}, R_e and L_{in} in the thermodynamic model in Section II-A, we collected data from a 4600 sq ft residential building in Scottsdale, Arizona. The building was equipped with a 5 ton two-stage and three 2.5 ton single-stage RHEEM/RUUD heat pumps, 4-setpoint thermostats, and 5-min data metering for energy consumption and interior and exterior temperature. In this experiment, we applied two different thermostat programming sequences for two non-consecutive summer days. On the first day, we applied a pre-cooling strategy which lowers the interior temperature to $23.9°C$ during the off-peak hours and allows the temperature to increase to $27.8°C$ during the on-peak hours 12:00 PM to 7:00 PM. In the second day, we applied the same pre-cooling strategy except that the temperature is again lowered to $23.9°C$ between 2:00 PM and 4:00 PM. We then used Matlab's least squares optimization algorithm to optimize the parameters such that the root-mean-squared error between the measured power consumption and the simulated power consumption (using Eqn. (4)) during the entire two days is minimized. The result was the following parameter values: $L_{in} = 0.4(m)$, $\alpha = 8.3 \times 10^{-7}(m^2/s)$, $R_e = 0.0015(K/W)$, $C_{in} = 45(Wm/K)$. In Fig. 3, we have compared the resulting simulated and measured consumed power for the entire two days.



3.1: Power consumption corresponding to a pre-cooling strategy for the interior temperature setting



3.2: Power consumption corresp. to a pre-cooling strategy with additional cooling from 14:00-16:00

Fig. 3. Simulated and measured power consumptions

C. Consumer-level problem I: optimal thermostat program

In this section, we define the problem of optimal thermostat programming for residential consumers. We first divide each day into three periods: off-peak hours from 12 AM to t_{on} with electricity price $p_{\text{off}}\,(\$/kWh)$; on-peak hours beginning at t_{on} and ending at $t_{\text{off}} > t_{\text{on}}$ with electricity price $p_{\text{on}}\,(\$/kWh)$; and off-peak hours from t_{off} to 12 AM with electricity price $p_{\text{off}}\,(\$/kWh)$. In addition to the on-peak and off-peak charges, we consider a monthly charge which is proportional to the maximum rate of consumption during the peak hours. The proportionality constant is called the *demand price* $p_d\,(\$/kW)$. Given $p := [p_{\text{on}}, p_{\text{off}}, p_d]$, the total cost of consumption (daily electricity bill) is divided as

$$J_t(\mathbf{u},T_1,p) = J_e(\mathbf{u},T_1,p_{\text{on}},p_{\text{off}}) + J_d(\mathbf{u},T_1,p_d), \tag{7}$$

where J_e is the energy cost, J_d is the demand cost and

$$\mathbf{u} := [u_0,\cdots,u_{N_f-1}] \in \mathbb{R}^{N_f}$$

is the vector temperature settings. The energy cost is

$$J_e(\mathbf{u},T_1,p_{\text{on}},p_{\text{off}}) = \Big(p_{\text{off}}\sum_{k\in S_{\text{off}}} g(k,u_k,T_1^k) + p_{\text{on}}\sum_{k\in S_{\text{on}}} g(k,u_k,T_1^k)\Big)\Delta t,$$

where $k \in S_{\text{on}}$ if $k\Delta t \in [t_{\text{on}}, t_{\text{off}}]$ and $k \in S_{\text{off}}$ otherwise. That is, S_{on} and S_{off} correspond to the set of on-peak and off-peak sampling times, respectively. The function g is a discretized version of q (in Eqn. (4)), i.e., g is the power consumed by the HVAC at time-step k:

$$g(k,u_k,T_1^k) := \frac{T_e^k - u_k}{R_e} + 2C_{in}\frac{T_1^k - u_k}{\Delta x}, \tag{8}$$

where T_e^k denotes the external temperature at time-step k. If demand charges are calculated monthly, the demand cost, J_d, for a single day can be considered as

$$J_d(\mathbf{u}, T_1, p_d) := \frac{p_d}{30} \max_{k \in S_{\text{on}}} g(k, u_k, T_1^k). \quad (9)$$

We now define the optimal thermostat programming (consumer-level) problem as minimization of the total cost of consumption, $J_e + J_d$, as defined in (7), subject to the building thermodynamics in (6) and interior temperature constraints:

$$J^\star(p) = \min_{u_k, \gamma \in \mathbb{R}, T^k \in \mathbb{R}^M} J_e(\mathbf{u}, T_1, p_{\text{on}}, p_{\text{off}}) + \frac{p_d}{30}\gamma$$
$$\begin{aligned} \text{subject to } & g(k, u_k, T_1^k) \le \gamma && \text{for } k \in S_{\text{on}} \\ & T^{k+1} = f(T^k, u_k) && \text{for } k \in S_{\text{on}} \cup S_{\text{off}} \\ & T_{\min} \le u_k \le T_{\max} && \text{for } k \in S_{\text{on}} \cup S_{\text{off}} \\ & T^0 = [T_{\text{init}}(\Delta x), \cdots, T_{\text{init}}(M\Delta x)]^T, && (10) \end{aligned}$$

where $T_{\min}, T_{\max}$ are the acceptable bounds on int. temperature.

