H_∞-Optimal Estimation in the PIE framework for Systems with Multiple Delays and Sensor Noise

Shuangshuang Wu, Matthew M. Peet, Sachin Shivakumar, Changchun Hua

Abstract— In this paper, we consider the problem of H_∞-optimal estimation for linear multi-delay systems with sensor noise and delayed output using recently proposed fundamental-state framework. We propose an extended Luenberger observer which can corrects both the estimate of present state and the history of the state. Our synthesis condition is defined as a Linear Operator Inequality (LOI) using the Partial Integral Equation (PIE) formulation of time-delay systems and is implemented using the PIETOOLS Matlab toolbox for manipulation of Partial Integral (PI) operators. Numerical examples show that synthesis condition we propose produces an estimator with provable H_∞ gain bound which is optimal to at least 4 decimal places as measure using comparison with Padé-based discretization.

I. Introduction

Time-delay system are often represented using Delay Differential Equations (DDEs). Alternative representations include Partial Differential Equations (PDE) coupled with Ordinary Differential Equations (ODEs) [1]. Asymptotic algorithms for stability analysis and control synthesis of time-delay systems based on the use of Lyapunov-Krosovikii (L-K) functionals include the work of [2], [3], and [4]. Many such results are based on direct construction of L-K functionals combined with the use of efficient bounding techniques - See [2], [3], and [5]. Algorithms based on the use of SOS include [6] and [4].

The problem of estimator design for time-delay systems has been considered in such works as [7], [8], and [10]. Most recently, the H$_\infty$-optimal estimator design problem for multi-delay systems was addressed using the SOS-operator framework in [9] - with remarkably accurate results. Unfortunately, however, none of these works consider the effect of sensor noise in designing the estimator. In practical implementations, of course, sensor noise is significant and inevitable - due to e.g. 60Hz noise, mechanical vibrations or quantization of the measured variable. Failure to accurately account for sensor noise can lead to chattering effects or even system failure.

Motivated by these observations, this paper investigates the estimator design problem for the following system,

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + \sum_{i=1}^{K} A_i x(t-\tau_i) + B_1 w(t)\\ y(t) &= C_{20}x(t) + \sum_{i=1}^{K} C_{2i}x(t-\tau_i) + D_2 w(t)\\ z(t) &= C_{10}x(t) + \sum_{i=1}^{K} C_{1i}x(t-\tau_i) + D_1 w(t), \qquad (1)\end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^r$ is an external disturbance input, $y(t) \in \mathbb{R}^q$ is the measured output, $z(t) \in \mathbb{R}^p$ is the regulated output. The delays $\tau_i > 0$ for $i \in [1, \ldots, K]$ are ordered by increasing magnitude and $A_0, A_i, B_1, C_{10}, C_{1i}, C_{20}, C_{2i}, D_1, D_2$ are constant matrices with appropriate dimensions.

Our goal is to design an optimal estimator which uses the measured output $y(t)$ to construct an estimate of $x(t)$ and $z(t)$ while minimizing $\gamma := \sup_{w\in L_2} \frac{\|z_e\|_{L_2}}{\|w\|_{L_2}}$ where $z_e(t) = \hat{z}(t) - z(t)$ is the difference between the real $z(t)$ and its estimate $\hat{z}(t)$.

The following approach is taken in this work. Firstly, we propose an operator-valued extension of a well-known LMI for H_∞-optimal estimator design. This formulation is valid for a general class of Distributed-Parameter Systems (DPS) represented using Partial Integral Equations. Next, we construct an equivalent PIE representation of Equation (1), applying the new fundamental-state-space framework presented in [11] to time delay systems. This new state-space model embeds the boundary condition into the system dynamics - simplifying the analysis and synthesis process. Next, a generalized PI (Partial Intergral) operator-valued version of LMI formulation for estimator design of system (1) is obtained. Finally, by means of SOS-based PIETOOLS proposed in [12], the results of numerical implementation are given.

A. Notation

Shorthand notation used throughout this paper includes the Hilbert spaces $L_2^m[X] := L_2(X;\mathbb{R}^m)$ of square integrable functions from X to $\mathbb{R}^m$ and $W_2^m[X] := W^{1,2}(X;\mathbb{R}^m) = H^1(X;\mathbb{R}^m) = \{x : x, \dot{x} \in L_2^m[X]\}$. We use L_2^m, W_2^m when domains are clear from context. We also use the extensions $W_2^{n\times m}[X] := W^{1,2}(X;\mathbb{R}^{n\times m})$ for matrix-valued functions. An operator $\mathcal{P} : Z \to Z$ is positive on a subset X of Hilbert space Z if $\langle x, \mathcal{P}x\rangle_Z \geq 0$ for all $x \in X$. $\mathcal{P}$ is coercive on X if $\langle x, \mathcal{P}x\rangle_Z \geq \epsilon\|x\|_Z^2$ for some $\epsilon > 0$ for all $x \in X$. I denotes the identity

S. Wu is with the Institute of Electrical Engineering, Yanshan University, Qinhuangdao, 066004, China. e-mail: ssw_0538_ysu@163.com

M. Peet is with the School for the Engineering of Matter, Transport and Energy, Arizona State University, Tempe, AZ, 85298 USA. e-mail: mpeet@asu.edu

C. Hua is with the Institute of Electrical Engineering, Yanshan University, Qinhuangdao, 066004, China. e-mail: cch@ysu.edu.cn

matrix. $0_{n\times m} \in \mathbb{R}^{n\times m}$ is the matrix of zeros matrix with shorthand $0_n := 0_{n\times n}$. The symmetric completion of a function of matrices or operators is denoted by $*$. If $\mathcal{P}^1$ and $\mathcal{P}^2$ are two linear operators then $(\mathcal{P}^1)^*$ stands for the adjoint of $\mathcal{P}^1$ and $\mathcal{P}^1\mathcal{P}^2$ represents composition of those operators in shown order.

