Representation of PDE Systems With Delay and Stability Analysis Using Convex Optimization

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***Abstract*—Partial Integral Equations (PIEs) have been used to represent both systems with delay and systems of Partial Differential Equations (PDEs) in one or two spatial dimensions. In this letter, we show that these results can be combined to obtain a PIE representation of any suitably well-posed 1D PDE model with constant delay. In particular, we represent these delayed PDE systems as coupled systems of 1D and 2D PDEs, obtaining a PIE representation of both subsystems. Taking the feedback interconnection of these PIE subsystems, we then obtain a 2D PIE representation of the 1D PDE with delay. Next, based on the PIE representation, we formulate the problem of stability analysis as convex optimization of positive operators which can be solved using the PIETOOLS software suite. We apply the result to PDE examples with delay in the state and boundary conditions.**

***Index Terms*—Distributed parameter systems, delay systems, stability of linear systems, LMIs.**

I. Introduction

We consider the problem of analysis of coupled systems of Partial Differential Equations (PDEs). In both modeling and control of PDE systems, the evolution of the system often depends on the internal state of the system at earlier points in time, giving rise to delays in the model. For example, these delays may be inherent to the dynamics of the system itself, appearing within the PDE (sub)system, as in the following wave equation adapted from [1]

$$\mathbf{u}_{tt}(t,x)=\mathbf{u}_{xx}(t,x)+\mu\mathbf{u}_t(t,x)-\mathbf{u}_t(t-\tau,x),$$
$$\mathbf{u}(t,0)=\mathbf{u}(t,1)=0.$$

Alternatively, delay may appear in the Boundary Conditions (BCs) of the PDE, as in the following wave equation from [2]

$$\mathbf{u}_{tt}(t,x)=\mathbf{u}_{xx}(t,x),$$
$$\mathbf{u}(t,0)=0,\ \mathbf{u}_x(t,1)=(1-\mu)\mathbf{u}_t(t,1)+\mu\mathbf{u}_t(t-\tau,1).$$

In each case, the presence of delays naturally complicates analysis of solution properties such as stability of the system, as the state of the system involves not only the current value of the state $\mathbf{u}(t)$, but also the value of $\mathbf{u}(s)$ for all $s\in[t-\tau,t]$.

To verify stability of PDEs with delay, one common approach involves testing for existence of a positive definite functional V that decays along solutions to the system – i.e., a Lyapunov-Krasovskii Functional (LKF) [3]. In particular, for a delayed PDE with state $\mathbf{u}(t)$ and delayed state $\boldsymbol{\phi}(t)$ defined by $\boldsymbol{\phi}(t,s):=\mathbf{u}(t-s\tau)$ for $s\in[0,1]$, stability can be tested by finding a functional $V(\mathbf{u},\boldsymbol{\phi})$ that satisfies $V(0)=0$, $V(\mathbf{u},\boldsymbol{\phi})>0$ for $(\mathbf{u},\boldsymbol{\phi})\neq 0$, and $\dot V(\mathbf{u}(t),\boldsymbol{\phi}(t))\leq 0$ along all solutions to the system. In practice, a candidate LKF $V>0$ is usually fixed a priori, often as some variation on the energy functional $V(\mathbf{u},\boldsymbol{\phi})=\|\mathbf{u}\|^2+\|\boldsymbol{\phi}\|^2$, and then proven to decay along solutions to a system of interest. Although stability properties of a variety of PDEs with delay have been proven this way, including for heat and wave equations with both time-varying and constant delay [4], [5], results obtained in this manner are difficult to extend to other systems. Specifically, a LKF that certifies stability for one system may not be valid for another, and identifying a suitable candidate LKF for a given system requires significant insight.

In order to find LKFs for more general PDEs with delay, a cone of positive candidate functionals $V>0$ is often parameterized by positive definite matrices $P\succ 0$. The challenge in testing stability then becomes that of enforcing decay of the functionals, $\dot V\leq 0$, as a Linear Matrix Inequality (LMI), $Q\preceq 0$, which can be efficiently solved using semidefinite programming. Unfortunately, enforcing $\dot V\leq 0$ along solutions to a PDE with delay as an LMI is complicated by the fact that PDE dynamics are defined by (unbounded) differential operators, and that solutions are constrained to satisfy BCs. As such, most prior work in this field focuses only on specific PDEs with delay, exploiting the structure of the PDE (parabolic, hyperbolic, elliptic) and the type of BCs (Dirichlet, Neumann, Robin) to enforce $\dot V\leq 0$. For example, stability tests for heat and wave equations were derived in [1], posing $\dot V\leq 0$ as an LMI using the Wirtinger inequality. LMIs for stability of linear and semi-linear diffusive PDEs with delay were similarly derived in [6], [7], [8], as well as for reaction-diffusion systems with delayed boundary inputs in [9].

The disadvantage of these approaches, however, is that the results are again valid only for a restricted class of systems, and rely on the use of specific inequalities (e.g., Wirtinger, Jensen, Poincaré) to enforce $\dot V\leq 0$. Extending these results to even slightly different models, then, may require significant expertise from the user.