D. Consumer-level problem II: 4-setpoint thermostat program

Many commercially available programmable thermostats only include four programming periods per-day, each period possessing a constant temperature. In this section, we account for this constraint. First, we partition the day into programming periods: $P_i := [t_{i-1}, t_i]$, $i = 1, \cdots, 4$ such that

$$\bigcup_{i=1}^{4} P_i = [0, 24], \quad t_{i-1} \le t_i, \quad t_0 = 0 \text{ and } t_4 = 24.$$

We call $t_0, \cdots, t_4$ *switching* times. Similar to the previous model, $u_i \in [T_{\min}, T_{\max}]$ denotes the temperature setting corresponding to the programming period P_i.

To simplify the mathematical formulation of our problem, we introduce some additional notation. Define the set S_i by $k \in S_i$ if $k\Delta t \in P_i$. Denote $L_i := \max_{k \in S_i} k$. For clarity, we have depicted L_i in Fig. 4. Moreover, for each P_i, we define $\overline{\Delta t}_i$ as the period between the last time-step of P_i and the end of P_i, i.e., $\overline{\Delta t}_i := t_i - L_i\Delta t$. See Fig. 4 for an illustration of $\overline{\Delta t}_i$. In this framework, the daily consumption charge is

$$I_t(\mathbf{u}, T_1, p) = I_e(\mathbf{u}, T_1, p_{\text{on}}, p_{\text{off}}) + I_d(\mathbf{u}, T_1, p_d),$$

where I_e is the energy cost

$$I_e(\mathbf{u}, T_1, p_{\text{on}}, p_{\text{off}}) =$$
$$\sum_{i=1}^{4} \left(\sum_{k \in S_i} \left(r(k) g(k, u_i, T_1^k)\Delta t \right) + r(L_i) g(k, u_i, T_1^{L_i})\overline{\Delta t}_i \right), \quad (11)$$

and I_d is the demand cost $I_d(\mathbf{u}, T_1, p_d) = \max_{k \in S_{\text{on}}} g(k, u, T_1^k)$, where $L_0 = 0$ and r is defined as

$$r(k) := \begin{cases} p_{\text{on}} & t_{\text{off}} \le k\Delta t < t_{\text{on}} \\ p_{\text{off}} & \text{otherwise.} \end{cases}$$

Assuming that the demand cost for a single day is $\frac{p_d}{30}\max_{k \in S_{\text{on}}} g(k, u, T_1^k)$, we define the 4-setpoint thermostat programming problem as

$$\min_{\substack{u_1, \cdots, u_4 \in \mathbb{R} \\ t_1, t_2, t_3, \gamma \in \mathbb{R}, T^k \in \mathbb{R}^M}} I_e(\mathbf{u}, T_1, p_{\text{on}}, p_{\text{off}}) + \frac{p_d}{30}\gamma \qquad \text{subject to}$$
$$\begin{aligned} & g(k, u_i, T_1^k) \le \gamma && \text{for } k \in S_{\text{on}},\ i \in \{1,2,3,4\} \\ & T^{k+1} = f(T^k, u_i) && \text{for } k \in S_i \text{ and } i \in \{1,2,3,4\} \\ & T_{\min} \le u_i \le T_{\max} && \text{for } i \in \{1,2,3,4\} \\ & 0 \le t_{i-1} \le t_i \le 24 && \text{for } i \in \{1,2,3,4\} \\ & T^0 = [T_{\text{init}}(\Delta x), \cdots, T_{\text{init}}(M\Delta x)]^T, && (12) \end{aligned}$$

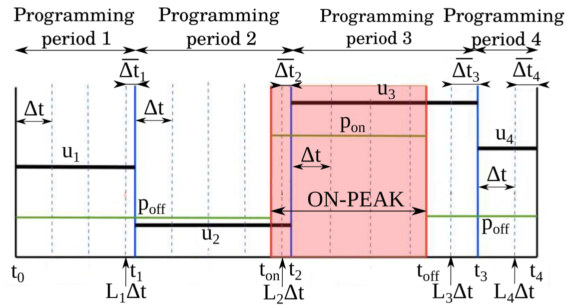


Fig. 4. An illustration for the programming periods of the 4-Setpoint thermostat problem, switching times t_i, pricing function r, L_i and $\overline{\Delta t}_i$.

where $t_0 = 0$ and $t_4 = 24$.