II. Optimal Estimator design for Distributed-Parameter Systems

Consider a general class of distributed-parameter system (DPS) given as

$$\begin{aligned}\mathcal{T}\dot{\mathbf{x}}_f(t) &= \mathcal{A}\mathbf{x}_f(t) + \mathcal{B}\omega(t)\\ z(t) &= \mathcal{C}_1\mathbf{x}_f(t) + D_1\omega(t)\\ y(t) &= \mathcal{C}_2\mathbf{x}_f(t) + \mathcal{D}_2\omega(t)\end{aligned} \tag{2}$$

where $\mathcal{T} : X_f \to Z$, $\mathcal{A} : X_f \to Z$, $\mathcal{B} : \mathbb{R} \to Z$, $\mathcal{C}_1 : X_f \to \mathbb{R}$, $\mathcal{C}_2 : X_f \to \mathbb{R}$, $\mathcal{D}_1 : \mathbb{R} \to \mathbb{R}$ and $\mathcal{D}_2 : \mathbb{R} \to \mathbb{R}$.

Consider an estimator with the following dynamics.

$$\begin{aligned}\mathcal{T}\dot{\hat{\mathbf{x}}}_f(t) &= \mathcal{A}\hat{\mathbf{x}}_f(t) + \mathcal{L}(\mathcal{C}_2\hat{\mathbf{x}}_f(t) - \mathbf{y}(t)) \qquad (3)\\ \hat{z}(t) &= \mathcal{C}_1\hat{\mathbf{x}}_f(t)\end{aligned}$$

where $\mathcal{L} : \mathbb{R} \to Z$. Define $\mathbf{e}_f(t) = \hat{\mathbf{x}}_f(t) - \mathbf{x}_f(t)$. The closed-loop error dynamics are now

$$\begin{aligned}\mathcal{T}\dot{\mathbf{e}}_f(t) &= (\mathcal{A} + \mathcal{L}\mathcal{C}_2)\mathbf{e}_f(t) - (\mathcal{B} + \mathcal{L}\mathcal{D}_2)\omega(t)\\ z_e(t) &= \mathcal{C}_1\mathbf{e}_f(t) - D_1\omega(t)\end{aligned} \tag{4}$$

where $\mathbf{e}_f(0) = 0$. The following result can be found in [4].

Theorem 1: Suppose $\mathcal{P}$ is a bounded, self-adjoint, coercive linear operator $\mathcal{P} : X \to X$. Then P^{-1} exists; is bounded; is self-adjoint; $\mathcal{P}^{-1} : X \to X$; and P^{-1} is coercive.

We now give a Linear Operator Inequality (LOI) for optimal estimation of the abstract DPS.

Theorem 2: Suppose there exists a scalar $\gamma > 0$ and bounded linear operators $\mathcal{P} : Z \to Z$ satisfying Theorem 1 and $\mathcal{Z} : \mathbb{R} \to Z$ such that

$$\begin{bmatrix} -\gamma I & -D_1^T & -(\mathcal{P}\mathcal{B} + \mathcal{Z}\mathcal{D}_2)^*\mathcal{T}\\ -D_1 & -\gamma I & \mathcal{C}_1\\ (\star)^* & (\mathcal{C}_1)^* & (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)^*\mathcal{T} + \mathcal{T}^*(\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\end{bmatrix} < 0 \tag{5}$$

Then $\mathcal{P}^{-1}$ is a bounded linear operator. If $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$, any solution of Eqn. (2)-(4) satisfies $\|z_e\|_{L_2} \le \gamma\|\omega\|_{L_2}$.

Proof: Define the storage functional $V(\mathbf{e}_f) = \langle \mathcal{T}\mathbf{e}_f, \mathcal{P}\mathcal{T}\mathbf{e}_f\rangle_Z \ge \delta\|\mathbf{e}_f\|^2$ which holds for some $\delta > 0$ since $\mathcal{P}$ is coercive. Define $\mathcal{Z} = \mathcal{P}\mathcal{L}$. Then

$$\begin{aligned}&\dot{V}(\mathbf{e}_f) - \gamma\|\omega\|^2 - \gamma\|\upsilon_e\|^2 + \langle \upsilon_e, z_e\rangle_Z + \langle z_e, \upsilon_e\rangle_Z\\ &= \langle \mathcal{T}\mathbf{e}_f, (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}_f\rangle_Z + \langle (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}_f, \mathcal{T}\mathbf{e}_f\rangle_Z\\ &- \langle \mathcal{T}\mathbf{e}_f, (\mathcal{P}\mathcal{B} + \mathcal{Z}\mathcal{D}_2)w\rangle_Z - \langle (\mathcal{P}\mathcal{B} + \mathcal{Z}\mathcal{D}_2)w, \mathcal{T}\mathbf{e}_f\rangle_Z\\ &- \gamma\|\omega\|^2 - \gamma\|\upsilon_e\|^2 + \langle \upsilon_e, \mathcal{C}_1\mathbf{e}_f\rangle + \langle \mathcal{C}_1\mathbf{e}_f, \upsilon_e\rangle\\ &- \langle \upsilon_e, D_1\omega\rangle - \langle D_1\omega, \upsilon_e\rangle,\end{aligned}$$

where $\upsilon_e(t) = \frac{1}{\gamma}z_e(t)$. If Eqn. (5) is satisfied, then

$$\dot{V}(\mathbf{e}_f) < \gamma\|\omega\|^2 - \frac{1}{\gamma}\|z_e\|^2.$$

Integration of this inequality with t yields

$$V(\mathbf{e}_f(t)) - V(\mathbf{e}_f(0)) + \frac{1}{\gamma}\int_0^t \|z_e(s)\|^2\, ds \le \gamma\int_0^t \|w(s)\|^2\, ds$$

As $V(\mathbf{e}_f(0)) = 0$ and $V(\mathbf{e}_f(t)) \ge 0$, if we let $t \to \infty$, we see $\|z_e\|_{L_2} \le \gamma\|\omega\|_{L_2}$. ■

III. Coupled Plant and Estimator Dynamics in PDE-ODE form

In this section, we give an equivalent PDE-ODE representation of the plant (1). Based on the PDE-ODE representation of the plant, we construct an estimator in PDE-ODE form.