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In this letter, we propose an alternative, LMI-based method for testing stability of a general class of linear PDE systems with constant delay, by representing them as Partial Integral Equations (PIEs). A PIE is an alternative representation of linear ODE-PDE systems, taking the form [10]

$$\mathcal{T}\mathbf{v}_t(t) = \mathcal{A}\mathbf{v}(t),$$

where the operators $\{\mathcal{T}, \mathcal{A}\}$ are Partial Integral (PI) operators. In [10] and [11], it was shown that the sets of 1D and 2D PI operators form *-algebras, meaning that the sum, composition, and adjoint of such PI operators is a PI operator as well. As such, parameterizing Lyapunov functionals $V(\mathbf{v}(t)) = \langle\mathcal{T}\mathbf{v}(t), \mathcal{P}\mathcal{T}\mathbf{v}(t)\rangle$ by PI operators $\mathcal{P} \succ 0$, the decay condition $\dot{V}(\mathbf{v}(t)) \le 0$ can be enforced as a Linear PI Inequality (LPI)

$$\mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} \preceq 0. \tag{1}$$

Such LPIs constitute a specific class of linear operator inequalities (introduced for stability analysis of PDEs with delay in [1]), wherein the operator variable $\mathcal{P}$ has the structure of a PI operator. Since the fundamental state $\mathbf{v}(t) \in L_2$ in the PIE representation is not constrained by, e.g., BCs, these LPI constraints need only be enforced on L_2. Parameterizing positive PI operators $\mathcal{P} \succeq 0$ by positive matrices $P \succeq 0$, then, LPIs can be readily tested as LMIs, allowing analysis, control, and estimation of PDEs to be performed using convex optimization (see [12] and references therein).

In [13], it was shown that a general class of linear Delay Differential Equations (DDEs) can be equivalently represented as PIEs. Similarly, in [14], it was shown that any suitably well-posed PDE system without delay can also be equivalently represented as a PIE. However, constructing a PIE representation for 1D PDE systems with delay is complicated by the fact that the delayed state $\boldsymbol{\phi}(t, s, x) = \mathbf{u}(t - s, x)$ in this case varies in two spatial variables. To address this problem, in this letter, we decompose the delayed PDE into a feedback interconnection of a 1D PDE and a 2D transport equation, where the interconnection signals are infinite-dimensional. We prove that each of these subsystems can be equivalently represented as an associated PIE with infinite-dimensional inputs and outputs, extending prior work on PIE input-output systems to the case of infinite-dimensional inputs and outputs. Next, we consider the feedback interconnection of PIEs with infinite-dimensional inputs and outputs, deriving explicit expressions for the operators defining the resulting closed-loop PIE. Finally, paramaterizing a LKF by PI operators, we establish stability conditions in terms of LPI constraints. These LPIs are then converted to semidefinite programming problems using the PIETOOLS software package and tested on several examples of delayed PDE systems.

II. Problem Formulation

A. Notation

For a given domain $\Omega \subset \mathbb{R}^d$, let $L_2^n[\Omega]$ and $L_\infty^n[\Omega]$ denote the sets of $\mathbb{R}^n$-valued square-integrable and bounded functions on Ω, respectively, where we omit the domain when clear from context. Define intervals $\Omega_0^1 := [0, 1]$ and $\Omega_a^b := [a, b]$, and let $\Omega_{0a}^{1b} := \Omega_0^1 \times \Omega_a^b$. For $\mathrm{k} = (k_1, k_2) \in \mathbb{N}^2$, define Sobolev subspaces $H_{k_1}^n[\Omega_a^b]$ and $H_{\mathrm{k}}^n[\Omega_{0a}^{1b}]$ of L_2^n as

$$H_{k_1}^n[\Omega_a^b] = \left\{\mathbf{v} \mid \partial_x^\alpha \mathbf{v} \in L_2^n[\Omega_a^b],\ \forall \alpha \in \mathbb{N} : \alpha \le k_1\right\},$$
$$H_{\mathrm{k}}^n[\Omega_{0a}^{1b}] = \left\{\mathbf{v} \mid \partial_s^{\alpha_1}\partial_x^{\alpha_2}\mathbf{v} \in L_2^n[\Omega_{0a}^{1b}],\ \forall \left[\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right] \in \mathbb{N}^2 : \begin{smallmatrix}\alpha_1 \le k_1\\ \alpha_2 \le k_2\end{smallmatrix}\right\}.$$