E. A simplified utility-level optimization problem

In this section, we use our consumer's model in Section II-C to defined a simplified model which relates TOU and prices to the costs to regulated utilities. This model will be used in section IV to explore the potential implications of combined TOU and demand pricing for the costs to regulated utilities. Regulated utilities must meet expected load while maintaining a balance between revenue and costs. Therefore, we define a simple utility optimization problem as minimization of the total cost of generation, transmission and distribution of electricity such that generation is equal to consumption, and the total cost is a fixed percentage of the revenue of the utility company. Note that in this paper, we solely focus on vertically-integrated utility companies - meaning that the company provides all aspects of electric services including generation, transmission, distribution, metering and billing services as a single firm. Therefore, we preclude the role of local distributors and market competition. Let $s(t)$ be the amount of electricity produced as a function of time and let $\mathbf{s} := [s_0, \cdots, s_{N_f-1}]$, where $s_k = s(k\Delta t)$. The vector $\mathbf{s}$ is determined by the electricity consumed by the consumers, which we model as a small number of consumer groups which are lumped according to different building models, temperature limits, and solar generating capacity so that aggregate consumer group i has N_i members. Next, we define $u_k^{\star,i}(p)$ to be the minimizing temperature setting for consumer group i at time k with prices p, and $T_j^{i,\star,k}(p)$ to be the minimizing interior wall temperatures for consumer group i at time k and discretization point j for prices p. The minimization is with respect to the consumer-level problem defined in (10). Then the model of electricity consumption by the rational consumer group i at time-step k for prices p is given by $g(k, u_k^{\star,i}(p), T_1^{\star,k,i}(p))$, g defined in (8). Thus the constraint that production equals consumption at all time implies

$$s_k = \sum_i N_i g(k, u_k^{\star,i}(p), T_1^{i,\star,k}(p)) \text{ for all } k = 0, \cdots, N_f - 1. \quad (13)$$

Now, since utility's revenue equals the amount paid by the consumers, the model for revenue from rational consumer i becomes $J_t(u^{\star,i}(p), T_1^{i,\star}(p), p)$, where J_t is defined in (7). We may now define the utility-level optimization problem as minimization of the total cost subject to equality of generation and consumption and proportionality of revenue and costs.

$$\begin{aligned} \min_{p_{\text{on}}, p_{\text{off}}, p_d \in \mathbb{R}} \quad & C(\mathbf{s}) \\ \text{s.t.} \quad & s_k = \sum_i N_i g(k, u_k^{\star,i}(p), T_1^{i,\star,k}(p)) \quad k = 0, \cdots, N_f - 1 \\ & C(\mathbf{s}) = \lambda \sum_i N_i J_t(u^{\star,i}(p), T_1^{i,\star}(p), p), \quad (14) \end{aligned}$$

where $\lambda \leq 1$ is usually determined by the company's assets, accumulated depreciation and allowed rate of return. We refer to the minimizers $p_{\text{on}}^{\star}, p_{\text{off}}^{\star}, p_d^{\star}$ which solve Problem (14) as *optimal electricity prices*.

a) Model of total cost, $C(\mathbf{s})$, to utility company: The algorithm defined in the following section was chosen so that only a black-box model of utility costs is required. However, for the case studies included in Section IV, we use two simplified models of utility costs based on ongoing discussions and collaboration with Arizona utility company SRP. In the first model, we consider a linear representation of both fuel and capacity costs.

$$C(\mathbf{s}) := a \sum_{k \in S_{\text{on}} \cup S_{\text{off}}} s_k \Delta t + b \max_{k \in S_{\text{on}}} s_k, \tag{15}$$

where $a\,(\$/kWh)$ is the marginal cost of producing the next kWh of energy and $b\,(\$/kW)$ is the marginal cost of installing and maintaining the next kW of capacity. Estimated values of the coefficients a and b for SRP can be found in [21] as $a = 0.0814\$/kWh$ and $b = 59.76\$/kW$. According to [21], these marginal costs include fuel, building, operation and maintenance of facilities, transmission and distribution costs. The advantage of this model is that the solution to the utility optimization problem does not depend on the number of consumers, but rather the fraction of consumers in each group.

Our second model for utility costs includes a quadratic term to represent fuel costs. The quadratic term reflects the increasing fuel costs associated with the required use of older, less-efficient generators when demand increases.

$$C(\mathbf{s}) := \tau \Big(\sum_{k \in S_{\text{on}} \cup S_{\text{off}}} s_k \Delta t \Big)^2 + \nu \sum_{k \in S_{\text{on}} \cup S_{\text{off}}} s_k \Delta t + b \max_{k \in S_{\text{on}}} s_k \tag{16}$$

This model was calibrated using artificially modified fuel, operation and maintenance data provided by SRP, yielding estimated τ =0.00401 \$/(MWh)^2 and ν =4.54351 \$/(MWh).