A. Plant Dynamics in PDE-ODE form

First, we give an equivalent PDE-ODE representation of Eqn. (1), which expresses Eqn. (1) in a form without delay but defined on t and s. Note that in [9], the history of states in the delay channels was represented as $\phi_i(t,s) := x(t+s)$ for $s \in [-\tau_i, 0]$ and the domain of s varied from channel to channel. In this paper, however, we use a uniformly defined domain

$$\phi_i(t,s) := x(t + \tau_i s), s \in [-1, 0]. \tag{6}$$

and $\phi_i(t,s) \in L_2^n[-1,0]$. Using the scaling approach, the history of state $x(t)$ in different delay channels can be rewritten as $\phi_i(t,s)$ - all within the same range $s \in [-1,0]$ - thereby simplifying the analysis and notation.

Applying the fundamental theorem of calculus, we have

$$\begin{aligned}\phi_i(t,s) &= x(t) - \int_s^0 \phi_{is}(t,s)ds\\ \phi_{it}(t,s) &= \frac{1}{\tau_i}\phi_{is}(t,s).\end{aligned}$$

Define

$$\phi(t,s) = \left[\phi_1^T(t,s), \phi_2^T(t,s), \cdots, \phi_K^T(t,s)\right]^T, \tag{7}$$

and

$$\mathbf{x}_p(t) = \begin{bmatrix} x(t)\\ \phi(t,s)\end{bmatrix} = \begin{bmatrix} x(t)\\ \tilde{I}x(t) - \int_s^0 \phi_s(t,s)ds\end{bmatrix}.$$

We represent the system dynamics (1) in PDE-ODE form as

$$\begin{aligned}\dot{\mathbf{x}}_p(t) &= \begin{bmatrix}(A_0 + \sum_i A_i)x(t) + \int_{-1}^0 \tilde{A}\phi_s(t,s)ds\\ H\phi_s(t,s)\end{bmatrix} + \begin{bmatrix} Bw(t)\\ 0\end{bmatrix}\\ z(t) &= (C_{10} + \sum_i C_{1i})x(t) - \tilde{C}_1\int_{-1}^0 \phi_s(t,s)ds + D_1w(t)\\ y(t) &= (C_{20} + \sum_i C_{2i})x(t) - \tilde{C}_2\int_{-1}^0 \phi_s(t,s)ds + D_2w(t),\end{aligned} \tag{8}$$

where

$$\tilde{I} = [\underbrace{I_n, I_n, \cdots, I_n}_{K}]^T$$

$$H = \begin{bmatrix} \frac{1}{\tau_1}I & 0 & \cdots & 0 \\ 0 & \frac{1}{\tau_2}I & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\tau_K}I \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} A_1 & A_2 & \cdots & A_K \end{bmatrix}$$

$$\tilde{C}_1 = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1K} \end{bmatrix}$$

$$\tilde{C}_2 = \begin{bmatrix} C_{21} & C_{22} & \cdots & C_{2K} \end{bmatrix}. \quad (9)$$

Lemma 3: Suppose w, x, y, and z satisfy Eqn. (1). Then if $\tilde{I}, \tilde{A}, \tilde{C}_1, \tilde{C}_2$ are as defined in Eqn. (9) and ϕ is as defined in Eqn. (7), then w, x, y, and z also satisfy Eqn. (8).

B. Estimator Dynamics in PDE-ODE form

For Eqn.(8), we construct the estimator dynamics as

$$\dot{\hat{\mathbf{x}}}_p(t) = \begin{bmatrix} (A_0 + \sum_i A_i)\hat{x}(t) + \int_{-1}^0 \tilde{A}\hat{\phi}_s(t,s)ds \\ H\hat{\phi}_s(t,s) \end{bmatrix} + \begin{bmatrix} L_1 y_e(t) \\ L_2(s) y_e(t) \end{bmatrix}$$

$$\hat{z}(t) = (C_{10} + \sum_i C_{1i})\hat{x}(t) - \tilde{C}_1 \int_{-1}^0 \hat{\phi}_s(t,s)ds$$

$$\hat{y}(t) = (C_{20} + \sum_i C_{2i})\hat{x}(t) - \tilde{C}_2 \int_{-1}^0 \hat{\phi}_s(t,s)ds. \quad (10)$$

where

$$\hat{\mathbf{x}}_p(t) = \begin{bmatrix} \hat{x}(t) \\ \tilde{I}\hat{x}(t) - \int_s^0 \hat{\phi}_s(t,s)ds \end{bmatrix}$$

$$\hat{\phi}(t,s) = \{\hat{\phi}_1(t,s), \hat{\phi}_2(t,s), \cdots, \hat{\phi}_K(t,s)\}$$

$$y_e(t) = \hat{y}(t) - y(t)$$

and $\hat{x}(t) \in \mathbb{R}^n$ as the estimate of state $x(t)$, $\hat{\phi}(t) \in L_2^{nK}$ as the estimate of state $\phi(t)$, $\hat{y}(t) \in \mathbb{R}^q$ as the estimate of $y(t)$.

In [9], the estimator was designed using both the current and history of output $y(t)$ of the real system, and the sensor noise was not considered. Note that aside from sensor noise and output delay, our new estimator has a simpler structure with only the current output information plugging in, and thus is easier implemented.