B. Objectives and Approach

In this letter, we propose a framework for testing exponential stability of linear, 1D, 2nd order PDEs, with delay in the dynamics. Specifically, we focus on systems of the form

$$\mathbf{u}_t(t, x) = A(x)\begin{bmatrix}\mathbf{u}(t, x)\\ \mathbf{u}_x(t, x)\\ \mathbf{u}_{xx}(t, x)\end{bmatrix} + A_d(x)\begin{bmatrix}\mathbf{u}(t-\tau, x)\\ \mathbf{u}_x(t-\tau, x)\\ \mathbf{u}_{xx}(t-\tau, x)\end{bmatrix}, \tag{2}$$
$$\mathbf{u}(t) \in X_B[\Omega_a^b],\ t \ge 0,\ \ x \in \Omega_a^b,$$

where $A, A_d \in L_\infty^{n\times 3n}[\Omega_a^b]$, and where the PDE domain X_B is constrained by boundary conditions and continuity constraints, and is defined by a matrix $B \in \mathbb{R}^{2n\times 4n}$ as

$$X_B[\Omega_a^b] := \left\{\mathbf{u} \in H_2^n[\Omega_a^b] \;\middle|\; B\begin{bmatrix}\mathbf{u}(a)\\ \mathbf{u}(b)\\ \mathbf{u}_x(a)\\ \mathbf{u}_x(b)\end{bmatrix} = 0\right\}, \tag{3}$$

where B must be of full row-rank, defining sufficient and independent boundary conditions (see also [10, Sec. 3.2]). Because of limited space, in this letter, we will not explicitly consider cases wherein there is a delayed term in, e.g., the boundary conditions, or in some ODE coupled to the PDE. However, the methodology presented in this letter can be readily adapted to those cases as well, as well as to cases with multiple delayed signals, and Nth order PDEs. More details on such generalizations can be found in the extended version of this letter [15].

In order to test stability of the delayed PDE (2), we will first derive an equivalent representation of the system as a Partial Integral Equation (PIE), taking the form

$$(\mathcal{T}\mathbf{w}_t)(t, s, x) = (\mathcal{A}\mathbf{w})(t, s, x),\ (s, x) \in \Omega_{0a}^{1b}, \tag{4}$$

wherein the state $\mathbf{w}(t) \in L_2[\Omega_a^b] \times L_2[\Omega_{0a}^{1b}]$ is free of boundary conditions, and where the operators $\mathcal{T}$ and $\mathcal{A}$ are Partial Integral (PI) operators, defined as in Block 1. Using these operators, stability of the PDE can then be tested as follows.

Proposition 1: Let $A, A_d \in L_\infty^{n\times 3n}$, $B \in \mathbb{R}^{2n\times 4n}$ and $\tau > 0$ define a delayed PDE as in (2). Define PI operators $\mathcal{T}, \mathcal{A}$ as in Block 1. Suppose that there exist constants $\epsilon > 0$, $\alpha \ge 0$, and a PI operator $\mathcal{P}$ such that $\mathcal{P} = \mathcal{P}^*$, $\mathcal{P} \succeq \epsilon^2 I$, and

$$\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} \preceq -2\alpha\mathcal{T}^*\mathcal{P}\mathcal{T}. \tag{5}$$

Finally, let $\zeta = \sqrt{\|\mathcal{P}\|_{\mathcal{L}}}$. Then, for any solution $\mathbf{u}$ to the PDE (2) with $\boldsymbol{\phi}(t, s) = \mathbf{u}(t - s\tau)$ for $s \in [0, 1]$, we have

$$\left\|\begin{bmatrix}\mathbf{u}(t)\\ \boldsymbol{\phi}(t)\end{bmatrix}\right\|_Z \le \frac{\zeta}{\epsilon}\left\|\begin{bmatrix}\mathbf{u}(0)\\ \boldsymbol{\phi}(0)\end{bmatrix}\right\|_Z e^{-\alpha t}, \quad \forall t \ge 0,$$

where $\left\|\left[\begin{smallmatrix}\mathbf{u}(t)\\ \boldsymbol{\phi}(t)\end{smallmatrix}\right]\right\|_Z^2 := \|\mathbf{u}(t)\|_{L_2}^2 + \int_0^1 \|\boldsymbol{\phi}(t, s)\|_{L_2}^2 ds$.

In the remainder of this letter, we show how we arrive at this result, explicitly proving it in Corollary 2. In particular, to derive this result, we take the following four steps:

1. First, in Section III, we represent the delayed PDE as the interconnection of a 1D PDE and a 2D PDE.
2. Then, in Sections III-A and III-B, we derive equivalent 1D and 2D PIE representations of the 1D and 2D PDE subsystems, respectively.
3. Next, in Section IV, we prove that the feedback interconnection of PIEs can be represented as a PIE as well,

Block 1 Operators $\mathcal{T}$ and $\mathcal{A}$ Defining the PIE (4) Associated to the Delayed PDE (2) With Boundary Conditions as in (3)