III. Solving Consumer- & Utility-level Problems

First, we solve the optimal thermostat programming problem using a variant of dynamic programming. This yields consumption as a function of prices $p_{\text{on}}, p_{\text{off}}, p_d$. Next, we embed this implicit function in the Nelder-Mead simplex algorithm in order to find prices which minimize the generation cost in the utility-level optimization problem as formulated in (14). We start the consumer-level problem by fixing the variable $\gamma \in \mathbb{R}^+$ and defining a *cost-to-go* function, V_k. At the final time $N_f \Delta t = 24$, we have

$$V_{N_f}(x) := p_d/30 \cdot \gamma. \tag{17}$$

Here for simplicity, we use $x = T \in \mathbb{R}^M$ to represent the discretized temperature distribution in the wall. We define the dilated vector of prices by $p_j = p_{\text{off}}$ if $j \in S_{\text{off}}$ and $p_j = p_{\text{on}}$ otherwise. Then, we construct the cost-to-go function inductively as

$$V_{j-1}(x) := \min_{u \in W_{\gamma,j-1}(x)} \big(p_{j-1}\, g(j-1,u,x_1)\Delta t + V_j(f(x,u)) \big) \tag{18}$$

for $j = 1, \cdots, N_f$, where $W_{\gamma,j}(x)$ is the set of allowable inputs (interior air temperatures) at time j and state x:

$$W_{\gamma,j}(x) := \begin{cases} \{u \in \mathbb{R} : T_{\min} \leq u \leq T_{\max}, g(j,u,x_1) \leq \gamma\}, & j \in S_{\text{on}} \\ \{u \in \mathbb{R} : T_{\min} \leq u \leq T_{\max}\}, & j \in S_{\text{off}}. \end{cases}$$

Now we present the main result.

Theorem 1. *Given $\gamma \in \mathbb{R}^+$, suppose that V_i satisfies* (17) *and* (18)*. Then $V_0(T^0) = J^\star(p)$, where*

$$\begin{aligned} J^\star(p) = &\min_{u \in \mathbb{R}^{N_f}, T^k \in \mathbb{R}^M} J_e(u, T_1, p_{on}, p_{off}) + \frac{p_d}{30}\gamma \\ \text{subject to } & g(k, u_k, T_1^k) \leq \gamma && \text{for } k \in S_{on} \\ & T^{k+1} = f(T^k, u_k) && \text{for } k \in S_{on} \cup S_{off} \\ & T_{\min} \leq u_k \leq T_{\max} && \text{for } k \in S_{on} \cup S_{off} \\ & T^0 = [T_{init}(\Delta x), \cdots, T_{init}(M\Delta x)]^T. \end{aligned} \tag{19}$$

Proof. The proof has been omitted but can be found in conference form in [20]. □

The optimal temperature set-points for Problem (19) can be found as the sequence of minimizing arguments in the value function (18). However, this is not a solution to the original consumer-level optimization problem in (10), as the solution only applies for a fixed consumption bound, γ. However, as this consumption bound is scalar, we may apply a bisection on γ to solve the original optimization problem as formulated in (10). Details are presented in Algorithm 1. The computational complexity of this algorithm is proportional to $N_f \cdot n_s^M \cdot n_u$, where N_f is the number of discretization points in time, M is the state-space dimension of the discretized system in (6), n_s is the number of possible discrete values for each state, T and n_u is the number of possible discrete values for the control input (interior air temperature). In all of the case studies in Section IV, we use $N_f = 73, M = 3, n_s = n_u = 13$. The execution time of our Matlab implementation of Algorithm 1 for solving the three-day consumer-level problem on a Core i7 processor with 8 GB of RAM was $<$ 4.5 minutes.

Finding a solution to the 4-Setpoint thermostat programming problem (12) is significantly more difficult due to the presence of the switching times t_1, t_2, t_3 as decision variables. However, for this specific problem, a simple approach is to use Algorithm 1 as an inner loop for fixed t_i and then use a Monte Carlo search over t_i. For fixed t_i, our Matlab implementation for Algorithm 1 solves the 4-Setpoint thermostat programming problem in less than 17 seconds on a Core i7 processor with 8 GB of RAM. Our experiments on the same machine show that the total execution time for a Monte Carlo search over 300 valid (i.e., $t_i \leq t_{i+1}$) random combinations of t_1, t_2, t_3 is less than 1.41 hours.