Lemma 4: Suppose the conditions of Lemma 3 are satisfied, and w, x, y, z, ϕ, $\hat{x}$, $\hat{y}$, $\hat{z}$ and $\hat{\phi}$ satisfy Eqn. (8) coupled with Eqn. (10). Then w, x, y, z, $\hat{x}$, $\hat{y}$, $\hat{z}$ and $\hat{\phi}$ also satisfy coupled Eqn. (1) coupled with Eqn. (10).

IV. Coupled Plant and Estimator Dynamics in PIE

To apply Theorem 2 to the case of multiple-delay systems (specifically to system (1)) in this section, we express the equivalent PDE-ODE form Eqn.(8) of system (1) in the abstract form (2) and the estimator Eqn. (10) in the abstract form (3), where all the infinitesimal generators are in the form of PI operators.

Firstly, we define the inner-product space $Z_{m,n} := \{\mathbb{R}^m \times L_2^n[-1,0]\}$, and the inner product on $Z_{m,n}$ as

$$\left\langle \begin{bmatrix} y \\ \psi \end{bmatrix}, \begin{bmatrix} x \\ \phi \end{bmatrix} \right\rangle_{Z_{m,n}} = y^T x + \int_{-1}^0 \psi(s)^T \phi(s) ds.$$

and we give the definition of a PI operator as follows,

Definition 5: A Partial Integral (PI) operator $\mathcal{P}\left[\begin{smallmatrix} P, & Q_1 \\ Q_2, & \{R_i\} \end{smallmatrix}\right] : Z_{m,n} \to Z_{p,q}$ is parameterized by a matrix $P \in \mathbb{R}^{p,m}$ and matrix-valued functions $Q_1 \in W_2^{p\times n}[-1,0]$, $Q_2 \in W_2^{q\times m}[-1,0]$, $R_0 \in W_2^{q\times n}[-1,0]$, $R_1, R_2 \in W_2^{q\times n}\left[[-1,0]\times[-1,0]\right]$ as

$$\left(\mathcal{P}\left[\begin{smallmatrix} P, & Q_1 \\ Q_2, & \{R_i\} \end{smallmatrix}\right] \begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) := \quad (11)$$

$$\begin{bmatrix} Px + \int_{-1}^0 Q_1(s)\phi(s)ds \\ \\ (Q_2^T(s)x + R_0(s)\phi(s) + \int_{-1}^s R_1(s,\theta)\phi(\theta)d\theta \\ + \int_s^0 R_2(s,\theta)\phi(\theta)d\theta) \end{bmatrix}.$$

Now we represent multi-delay system into the DPS format using PI operators. Define the fundamental state of system (8) as

$$\mathbf{x}_f(t,s) = \begin{bmatrix} x(t) \\ \phi_s(t,s) \end{bmatrix} \in X_f,$$

and correspondingly, define the fundamental state of system (10) as

$$\hat{\mathbf{x}}_f(t,s) = \begin{bmatrix} \hat{x}(t) \\ \hat{\phi}_s(t,s) \end{bmatrix} \in X_f.$$

where $X_f = Z_{n,nK}$.

We represent the infinitesimal generators $\mathcal{T} : Z_{n,nK} \to Z_{n,nK}$, $\mathcal{A} : Z_{n,nK} \to Z_{n,nK}$, $\mathcal{B} : \mathbb{R}^r \to Z_{n,nK}$, $\mathcal{C}_1 : Z_{n,nK} \to \mathbb{R}^p$, $\mathcal{C}_2 : Z_{n,nK} \to \mathbb{R}^q$, and $\mathcal{L} : \mathbb{R}^q \to Z_{n,nK}$ in PI operator format as follows

$$\mathcal{T} := \mathcal{P}\left[\begin{smallmatrix} I_n, & 0 \\ \tilde{I}, & \{0,0,-I_{nK}\} \end{smallmatrix}\right]$$

$$\mathcal{A} := \mathcal{P}\left[\begin{smallmatrix} A_0 + \sum_i A_i, & -\tilde{A} \\ 0, & \{H,0,0\} \end{smallmatrix}\right]$$

$$\mathcal{B} := \mathcal{P}\left[\begin{smallmatrix} B, & 0 \\ 0, & \{0\} \end{smallmatrix}\right]$$

$$\mathcal{C}_1 := \mathcal{P}\left[\begin{smallmatrix} C_{10} + \sum_i C_{1i}, & -\tilde{C}_1 \\ 0, & \{0\} \end{smallmatrix}\right]$$

$$\mathcal{C}_2 := \mathcal{P}\left[\begin{smallmatrix} C_{20} + \sum_i C_{2i}, & -\tilde{C}_2 \\ 0, & \{0\} \end{smallmatrix}\right]$$

$$\mathcal{D}_2 := \mathcal{P}\left[\begin{smallmatrix} D_2, & 0 \\ 0, & \{0\} \end{smallmatrix}\right]$$

$$\mathcal{L} := \mathcal{P}\left[\begin{smallmatrix} L_1, & 0 \\ L_2, & \{0\} \end{smallmatrix}\right] \quad (12)$$

Now let us now turn to the other operators used in Theorem 2. We define $\mathcal{P}$ to have the structure $\mathcal{P} := \mathcal{P}\left[\begin{smallmatrix} P, & Q_1 \\ Q_2, & \{R_i\} \end{smallmatrix}\right]$ and we parameterize the decision operator variable $\mathcal{Z}$ as

$$\mathcal{Z} := \mathcal{P}\left[\begin{smallmatrix} Z_1, & 0 \\ Z_2, & \{0\} \end{smallmatrix}\right], \quad (13)$$

which gives $\mathcal{L}$ the structure as defined in Eqn. (12) if $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$.