Define $\mathcal{T} := \begin{bmatrix}\mathcal{T}_1 & 0\\ \mathcal{T}_1 & \mathcal{T}_2\end{bmatrix}$, $\mathcal{A} := \begin{bmatrix}\mathcal{A}_{11}+\mathcal{A}_{11,\mathrm{d}} & \mathcal{A}_{12}\\ 0 & \mathcal{A}_{22}\end{bmatrix}$, where $\mathcal{T}_1 := \mathcal{P}_{\mathrm{T}}$, $\mathcal{A}_{11} := \mathcal{P}_{\mathrm{A}}$, and $\mathcal{A}_{11,d} := \mathcal{P}_{\mathrm{A}_d}$ are 3-PI operators (see Defn. 2), and where for $\hat{\mathbf{v}} \in L_2[\Omega_{0a}^{1b}]$,

$$(\mathcal{T}_2\hat{\mathbf{v}})(s) := \int_0^s(\mathcal{T}_1\hat{\mathbf{v}}(\theta))d\theta, \qquad (\mathcal{A}_{12}\hat{\mathbf{v}}) := \int_0^1(\mathcal{A}_{11,\mathrm{d}}\hat{\mathbf{v}}(s))ds, \qquad (\mathcal{A}_{22}\hat{\mathbf{v}})(s) := -\tfrac{1}{\tau}(\mathcal{T}_1\hat{\mathbf{v}}(s)),$$

with parameters

$$\begin{aligned}&\mathrm{T} := \{0, T_1, T_2\}\\ &T_1(x,\theta) := (x-\theta)I_n + T_2(x,\theta),\\ &T_2(x,\theta) := -K(x)(BH)^{-1}BQ(x,\theta),\\ &K(x) := \begin{bmatrix}I_n & (x-a)I_n\end{bmatrix},\end{aligned} \quad H := \begin{bmatrix}I_n 0\\ I_n(b-a)I_n\\ 0I_n\\ 0I_n\end{bmatrix}, \quad Q(x,\theta) := \begin{bmatrix}0\\ (b-\theta)I_n\\ 0\\ I_n\end{bmatrix}, \quad \begin{aligned}&A_0(x) := A(x)\begin{bmatrix}0\\0\\I_n\end{bmatrix},\\ &A_{d,0}(x) := A_d(x)\begin{bmatrix}0\\0\\I_n\end{bmatrix},\end{aligned}$$

$$\begin{aligned}&\mathrm{A} := \{A_0, A_1, A_2\},\\ &\mathrm{A}_d := \{A_{d,0}, A_{d,1}, A_{d,2}\},\end{aligned} \; A_j(x,\theta) := A(x)\begin{bmatrix}T_j(x,\theta)\\ \partial_x T_j(x,\theta)\\ 0\end{bmatrix}, \qquad A_{d,j}(x,\theta) := A_d(x)\begin{bmatrix}T_j(x,\theta)\\ \partial_x T_j(x,\theta)\\ 0\end{bmatrix}, \qquad j \in \{1,2\}.$$

and take the interconnection of the 1D and 2D PIEs to obtain a PIE representation for the delayed PDE.

4. Finally, in Section V, we provide a Lyapunov functional based stability test for the PIE representation, allowing us to test stability of the original delayed PDE.

III. A PIE Representation of Delayed PDEs

In order to test stability of the Delayed PDE (DPDE) (2), we first represent the system in a format free of explicit delay. In particular, let $\boldsymbol{\phi}(t,s)$ represent $\mathbf{u}(t-s\tau)$ for $s \in [0,1]$. Then, $\mathbf{u}(t)$ satisfies the DPDE (2) if and only if $(\mathbf{u}(t), \boldsymbol{\phi}(t))$ satisfies

$$\begin{aligned}\mathbf{u}_t(t) &= \mathrm{M}_A\begin{bmatrix}\mathbf{u}(t)\\ \mathbf{u}_x(t)\\ \mathbf{u}_{xx}(t)\end{bmatrix} + \mathrm{M}_{A_\mathrm{d}}\begin{bmatrix}\boldsymbol{\phi}(t,1)\\ \boldsymbol{\phi}_x(t,1)\\ \boldsymbol{\phi}_{xx}(t,1)\end{bmatrix}, \mathbf{u}(t) \in X_B, \\ \boldsymbol{\phi}_t(t) &= -(1/\tau)\boldsymbol{\phi}_s(t), \boldsymbol{\phi}(t) \in Y_{\mathbf{u}(t)},\end{aligned} \tag{6}$$

where M_A denotes the multiplier operator associated to $A \in L_\infty[\Omega_a^b]$, and where we define the domain of $\boldsymbol{\phi}(t)$ as

$$Y_{\mathbf{u}} := \left\{\boldsymbol{\phi} \in H^n_{(1,2)}[\Omega_{0a}^{1b}] \,\middle|\, \boldsymbol{\phi}(0,x) = \mathbf{u}(x),\ \ \boldsymbol{\phi}(s,.) \in X_B\right\}. \tag{7}$$

We define solutions to the DPDE in terms of this format.