To solve the utility-level problem in (14), we used Algorithm 1 as an inner loop for the Nelder-Mead simplex algorithm [22]. The Nelder-Mead simplex algorithm is a heuristic optimization algorithm which is typically applied to problems where the derivatives of the objective function and/or constraint functions are unknown. Each iteration is defined by a reflection step and possibly a contraction or expansion step. The reflection begins by evaluation of the inner loop (Algorithm 1) at each of 4 vertices of a polytope. The polytope is then reflected about the hyperplane defined by the vertices with the best three objective values. The polytope is then either dilated or contracted depending on the objective value of the new vertex. In all of our case studies in Section IV, this hybrid algorithm achieved an error convergence of $< 10^{-4}$ in less than 15 iterations. Using a Core i7 machine with 8 GB of

RAM, the execution time of the hybrid algorithm for solving the utility-level problem was less than 2.25 hours.

Algorithm 1: A bisection/dynamic programming algorithm for optimal thermostat programming

Inputs: p_{on}, p_{off}, p_{d}, T_e, t_{on},t_{off}, R_e, C_{in}, T_{init}, Δt, Δx, $T_{\min}$,$T_{\max}$, maximum number of bisection iterations $b_{\max}$, lower bound γ_l and upper bound γ_u for bisection search.
Main loop:
Set $k = 0$.
while $k < b_{\max}$ **do**
  Set $\gamma = \frac{\gamma_u+\gamma_l}{2}$.
  if V_0 *in* (18) *exists* **then**
    Calculate $u_0, \cdots, u_{N_f-1}$ as the minimizers of the RHS of (18) using a policy iteration technique.
    Set $\gamma_u = \gamma$. Set $u_i^\star = u_i$ for $i = 0, \cdots, N_{f-1}$.
  else
    Set $\gamma_l = \gamma$.
  Set $k = k + 1$.

Outputs: Optimal interior temperature setting: $u_0^\star, \cdots, u_{N_f-1}^\star$.

IV. Numerical Testing and Analysis

In this section, we evaluate and apply the algorithms in Section III. In Case I, we compare our optimal thermostat program with other HVAC programming strategies to determine the average reduction in consumer electricity bills using SRP rates. In Case II, we apply the Nelder-Mead simplex algorithm to the utility-level problem in (14) to determine optimal electricity prices using a simplified model of generation costs. In Case III, we study the effect of the presence of solar consumers on optimal electricity prices using a simplified model of generation costs.

In all cases, the algorithm was run for three consecutive days with demand charge prorated from a one month billing cycle. We used a time-step of $\Delta t = 1\, hr$, spatial-step $\Delta x = 0.1\, m$ and unless otherwise indicated, we used building parameters as listed in Section II-B. These parameters were determined using the model calibration procedure described in Section II-B. We used an external temperature profile measured for three typical summer days in Phoenix, Arizona (see Fig. 5). For each day, the on-peak period starts at 12 PM and ends at 7 PM. We used min and max interior temperatures as $T_{\min} = 22°C$ and $T_{\max} = 28°C$.

A. Case I: Effect of Optimal Thermostat Programming on Electricity Bills

In this case, we first applied Algorithm 1 to the continuous and 4-Setpoint thermostat programming problems (See (10) and (12)) for a non-solar consumer using electricity prices for Arizona utility APS [5] ($p_{\text{off}} = 0.044\frac{\$}{kWh}$, $p_{\text{on}} = 0.089\frac{\$}{kWh}$ and $p_d = 13.50\frac{\$}{kW}$). The resulting electricity bills are given in Table I as the total cost paid for three days with demand charge prorated from a one month billing cycle with the external temperature profile shown in Fig. 5. That is, we reduce the period as defined in Problems (10) and (12) to three days and reduce the demand price by a factor of $\frac{1}{10}$ to $\frac{1}{10}p_d = 1.35\frac{\$}{kW}$. A period of 3 days is used to simplify the presentation of results. To obtain the expected *monthly* savings, all costs and savings should be multiplied by a factor of 10. For comparison, we include a solution obtained using the general-purpose optimization solver GPOPS [23] and also

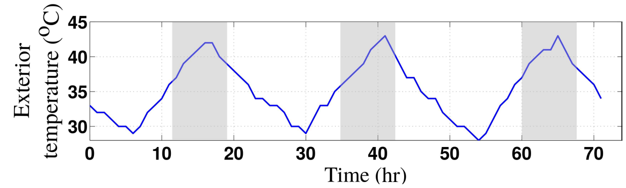


Fig. 5. External temperature of three typical summer days in Phoenix, Arizona. Shaded areas correspond to on-peak hours.