V. Inverting the Operator and Constructing Estimator Gains

Because the observer gains are of the form $\mathcal{L}=\mathcal{P}^{-1}\mathcal{Z}$, we need an expression for $\mathcal{P}\left[\begin{smallmatrix}P, & Q\\ Q^T, & \{R_i\}\end{smallmatrix}\right]^{-1}$. Unfortunately, it still is an open problem how to get an analytic form for this inverse. So we compromise here by letting $\mathcal{P}:=\mathcal{P}\left[\begin{smallmatrix}P, & Q\\ Q^T, & \{R_0,R_1,R_1\}\end{smallmatrix}\right]$. The inverse of the slightly more structured operator $\mathcal{P}\left[\begin{smallmatrix}P, & Q\\ Q^T, & \{R_0,R_1,R_1\}\end{smallmatrix}\right]$ has previously been utilized in [4], [9] and has a known analytic inverse.

The following lemma presents an analytical expression for the inverse of the operator $\mathcal{P}\left[\begin{smallmatrix}P, & Q\\ Q^T, & \{R_0,R_1,R_1\}\end{smallmatrix}\right]$.

Lemma 6: Suppose that $Q(s)=HZ(s)$ and $R_1(s,\theta)=Z(s)^T\Gamma Z(\theta)$ where $Z(s)$ is the column base function with degree d and $\mathcal{P}:=\mathcal{P}\left[\begin{smallmatrix}P, & Q\\ Q^T, & \{R_0,R_1,R_1\}\end{smallmatrix}\right]$ is a coercive and self-adjoint operator where $\mathcal{P}:X\to X$. Then we get $\mathcal{P}^{-1}:=\mathcal{P}\left[\begin{smallmatrix}\hat{P}, & \hat{Q}\\ \hat{Q}^T, & \{\hat{R}_0,\hat{R}_1,\hat{R}_1\}\end{smallmatrix}\right]$ where

$$\hat{P}=\left(I-\hat{H}KH^T\right)P^{-1},\qquad \hat{Q}(s)=\hat{H}Z(s)R_0(s)^{-1}$$
$$\hat{R}_0(s)=R_0(s)^{-1},\quad \hat{R}_1(s,\theta)=\hat{R}_0^T(s)Z(s)^T\hat{\Gamma}Z(\theta)\hat{R}_0(\theta),$$

if we define

$$K=\int_{-1}^{0}Z(s)R_0(s)^{-1}Z(s)^Tds$$
$$\hat{H}=P^{-1}H\left(KH^TP^{-1}H-I-K\Gamma\right)^{-1}$$
$$\hat{\Gamma}=-(\hat{H}^TH+\Gamma)(I+K\Gamma)^{-1}$$

and $\mathcal{P}^{-1}$ is self-adjoint where $\mathcal{P}^{-1}:X\to X$, and $\mathcal{P}^{-1}\mathcal{P}\mathbf{x}=\mathcal{P}\mathcal{P}^{-1}\mathbf{x}=\mathbf{x}$ for any $\mathbf{x}\in Z_{m,n}$.

Proof: See [14] for a proof. ∎

Armed with this inverse, we construct the observer gains as in Lemma 7.

Lemma 7: If $\mathcal{L}=\mathcal{P}^{-1}\mathcal{Z}$ where $\mathcal{P}:=\mathcal{P}\left[\begin{smallmatrix}P, & Q\\ Q^T, & \{R_0,\hat{R}_1,R_1\}\end{smallmatrix}\right]$ and $\mathcal{P}^{-1}$ is defined as in Lemma 6 and $\mathcal{Z}$ is as in Eqn. (13) where $Z_2(s)$ is a polynomial representated as $Z_2(s)=Z^T(s)W$, then we get $\mathcal{L}$ as in Eqn. (12), where

$$L_1=\hat{P}Z_1+\hat{H}TW$$
$$L_2(s)=X(s)\left(\hat{H}^TZ_1+W+\hat{\Gamma}TW\right)$$

if we define

$$X(s)=\hat{S}(s)Z(s)^T,$$
$$T=\int_{-1}^{0}Z(s)\hat{S}(s)Z(s)^Tds.$$

Proof: The proof follows from the formula for composition of operators $\mathcal{P}^{-1}$ and $\mathcal{Z}$ - see Appendix for the formula for the composition operation. ∎

VI. Theorem 2 applied to Multi-delay systems

In this section, we apply the conditions of Theorem 2 to multi-delay systems and obtain an LOI for optimal observer synthesis.

Theorem 8: Suppose there exists positive scalar γ, matrix $P\in\mathbb{R}^{n\times n}$, functions $Q\in W_2^{n\times nK}[-1,0], R_0\in W_2^{nK\times nK}[-1,0]$, $R_1\in W_2^{nK\times nK}\left[[-1,0]\times[-1,0]\right]$, matrix $Z_1\in\mathbb{R}^{n\times q}$, functions $Z_2\in W_2^{nK\times q}[-1,0]$ such that the operator $\mathcal{P}:=\mathcal{P}\left[\begin{smallmatrix}P, & Q\\ Q^T, & \{R_0,\hat{R}_1,R_1\}\end{smallmatrix}\right]$ satisfies Theorem 1, and the operator $\mathcal{P}:=\mathcal{P}\left[\begin{smallmatrix}P, & Q\\ Q^T, & \{R_0,\hat{R}_1,R_1\}\end{smallmatrix}\right]$ and $\mathcal{Z}:=\mathcal{P}\left[\begin{smallmatrix}Z_1, & 0\\ Z_2, & \{0\}\end{smallmatrix}\right]$ satisfy

$$\begin{bmatrix}-\gamma I & -D_1^T & -(\mathcal{P}\mathcal{B}+\mathcal{Z}\mathcal{D}_2)^*\mathcal{T}\\ -D_1 & -\gamma I & \mathcal{C}_1\\ (\star)^* & (\mathcal{C}_1)^* & (\mathcal{P}\mathcal{A}+\mathcal{Z}\mathcal{C}_2)^*\mathcal{T}+\mathcal{T}^*(\mathcal{P}\mathcal{A}+\mathcal{Z}\mathcal{C}_2)\end{bmatrix}<0 \tag{14}$$