Definition 1 (Solution to the DPDE): For a given initial state $(\mathbf{u}_0, \boldsymbol{\phi}_0) \in X_B \times Y_{\mathbf{u}_0}$, we say that $(\mathbf{u}, \boldsymbol{\phi})$ is a solution to the DPDE defined by $\{A, A_\mathrm{d}, B, \tau\}$ if $(\mathbf{u}, \boldsymbol{\phi})$ is Frechét differentiable, $(\mathbf{u}(0), \boldsymbol{\phi}(0)) = (\mathbf{u}_0, \boldsymbol{\phi}_0)$, and for all $t \geq 0$, $\big(\mathbf{u}(t), \boldsymbol{\phi}(t)\big)$ satisfies (6).

Although the representation in (6) no longer involves explicit time-delay, stability analysis is still complicated by the auxiliary constraints $\begin{bmatrix}\mathbf{u}(t)\\ \boldsymbol{\phi}(t)\end{bmatrix} \in \begin{bmatrix}X_B\\ Y_{\mathbf{u}(t)}\end{bmatrix}$. Therefore, in the following subsections, we will separately consider the dynamics of $\mathbf{u}(t)$ and $\boldsymbol{\phi}(t)$, representing these dynamics in an equivalent format free of auxiliary constraints – as PIEs.

A. A PIE Representation of 1D PDEs

Consider the 1D subsystem of the coupled PDE in (6),

$$\mathbf{u}_t(t) = \mathrm{M}_A\begin{bmatrix}\mathbf{u}(t)\\ \mathbf{u}_x(t)\\ \mathbf{u}_{xx}(t)\end{bmatrix} + \mathbf{p}(t), \qquad \mathbf{u}(t) \in X_B, \tag{8}$$

where now $\mathbf{p}(t) = \mathrm{M}_{A_\mathrm{d}}\begin{bmatrix}\boldsymbol{\phi}(t,1)\\ \boldsymbol{\phi}_x(t,1)\\ \boldsymbol{\phi}_{xx}(t,1)\end{bmatrix} \in L_2^n\big[\Omega_a^b\big]$ is considered to be an input. In this system, we note that the 2nd-order derivative $\mathbf{v}(t) := \mathbf{u}_{xx}(t) \in L_2^n[\Omega_a^b]$ does not have to satisfy any boundary conditions or continuity constraints. Accordingly, we refer to $\mathbf{v}(t)$ as the *fundamental state* associated to the PDE, and we will derive an equivalent representation of the 1D PDE subsystem in (8) in terms of this state, as a PIE. To this end, we first recall the definition of a 3-PI operator.

Definition 2 (3-PI Operators (Π_3)): For $m, n \in \mathbb{N}$, define

$$\mathcal{N}_3^{m\times n}[\Omega_a^b] := L_2^{m\times n}[\Omega_a^b] \times L_2^{m\times n}[\Omega_a^b \times \Omega_a^b] \times L_2^{m\times n}[\Omega_a^b \times \Omega_a^b].$$

Then, for given parameters $\mathrm{R} := \{R_0, R_1, R_2\} \in \mathcal{N}_3$, we define the *3-PI operator* $\mathcal{R} = \mathcal{P}_\mathrm{R}$ for $\mathbf{u} \in L_2^n[\Omega_a^b]$ as

$$\big(\mathcal{R}\mathbf{u}\big)(x) = R_0(x)\mathbf{u}(x) + \int_a^x R_1(x,\theta)\mathbf{u}(\theta)d\theta + \int_x^b R_2(x,\theta)\mathbf{u}(\theta)d\theta.$$

We say $\mathcal{R} \in \Pi_3$ if $\mathcal{R} := \mathcal{P}_\mathrm{R}$ for some $\mathrm{R} \in \mathcal{N}_3$.

Defining 3-PI operators in this manner, it has been shown that Π_3 forms a *-algebra – i.e., is closed under summation, composition, scalar-multiplication and adjoint with respect to L_2 [14]. Moreover, under mild assumptions on the boundary conditions B, we can define a continuous, bijective map $\mathcal{T}_1 : L_2^n \to X_B$ from the fundamental to the PDE state space as a 3-PI operator, as shown in the following result from [10].

Lemma 1: Let X_B be as defined in (3), for some $B \in \mathbb{R}^{2n\times 4n}$ such that $BH \in \mathbb{R}^{2n\times 2n}$ is invertible with H as in Block 1. If $\mathcal{T}_1 \in \Pi_3$ is as defined in Block 1, then,

$$\mathbf{u} = \mathcal{T}_1(\partial_x^2\mathbf{u}),\ \forall \mathbf{u} \in X_B \quad \text{and} \quad \mathbf{v} = \partial_x^2(\mathcal{T}_1\mathbf{v}),\ \forall \mathbf{v} \in L_2^n.$$