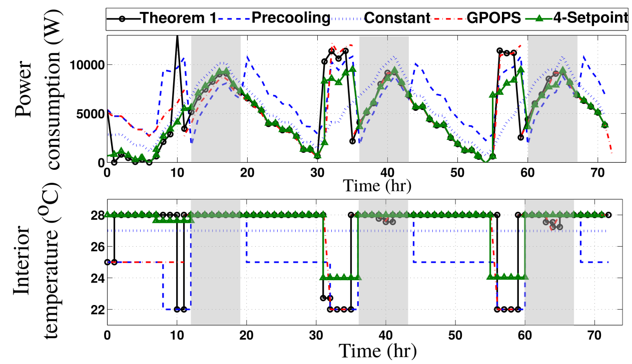


Fig. 6. CASE I: Comparison of power consumption and temperature settings for optimal (using APS rates) and heuristic strategies.

compare the results to a naive strategy of setting the thermostat to $T_{\max}$ at all times. We also compare with a simple pre-cooling strategy with thermostat setting: $u = 25°C$ from 12 AM to 8 AM; $u = T_{\min} = 22°C$ from 8 AM to 12 PM; $u = T_{\max} = 28°C$ from 12 PM to 8 PM; $u = 25°C$ from 8 PM to 12 AM. As can be seen from Table I, our algorithm offers significant improvement over pre-cooling and constant strategies (more than \$28 saving per month). The power consumption and the temperature setting as a function of time for each strategy can be found in Fig. 6. For convenience, the on-peak and off-peak intervals are indicated on the figure.

To more rigorously evaluate the algorithm across a range of building types, we applied Algorithm 1 to 147 scenarios using a wide range of building parameters and several different levels of solar generation. The range of parameter values can be found in Table II. Prices are SRP summer peak prices [4]: $p_{\text{off}} = 0.0423\frac{\$}{kWh}$, $p_{\text{on}} = 0.0633\frac{\$}{kWh}$ and $p_d = 17.82\frac{\$}{kW}$. The results show that the algorithm can reduce the monthly bill by up to 25% (corresponding to the case where the building has the largest int. thermal mass) as compared to the constant $28°C$ strategy. On average, the algorithm reduces the bill by 9.2%.

To examine the impact of changes in the demand charge on peak demand, we compared three pricing policies corresponding to high, medium and low demand charges. Again, in each case, the results (optimal and 4-setpoint) are compared to GPOPS and a pre-cooling strategy. The results are summarized in Table III. For each pricing strategy, the smallest utility cost and associated demand peak are listed in bold. The power consumption and the temperature settings as a function of time for the optimal and 4-Setpoint strategies can be found in Figs. 8 and 7. The results show that by increasing the demand charge, the demand peak can be reduced by at least 29% with respect to the naive strategy. However this policy may be suboptimal at reducing costs to the utility.

B. Case II: optimal thermostat programming with optimized electricity prices

In this case, we consider the quadratic model of the fuel cost provided by SRP and defined in Section II-E. A typical pricing

TABLE I
CASE I: COMPARISON OF ELECTRICITY BILLS (**FOR THREE DAYS ONLY**) USING APS PRICES (MULTIPLY BY 10X FOR THE MONTHLY BILL).

Temperature setting	Electricity bill ($)	Demand peak (kW)
Optimal (Theorem 1)	**36.58**	9.222
GPOPS [23]	37.03	9.155
4-Setpoint (Theorem 1)	37.71	9.401
Pre-cooling	39.23	**8.803**
Constant	39.42	10.462

TABLE II
CASE I: RANGE OF BUILDING PARAMETERS AND SOLAR GENERATION. PARAMETERS ARE DEFINED IN SECTION II-A.

Parameter	Interval Range	Parameter	Interval Range
$L_e(m)$	[0.2, 0.7]	$k_e(W/mK)$	[1.75, 4.5]
$A_{in}(m^2)$	[40, 200]	$\rho(kg/m^3)$	[300, 2000]
$A_e(m)$	[50, 150]	$C_p(J/kg°C)$	[500, 2300]
$k_{in}(W/mK)$	[0.1, 1]	Peak solar (kW)	[0, 10.2]

strategy for SRP and other utilities is to set prices proportional to marginal generation costs. SRP estimates the mean marginal fuel cost at $a = 0.0814\$/kWh$ (See (15)). Linearizing the quadratic model of fuel cost and equating to this estimate of the marginal cost yields an estimate of the mean load. Dividing this mean load by the aggregate consumer defined in Case I yields an estimate of the mean number of consumers of this class at $N = 24,405$.

To compare the marginal pricing strategy with the optimal pricing strategy, we use this mean number of consumers in the utility optimization problem under the assumption that the building parameters in Section II-B represent a single aggregate rational consumer. The resulting optimal prices, associated generation cost, and associated peak demand are listed in Table IV. For comparison, we also included in Table IV the generation cost and demand peak for the same rational consumer subject to prices based solely on the marginal costs. Note that, as discussed in Section II-E, the prices are scaled so that revenue equals a fixed fraction of total costs. However, this scaling factor does not affect utility costs or demand peak.