where the operators $\mathcal{A},\mathcal{B},\mathcal{C}_1,\mathcal{C}_2,\mathcal{T},\mathcal{D}_2$ are all defined as Eqn.(12). Then if $\mathcal{L}=\mathcal{P}^{-1}\mathcal{Z}$, $\mathcal{L}$ has the form $\mathcal{L}=\mathcal{P}\left[\begin{smallmatrix}L_1, & 0\\ L_2, & \{0\}\end{smallmatrix}\right]$, Furthermore, for any $w\in L_2$, if $z_e(t)=\hat{z}(t)-z(t)$, where $z(t)$ and $\hat{z}(t)$ satisfy Eqn. (1) coupled with Eqn. (10), then z_e satisfies $\|z_e\|_{L_2}\le\gamma\|\omega\|_{L_2}$.

Proof: From Theorem 1, $\mathcal{P}^{-1}$ exists and is bounded coercive operator. From Lemma 6, we can get $\mathcal{P}^{-1}:=\mathcal{P}\left[\begin{smallmatrix}\hat{P}, & \hat{Q}\\ \hat{Q}^T, & \{\hat{R}_0,\hat{R}_1,\hat{R}_1\}\end{smallmatrix}\right]$. Then from Lemma 7, we get $\mathcal{L}=\mathcal{P}\left[\begin{smallmatrix}L_1, & 0\\ L_2, & \{0\}\end{smallmatrix}\right]$, which has the same structure used to obtain the estimator form defined in Eqn. (10).

For any solution of Eqn. (8) coupled with Eqn. (10), define $z_e(t)=\hat{z}(t)-z(t)$, $e(t)=\hat{x}(t)-x(t)$, and $\phi_{es}(t,s)=\hat{\phi}_s(t,s)-\phi_s(t,s)$. Then if the operators $\mathcal{A},\mathcal{B},\mathcal{C}_1,\mathcal{C}_2,\mathcal{T},\mathcal{D}_2$ are defined as Eqn.(12), Eqn. (4) is satisfied with $\mathbf{e}_f(t):=\begin{bmatrix}e(t)\\ \phi_{es}(t,s)\end{bmatrix}$. From Theorem 2, we get that for any $w\in L_2$ and any solution of Eqn. (8) coupled with Eqn. (10) we have $\|z_e\|_{L_2}\le\gamma\|\omega\|_{L_2}$. Then from Lemma 4, we get for any $w\in L_2$ and any solution of Eqn. (1) coupled with Eqn. (10), we have that $\|z_e\|_{L_2}\le\gamma\|\omega\|_{L_2}$. ∎

VII. Numerical Implementation

We implement our algorithm in Matlab using the PIETOOLS toolbox. This toolbox is available online for validation or download from Code Ocean [12]. The corresponding optimization problem is to minimize γ which satisfies the conditions in Theorem 8 and the corresponding almost complete MATLAB code is as follows,

```
» pvar s th gam;
» opvar T A B C1 C2 D1 D2;
» T=···; A=···; B=···; C1=···; C2=···;
» D1=···; D2=···; X=[−1, 0];
» prog=sosprogram([s,th],gam);
» [prog, P] = sos_posopvar(prog, [nx1,nx2],X,s,th,
                    degree,options1);
» [prog, Z] = sos_opvar(prog,[nx1 ny;nx2 0],X,s,;
                    theta,degree);
» E=(P*A+Z*C2)'*T+T'*(P*A+Z*C2);
```

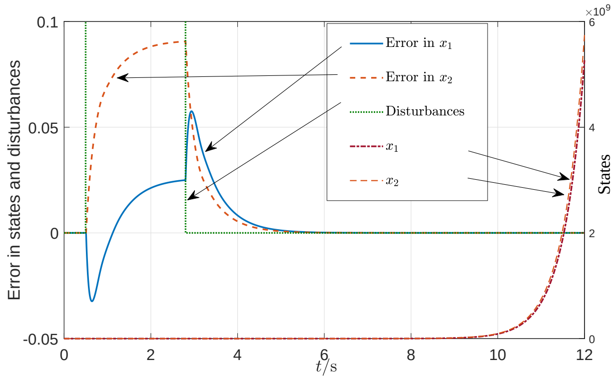

Fig. 1. Error and state dynamics to a step disturbance for E1

```
» Dop=[-gam*eye(nw)      -D1'     -(P*B+Z*D)'*T;
»              -D1       -gam*eye(nz)     C1;
»        -T'*(P*B+Z*D2)       C1'         E ];
» [prog, De1] = sos_posopvar(prog, [nw+nz+nx1 ,nx2],
                          X,s,theta,degree,options2);
» [prog, De2] = sos_posopvar(prog, [nw+nz+nx1 ,nx2],
                          X,s,theta,degree,options3);
» prog = sosopeq(prog,Dop+De1+De2);
» prog = sossetobj(prog, gam);
» prog = sosslove(prog);
```

For simulation, a fixed-step forward-difference-based discretization method is used, with a different set of states representing each delay channel. In the simulation results given below, 100 spatial discretization points are used for each delay channel.

We now apply the observer synthesis algorithm to several problems. In each case, the results are compared to an H_∞ optimal estimator designed for a discretized model obtained using a 10th order Padé approximation of the delay terms.

a) Example 1: Consider the following system,

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}-10 & 10\\ 0 & 1\end{bmatrix}x(t) + \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}x(t-0.3)\\ &\quad + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}w(t)\\ z(t) &= \begin{bmatrix}1.2 & 0\\ 0 & 1.2\end{bmatrix}x(t) + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}w(t)\\ y(t) &= \begin{bmatrix}0 & 10\end{bmatrix}x(t) + \begin{bmatrix}0 & 2\end{bmatrix}w(t)\end{aligned}$$

using Padé method and Thm. 8 ($d = 2$) gives us the same H_∞ norm of the optimal estimator out to 4 significant figures - $\gamma_{min} = 1.191$. Figure 1 displays the step disturbance $w(t)$ and error in states $e(t) = \hat{x}(t) - x(t)$ under the estimator we design. The real L_2 gain on the effect of the step disturbance to the error in regulated output under the estimator we design is $\gamma_{real} = 0.9502$.