Proof: Defining K, H, Q, T_j as in Block 1, and using Cauchy's formula for repeated integration, we can show that

$$\begin{aligned}\mathbf{u} &= \mathrm{M}_K\begin{bmatrix}\mathbf{u}(a)\\ \mathbf{u}_x(a)\end{bmatrix} + \mathcal{P}_{\{0,T_1-T_2,0\}}\mathbf{u}_{xx},\\ \begin{bmatrix}\mathbf{u}(a)\\ \mathbf{u}(b)\\ \mathbf{u}_x(a)\\ \mathbf{u}_x(b)\end{bmatrix} &= \begin{bmatrix}\mathbf{u}(a)\\ \mathbf{u}_x(a)\end{bmatrix} + \mathcal{P}_{\{0,Q,Q\}}\mathbf{u}_{xx},\ \forall \mathbf{u} \in H_2^n[\Omega_a^b].\end{aligned}$$

Imposing the boundary conditions in (3), it then follows that $\mathbf{u} = (\mathcal{P}_{\{0,T_1-T_2,0\}} - \mathrm{M}_K(BH)^{-1}B\mathcal{P}_{\{0,Q,Q\}})\mathbf{u}_{xx} = \mathcal{T}_1\mathbf{u}_{xx}$. A full proof is given in [10]. ■

Lemma 1 proves that, given sufficiently well-posed boundary conditions, any $\mathbf{u} \in X_B$ is uniquely defined by its highest-order partial derivative $\mathbf{v} = \mathbf{u}_{xx} \in L_2^n$ as $\mathbf{u} = \mathcal{T}_1\mathbf{v}$. Using the Leibniz integral rule, we can then also express

$$\mathrm{M}_A\begin{bmatrix}\mathbf{u}\\ \mathbf{u}_x\\ \mathbf{u}_{xx}\end{bmatrix} = \mathrm{M}_A\begin{bmatrix}\mathcal{T}_1\mathbf{u}_{xx}\\ \partial_x\mathcal{T}_1\mathbf{u}_{xx}\\ \mathbf{u}_{xx}\end{bmatrix} = \mathrm{M}_A\begin{bmatrix}\mathcal{P}_{\{0,T_1,T_2\}}\mathbf{v}\\ \mathcal{P}_{\{0,\partial_x T_1,\partial_x T_2\}}\mathbf{v}\\ \mathcal{P}_{\{I_n,0,0\}}\mathbf{v}\end{bmatrix} = \mathcal{A}_{11}\mathbf{v},$$

for $\mathcal{A}_{11} \in \Pi_3$ as in Block 1. It follows that $\mathbf{u}(t)$ satisfies the PDE (8) if and only if $\mathbf{v}(t) = \mathbf{u}_{xx}(t)$ satisfies the PIE

$$\mathcal{T}_1\mathbf{v}_t(t) = \mathcal{A}_{11}\mathbf{v}(t) + \mathbf{p}(t), \qquad \mathbf{v}(t) \in L_2^n[\Omega_a^b]. \tag{9}$$

Lemma 2: Suppose that $A \in L_\infty^{n\times 3n}[\Omega_a^b]$ and $B \in \mathbb{R}^{2n\times 4n}$ satisfies the conditions of Lemma 1. Define operators $\mathcal{T}_1, \mathcal{A}_1 \in \Pi_3$ as in Block 1. Then, for any given input $\mathbf{p}(t) \in L_2^n[\Omega_a^b]$, $\mathbf{v}$ is a solution to the PIE (9) with initial state $\mathbf{v}_0 \in L_2^n[\Omega_a^b]$ if and only if $\mathbf{u} = \mathcal{T}_1\mathbf{v}$ is a solution to the PDE (8) with initial state $\mathbf{u}_0 = \mathcal{T}_1\mathbf{v}_0$. Conversely, $\mathbf{u}$ is a solution to the PDE (8) with initial state $\mathbf{u}_0 \in X_B$ if and only if $\mathbf{v} = \partial_x^2\mathbf{u}$ is a solution to the PIE (9) with initial state $\mathbf{v}_0 = \partial_x^2\mathbf{u}_0$.

Proof: We refer to [10] for a proof. ∎

B. A PIE Representation of 2D Transport Equations

Consider now the 2D subsystem of the coupled PDE in (6),

$$\begin{aligned} &\boldsymbol{\phi}_t(t) = -(1/\tau)\boldsymbol{\phi}_s(t), \boldsymbol{\phi}(t) \in Y_{\mathcal{T}_1\mathbf{v}(t)}, \\ &\mathbf{p}(t) = \mathrm{M}_{A_d}\begin{bmatrix} \boldsymbol{\phi}(t,1) \\ \boldsymbol{\phi}_x(t,1) \\ \boldsymbol{\phi}_{xx}(t,1)\end{bmatrix}, \end{aligned} \tag{10}$$

wherein we consider $\mathbf{v}(t) = \mathbf{u}_{xx}(t) \in L_2^n[\Omega_a^b]$ as an input, and $\mathbf{p}(t) \in L_2^n[\Omega_a^b]$ as an output. Although a framework for constructing PIE representations for 2D PDEs has been developed in [11], in this case, we can significantly simplify this construction by exploiting the structure of the 2D subsystem. In particular, by definition of the space $Y_{\mathbf{u}(t)}$, any $\boldsymbol{\phi}(t) \in Y_{\mathbf{u}(t)}$ must satisfy the same boundary conditions as $\mathbf{u}(t)$. As such, we can use the same operator $\mathcal{T}_1$ as in Lemma 1 to also express $\boldsymbol{\phi}(t)$ in terms of its associated fundamental state $\boldsymbol{\phi}_{sxx}(t)$.