From Table IV, optimized pricing results in a slight reduction ($27,402 per day) in generation costs. The discrepancy between optimal prices and marginal costs may be surprising given that both the consumer and utility are trying to minimize the cost of electricity. However, there are several reasons for this difference. The first and most obvious reason is that the price structure for the consumer and the cost structure for the utility are not perfectly aligned. In the first place, the utility has a quadratic in consumption model for costs, where the consumer has a linear model. The second misalignment is that the capacity cost for the utility is calculated as a maximum over 24 hours and the demand charge for the consumer is calculated only during peak hours. A more fundamental reason that marginal costs are not optimal is nonlinearity of the cost function and heterogeneity of the consumers. To see this, suppose that cost function exactly equaled the price function for each consumer. The problem in this case is that the sum of the individual bills is NOT equal to the total generation cost. This can be seen in the demand charge, where $\sup_x f(x) + \sup_x g(x) \neq \sup_x (f(x) + g(x))$.

C. Case III: Impact of solar power on non-solar consumers

We now study the impact of solar integration on the bills of non-solar consumers in a regulated electricity market. We consider a network consisting of a utility company and two aggregate consumers - one solar and one non-solar. For the non-solar consumer, we define optimal thermostat programming as in (10). For the solar consumer, the optimal thermostat programming problem is as defined in (10), where we have now redefined the consumption function as

$$g(k, u_k, T_1^k) := \frac{T_e^k - u_k}{R_e} + 2C_{in}\frac{T_1^k - u_k}{\Delta x} - Q_k, \qquad (20)$$

where Q_k is the power supplied locally by solar panels. We assume that solar penetration is 50%, so that both aggregate consumers contribute equally to revenue and costs to the utility. For Q_k, we used data generated on June 4-7 from a typical 13kW south-facing rooftop PV array in Scottsdale, AZ. We applied our hybrid Nelder-Mead DP algorithm separately to each consumer, while considering (15) as the utility cost model. The results are presented in Table V. For comparison, we have also included optimal prices, prorated electricity bills over three days and demand peaks for the cases where all of the consumers are either solar or non-solar. From Table V we observe that the increase in price of the electricity bill of a non-solar consumer is $\cong$ 2%. It is also interesting to note that under optimized pricing, the monthly bill for a solar consumer decreases by $\cong$ 8%. The corresponding utility-generated power, solar-generated power and optimal temperature settings are shown in Fig. 9.

TABLE III
CASE I: UTILITY COST AND PEAK DEMAND OVER 3 DAYS FOR 3 DIFFERENT LEVELS OF DEMAND CHARGE. PRICES ARE CHOSEN ARBITRARILY. UTILITY COSTS ARE CALCULATED USING THE SRP MODEL DEFINED IN SECTION II-E (τ =0.00401 \$/(MWH)^2, ν =4.54351 \$/(MWH))

	Prices $[p_{\text{off}}, p_{\text{on}}, p_d]$	Demand Charge	Utility costs	Demand peak
Optimal	[0.015, 0.0214, 29.177]	high	**46.78**\$ (0.086 $\frac{\$}{kWh}$)	**7.4132** kW
	[0.015, 0.045, 13.573]	medium	**51.56**\$ (0.116 $\frac{\$}{kWh}$)	**8.2898** kW
	[0.015, 0.0219, 3.1092]	low	**59.42**\$ (0.168 $\frac{\$}{kWh}$)	9.6749 kW
4-Setpoint	[0.015, 0.0214, 29.177]	high	53.47\$ (0.114 $\frac{\$}{kWh}$)	8.5914 kW
	[0.015, 0.045, 13.573]	medium	55.19\$ (0.130 $\frac{\$}{kWh}$)	8.910 kW
	[0.015, 0.0219, 3.1092]	low	61.24\$ (0.169 $\frac{\$}{kWh}$)	9.974 kW
GPOPS	[0.015, 0.0214, 29.177]	high	49.53\$ (0.109 $\frac{\$}{kWh}$)	7.9440 kW
	[0.015, 0.045, 13.573]	medium	56.48\$ (0.142 $\frac{\$}{kWh}$)	9.1486 kW
	[0.015, 0.0219, 3.1092]	low	59.19\$ (0.159 $\frac{\$}{kWh}$)	9.6221 kW
Precooling	[0.015, 0.0214, 29.177]	high	54.75\$ (0.116 $\frac{\$}{kWh}$)	8.8031 kW
	[0.015, 0.045, 13.573]	medium	54.75\$ (0.116 $\frac{\$}{kWh}$)	8.8031 kW
	[0.015, 0.0219, 3.1092]	low	**54.75**\$ (0.116 $\frac{\$}{kWh}$)	**8.8031** kW

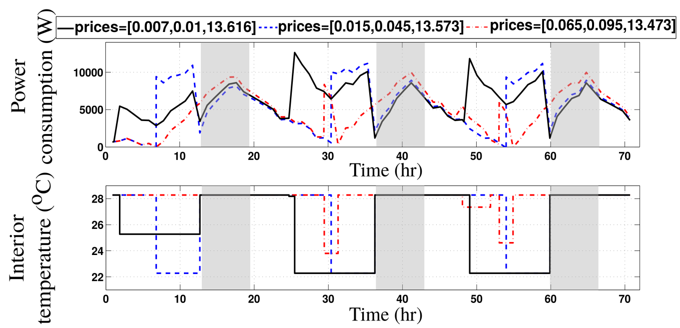


Fig. 7. CASE I: Power consumption and temperature settings for high, medium and low demand penalties using 4-Setpoint thermostat programming.