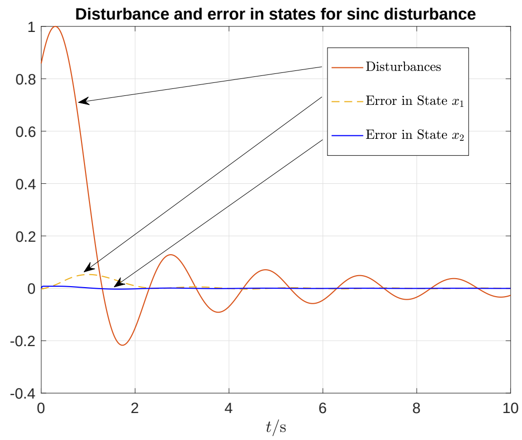

Fig. 2. Error dynamics to a sinc disturbance for E2

b) Example 2: Consider an example slightly changed from [15].

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}0 & 3\\ -4 & -5\end{bmatrix}x(t) + \begin{bmatrix}-0.1 & 0\\ 0.2 & -0.2\end{bmatrix}x(t-0.3)\\ &\quad + \begin{bmatrix}0 & 0.1\\ -0.2 & -0.3\end{bmatrix}x(t-0.5) + \begin{bmatrix}-0.4545 & 0\\ 0 & 0.9090\end{bmatrix}w(t)\\ y(t) &= \begin{bmatrix}0 & 100\end{bmatrix}x(t) + \begin{bmatrix}0 & 10\end{bmatrix}x(t-0.3)\\ &\quad + \begin{bmatrix}0 & 2\end{bmatrix}x(t-0.5) + \begin{bmatrix}1 & 1\end{bmatrix}w(t)\\ z(t) &= \begin{bmatrix}0 & 100\end{bmatrix}x(t)\end{aligned}$$

For comparison, we use a Padé approximation to get an estimator with estimated H_∞ disturbance rejection bound of 0.9592. Applying Theorem 8, our proposed algorithm obtains an H_∞ disturbance rejection bound of $\gamma_{min} = 0.9629$ for $d = 2$ and $\gamma_{min} = 0.9592$ for $d = 4$. The latter one for $d = 4$ is exactly the value as using Padé to 4 significant figures. When the observer gains are set as obtained for $d = 4$, a MATLAB simulation is shown in Fig. 2. This figure displays the sinc disturbance $w(t)$ and error in states $e(t) = \hat{x}(t) - x(t)$. The real L_2 gain on the effect of the sinc disturbance to the error in regulated output is $\gamma_{real} = 0.5792$.

c) Example 3: To test the computation load of our method, we consider the following unstable n-D system with K delays, a single disturbance $w(t)$ and a single regulated $z(t)$ and a single sensed output $y(t)$.

$$\begin{aligned}\dot{x}(t) &= -\sum_{i=1}^{K}\frac{x(t-i/K)}{K} + \mathbf{1}w(t)\\ z(t) &= y(t) = \mathbf{1}^T x(t) + \mathbf{1}^T w(t)\end{aligned}$$

The computational complexity is approximately a function of the product of the number of delays and number of states. Table I lists the detailed computation time as CPU sec on a Intel i7-5960X processor omitting preprocessing and postprocessing times.

VIII. Conclusion

We have investigated the problem of H_∞-optimal estimation problem of systems with multiple delays

TABLE I

K \ n	1	2	3	4	6
1	0.3610	0.4630	8.488	1.887	16.50
2	0.4380	1.573	11.94	77.94	950.8
3	0.9000	10.14	167.0	913.9	9827
4	1.331	82.92	912.6	4263	24030
6	12.10	967.2	9650	23980	N/A

and sensor noise. The commonly used delay-system equation is revalued into the fundamental-state-space representation and a convex optimization condition for the estimation synthesis problem in the form of a (Linear Operator Inequality) LOI is given. The effectiveness and non-conservative nature of our method has been verified using numerical simulation.

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Appendix

Details on the PIETOOLS toolbox can be found in [12] -including formulae for the addition, composition, concatenation and adjoint of PI operators. Thus in this section, we will only recall formulae used directly in the analysis - namely composition and parameterization of positive PI operators $\mathcal{P}\left[\begin{smallmatrix}P, & Q\\ Q^T, & \{R_i\}\end{smallmatrix}\right]$ on $Z_{n,nK}$ using positive matrices.

A. Enforcing Positivity of the operator

This lemma gives a map from positive matrices to positive PI operators.

Lemma 9: For any functions $Z_1 : [a,b] \to \mathbb{R}^{d_1\times n}, Z_2 : \left[[a,b]\times[a,b]\right] \to \mathbb{R}^{d_2\times n}$, suppose there exists a matrix $T \geq 0$ and $g(s) \geq 0$ for any $s \in [a,b]$ such that