Lemma 3: Let $Y_\mathbf{u}$ be as defined in (7), with the set X_B as defined in (3) for some $B \in \mathbb{R}^{2n\times 4n}$ satisfying the conditions of Lemma 1. If $\mathcal{T}_1 \in \Pi_3$ and $\mathcal{T}_2$ are as defined in Block 1 and $\mathbf{u} \in X_B$, then, for every $\boldsymbol{\phi} \in Y_\mathbf{u}$ and every $\boldsymbol{\psi} \in L_2^n[\Omega_{0a}^{1b}]$,

$$\boldsymbol{\phi} = \mathbf{u} + \mathcal{T}_2(\partial_s\partial_x^2\boldsymbol{\phi}), \text{ and } \boldsymbol{\psi} = \partial_s\partial_x^2(\mathbf{u} + \mathcal{T}_2\boldsymbol{\psi}).$$

Proof: Fix arbitrary $\mathbf{u} \in X_B$ and $\boldsymbol{\phi} \in Y_\mathbf{u}$. By definition of the set $Y_\mathbf{u}$, we have $\boldsymbol{\phi}(0) = \mathbf{u}$ and $\boldsymbol{\phi}(s) \in X_B$ for all $s \in [0, 1]$. By Lemma 1, then, $\boldsymbol{\phi}(s) = \mathcal{T}_1(\partial_x^2\boldsymbol{\phi}(s))$ for all $s \in [0, 1]$, implying that also

$$\partial_s\boldsymbol{\phi}(s) = \partial_s\mathcal{T}_1(\partial_x^2\boldsymbol{\phi}(s)) = \mathcal{T}_1(\partial_s\partial_x^2\boldsymbol{\phi}(s)).$$

Invoking the fundamental theorem of calculus, and using the definition of the operator $\mathcal{T}_2$, it follows that

$$\begin{aligned} \boldsymbol{\phi}(s) &= \boldsymbol{\phi}(0) + \int_0^s \partial_s\boldsymbol{\phi}(\theta)d\theta \\ &= \mathbf{u} + \int_0^s \mathcal{T}_1(\partial_s\partial_x^2\boldsymbol{\phi}(\theta))d\theta = \mathbf{u} + \big(\mathcal{T}_2(\partial_s\partial_x^2\boldsymbol{\phi})\big)(s). \end{aligned}$$

Now, fix arbitrary $\boldsymbol{\psi} \in L_2^n[\Omega_{0a}^{1b}]$. Then, for all $s \in [0, 1]$,

$$\partial_s\partial_x^2\big(\mathbf{u} + (\mathcal{T}_2\boldsymbol{\psi})(s)\big) = \partial_x^2\partial_s\left(\int_0^s \mathcal{T}_1\big(\boldsymbol{\psi}(\theta)\big)d\theta\right) = \partial_x^2\mathcal{T}_1(\boldsymbol{\psi}(s)).$$

Here, by Lemma 1, $\partial_x^2\mathcal{T}_1(\boldsymbol{\psi}(s)) = \boldsymbol{\psi}(s)$ for all $s \in [0, 1]$. ∎

By Lemma 3, $\boldsymbol{\phi}(t) = \mathcal{T}_1\mathbf{v}(t) + \mathcal{T}_2\boldsymbol{\psi}(t)$ with $\boldsymbol{\psi}(t) = \boldsymbol{\phi}_{sxx}(t)$ for any $\boldsymbol{\phi}(t) \in Y_{\mathcal{T}_1\mathbf{v}(t)}$. Defining operators $\{\mathcal{A}_{22}, \mathcal{A}_{11,d}, \mathcal{A}_{12}\}$ as in Block 1, then, we can show that $(\boldsymbol{\phi}, \mathbf{v}, \mathbf{p})$ satisfies the 2D PDE (10) if and only if $(\boldsymbol{\psi}, \mathbf{v}, \mathbf{p})$ satisfies the 2D PIE

$$\begin{aligned} &\mathcal{T}_1\mathbf{v}_t(t) + \mathcal{T}_2\boldsymbol{\psi}_t(t) = \mathcal{A}_{22}\boldsymbol{\psi}(t), \boldsymbol{\psi}(t) \in L_2^n[\Omega_{0a}^{1b}], \\ &\mathbf{p}(t) = \mathcal{A}_{11,d}\mathbf{v}(t) + \mathcal{A}_{12}\boldsymbol{\psi}(t). \end{aligned} \tag{11}$$