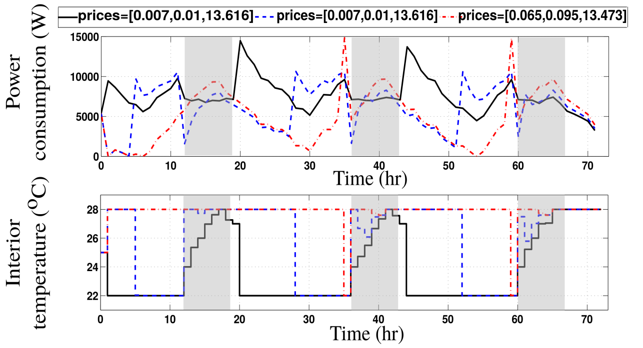


Fig. 8. CASE I: Power consumption & optimal temperature settings for high, medium and low demand penalties. Shaded areas correspond to on-peak hours.

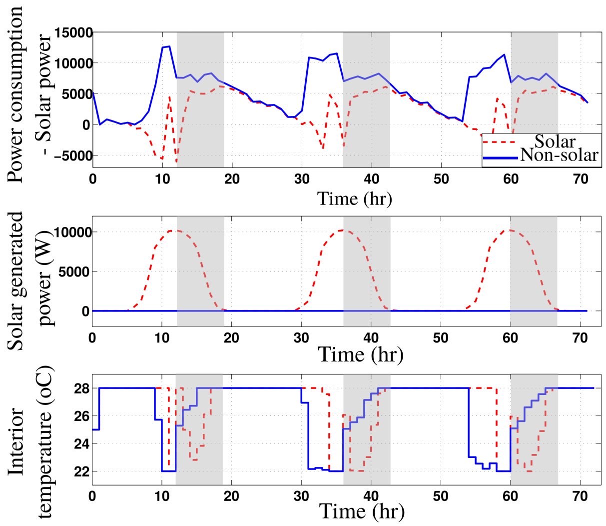


Fig. 9. CASE III: Power consumption, solar generated power and optimal temperature settings for the non-solar and solar consumers.

V. Conclusion

We proposed a dynamic-programming-based algorithm for solving the optimal control problem associated with thermostat programming in the presence of thermal energy storage for consumers with both time-of-use and demand charges using both continuously variable and 4-setpoint thermostats. Our model of thermal storage is based on a discretized version of the heat equation (a PDE) and captures the latency inherent in thermal diffusion processes. We applied this algorithm to a variety of building types to obtain an average cost savings of 9.6% for consumers. We also proposed a simple algorithm for optimizing electricity prices and demonstrated that optimal thermostat programming implies that current strategies of pricing based on marginal costs may be sub-optimal. Finally, our analysis suggests that when: demand charges are present; consumers are rational; and prices are optimal, solar consumers (even at 50% penetration) only increase the monthly bill of non-solar consumers by $\cong$ 2%.

VI. Acknowledgements

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TABLE IV
CASE II: Generation costs (for three days) and demand peaks associated with regulated optimal prices and prices from SRP. Coeffs. of utility cost: $\tau = 0.004$ \$/(MWh)^2, $\nu = 4.543$ \$/(MWh)

Strategy	$[p_{\text{off}}(\frac{\$}{kWh}), p_{\text{on}}(\frac{\$}{kWh}), p_d(\frac{\$}{kW})]$	Utility costs	Demand peak
Optimal	[0.0564, 0.0667, 51.1859]	**1,595,309** \$	195.607 *MW*
SRP	[0.0668, 0.0668, 49.0018]	1,677,516 \$	211.79 *MW*

TABLE V
CASE III: Optimal electricity prices, bills (for three days) and demand peaks for various consumers.

Consumers	$[p^\star_{\text{off}}(\frac{\$}{kWh}), p^\star_{\text{on}}(\frac{\$}{kWh}), p^\star_d(\frac{\$}{kW})]$	Elect. Bill	Demand peak
Solar &	[0.089, 0.115, 51.988]	\$ 50.052	6.1947 *kW*
Non-solar		\$ 84.717	8.6787 *kW*
Single Non-solar	[0.081, 0.108, 54.004]	\$ 83.333	8.3008 *kW*
Single Solar	[0.088, 0.118, 58.556]	\$ 54.311	6.1916 *kW*

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