$$
\begin{aligned}
P &= T_{11}\int_a^b g(s)ds\\
Q(\eta) &= g(\eta)T_{12}Z_1(\eta) + \int_\eta^b g(s)T_{13}Z_2(s,\eta)ds\\
&\quad + \int_a^\eta g(s)T_{14}Z_2(s,\eta)ds\\
R_1(s,\eta) &= g(s)Z_1(s)^T T_{23}Z_2(s,\eta) + g(\eta)Z_2(\eta,s)^T T_{42}Z_1(\eta)\\
&\quad + \int_s^b g(\theta)Z_2(\theta,s)^T T_{33}Z_2(\theta,\eta)d\theta\\
&\quad + \int_\eta^s g(\theta)Z_2(\theta,s)^T T_{43}Z_2(\theta,\eta)d\theta\\
&\quad + \int_a^\eta g(\theta)Z_2(\theta,s)^T T_{44}Z_2(\theta,\eta)d\theta\\
R_2(s,\eta) &= g(s)Z_1(s)^T T_{32}Z_2(s,\eta) + g(\eta)Z_2(\eta,s)^T T_{24}Z_1(\eta)\\
&\quad + \int_\eta^b g(\theta)Z_2(\theta,s)^T T_{33}Z_2(\theta,\eta)d\theta\\
&\quad + \int_s^\eta g(\theta)Z_2(\theta,s)^T T_{34}Z_2(\theta,\eta)d\theta\\
&\quad + \int_a^s g(\theta)Z_2(\theta,s)^T T_{44}Z_2(\theta,\eta)d\theta\\
R_0(s) &= g(s)Z_1(s)^T T_{22}Z_1(s)
\end{aligned}
$$

where

$$
T = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14}\\ T_{21} & T_{22} & T_{23} & T_{24}\\ T_{31} & T_{32} & T_{33} & T_{34}\\ T_{41} & T_{42} & T_{43} & T_{44}\end{bmatrix},
$$

then the operator $\mathcal{P}\left[\begin{smallmatrix}P, & Q\\ Q^T, & \{R_i\}\end{smallmatrix}\right]$ as defined in Eqn. 11 satisfies $\left\langle \begin{bmatrix}x\\ \phi\end{bmatrix}, \mathcal{P}\left[\begin{smallmatrix}P, & Q\\ Q^T, & \{R_i\}\end{smallmatrix}\right]\begin{bmatrix}x\\ \phi\end{bmatrix}\right\rangle_{Z_{m,n}} \geq 0$ for all $\begin{bmatrix}x\\ \phi\end{bmatrix} \in Z_{m,n}$.

Proof: Please see [13] for the proof. Specially, when we set $T_{i4} = T_{i3}$, $T_{4i} = T_{3i}$ for $i = 1,2$ and $T_{44} = T_{43} = T_{34} = T_{33}$ in Lemma 9, we get $R_1(s,\theta) = R_2(s,\theta)$, which turns $\mathcal{P}\left[\begin{smallmatrix}P, & Q\\ Q^T, & \{R_0, R_1, R_1\}\end{smallmatrix}\right]$ as

$$
\mathcal{P}\begin{bmatrix}x\\ \phi\end{bmatrix}(s) := \begin{bmatrix} Px + \int_{-1}^0 Q(\theta)\phi(\theta)d\theta \\ Q(s)^T x + R_0(s)\phi(s) + \int_{-1}^0 R_1(s,\theta)\phi(\theta)d\theta\end{bmatrix}. \tag{15}
$$

■

B. Composition

This Lemma gives a formula for composition of 2 PI operators.

Lemma 10: For any matrices $A, P \in \mathbb{R}^{m\times m}$ and bounded functions $B_1, Q_1 : [a,b] \to \mathbb{R}^{m\times n}$, $B_2, Q_2 : [a,b] \to \mathbb{R}^{n\times m}$, $C_0, R_0 : [a,b] \to \mathbb{R}^{n\times n}$, $C_i, R_i : [a,b]\times[a,b] \to \mathbb{R}^{n\times n}$ with $i \in \{1,2\}$, the following identity holds.

$$\mathcal{P}\left[\begin{smallmatrix} A, & B1\\ B2, & \{Ci\}\end{smallmatrix}\right]\mathcal{P}\left[\begin{smallmatrix} P, & Q1\\ Q2, & \{Ri\}\end{smallmatrix}\right] = \mathcal{P}\left[\begin{smallmatrix} \hat{P}, & \hat{Q}_1\\ \hat{Q}_2, & \{\hat{R}_i\}\end{smallmatrix}\right]$$

where

$$\begin{aligned}
\hat{P} &= AP + \int_0^L B_1(s)Q_2(s)\mathrm{d}s,\\
\hat{Q}_1(s) &= AQ_1(s) + B_1(s)R_0(s) + \int_s^L B_1(\eta)R_1(\eta,s)\mathrm{d}\eta\\
&\quad + \int_0^s B_1(\eta)R_2(\eta,s)\mathrm{d}\eta,\\
\hat{Q}_2(s) &= B_2(s)P + C_0(s)Q_2(s) + \int_0^s C_1(s,\eta)Q_2(\eta)\mathrm{d}\eta\\
&\quad + \int_s^L C_2(s,\eta)Q_2(\eta)\mathrm{d}\eta,\\
\hat{R}_0(s) &= C_0(s)R_0(s),\\
\hat{R}_1(s,\eta) &= B_2(s)Q_1(\eta) + C_0(s)R_1(s,\eta) + C_1(s,\eta)R_0(\eta)\\
&\quad + \int_0^\eta C_1(s,\theta)R_2(\theta,\eta)\mathrm{d}\theta + \int_\eta^s C_1(s,\theta)R_1(\theta,\eta)\mathrm{d}\theta\\
&\quad + \int_s^L C_2(s,\theta)R_1(\theta,\eta)\mathrm{d}\theta,\\
\hat{R}_2(s,\eta) &= B_2(s)Q_1(\eta) + C_0(s)R_2(s,\eta) + C_2(s,\eta)R_0(\eta)\\
&\quad + \int_0^s C_1(s,\theta)R_2(\theta,\eta)\mathrm{d}\theta + \int_s^\eta C_2(s,\theta)R_2(\theta,\eta)d\theta\\
&\quad + \int_\eta^L C_2(s,\theta)R_1(\theta,\eta)\mathrm{d}\theta.
\end{aligned}$$