Lemma 4: Suppose that $A_d \in L_\infty^{n\times 3n}[\Omega_a^b]$ and $\tau > 0$, and that $B \in \mathbb{R}^{2n\times 4n}$ satisfies the conditions of Lemma 1. Define PI operators $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{A}_{22}, \mathcal{A}_{11,d}, \mathcal{A}_{12}\}$ as in Block 1. Then, for any given input $\mathbf{v}(t) \in L_2^n[\Omega_a^b]$, $(\boldsymbol{\psi}, \mathbf{p})$ solves the PIE (11) with initial state $\boldsymbol{\psi}_0 \in L_2^n[\Omega_{0a}^{1b}]$ if and only if $\boldsymbol{\phi} = \mathcal{T}_1\mathbf{v} + \mathcal{T}_2\boldsymbol{\psi}$ solves the PDE (10) with initial state $\boldsymbol{\phi}_0 = \mathcal{T}_1\mathbf{v}(0) + \mathcal{T}_2\boldsymbol{\psi}_0$. Conversely, $(\boldsymbol{\phi}, \mathbf{p})$ solves the PDE (10) with initial state $\boldsymbol{\phi}_0 \in Y_{\mathcal{T}_1\mathbf{v}(0)}$ if and only if $\boldsymbol{\psi} = \partial_s\partial_x^2\boldsymbol{\phi}$ solves the PIE (11) with initial state $\boldsymbol{\psi}_0 = \partial_s\partial_x^2\boldsymbol{\phi}_0$.

Proof: Fix arbitrary $\boldsymbol{\psi}_0 \in L_2^n[\Omega_{0a}^{1b}]$ and $\mathbf{v}(t) \in L_2^n[\Omega_a^b]$ for $t \geq 0$. Let $\boldsymbol{\psi}(t) \in L_2[\Omega_{0a}^{1b}]$ and define $\boldsymbol{\phi}(t) = \mathcal{T}_1\mathbf{v}(t) + \mathcal{T}_2\boldsymbol{\psi}(t)$. By Lemma 3, $\boldsymbol{\phi}(t) \in Y_{\mathcal{T}_1\mathbf{v}(t)}$. In addition, it is clear that $\boldsymbol{\psi}(0) = \boldsymbol{\psi}_0$ if and only if $\boldsymbol{\phi}(0) = \mathcal{T}_1\mathbf{v}(0) + \mathcal{T}_2\boldsymbol{\psi}_0$. Moreover, since $\mathcal{T}_1\mathbf{v}(t)$ does not vary in $s \in [0, 1]$,

$$\boldsymbol{\phi}_s(t) = \partial_s\mathcal{T}_2\boldsymbol{\psi}(t) = \partial_s\int_0^s \big(\mathcal{T}_1\boldsymbol{\psi}\big)(t, \theta)d\theta = \mathcal{T}_1\boldsymbol{\psi}(t) = -\tau\mathcal{A}_{22}\boldsymbol{\psi}(t),$$

noting that $\mathcal{A}_{22} = -\frac{1}{\tau}\mathcal{T}_1$. It follows that

$$\mathcal{T}_1\mathbf{v}_t(t) + \mathcal{T}_2\boldsymbol{\psi}_t(t) - \mathcal{A}_{22}\boldsymbol{\psi}(t) = \boldsymbol{\phi}_t(t) + (1/\tau)\boldsymbol{\phi}_s(t). \tag{12}$$

Furthermore, we note that, for any $\mathbf{w} \in L_2^n[\Omega_a^b]$,

$$\begin{aligned} \partial_x\big(\mathcal{T}_1\mathbf{w}\big) &= \partial_x\int_a^x T_1(x, \theta)\mathbf{w}(\theta)d\theta + \partial_x\int_x^b T_2(x, \theta)\mathbf{w}(\theta)d\theta \\ &= T_1(x, x)\mathbf{w}(x) + \int_a^x \partial_x T_1(x, \theta)\mathbf{w}(\theta)d\theta \\ &\quad - T_2(x, x)\mathbf{w}(x) + \int_x^b \partial_x T_2(x, \theta)\mathbf{w}(\theta)d\theta \\ &= \int_a^x \partial_x T_1(x, \theta)\mathbf{w}(\theta)d\theta + \int_x^b \partial_x T_2(x, \theta)\mathbf{w}(\theta)d\theta. \end{aligned}$$

By definition of the operators $\mathcal{A}_{11,d}$ and $\mathcal{A}_{12}$, it follows that

$$\begin{aligned} &\big(\mathcal{A}_{11,d}\mathbf{v}\big)(t, x) + \big(\mathcal{A}_{12}\boldsymbol{\psi}\big)(t, x) \\ &= A_d(x)\left(\begin{bmatrix} I \\ \partial_x \\ \partial_x^2\end{bmatrix}\big(\mathcal{T}_1\mathbf{v}\big)(t, x) + \int_0^1 \begin{bmatrix} I \\ \partial_x \\ \partial_x^2\end{bmatrix}\big(\mathcal{T}_1\boldsymbol{\psi}\big)(t, s, x)ds\right) \\ &= A_d(x)\begin{bmatrix} I \\ \partial_x \\ \partial_x^2\end{bmatrix}\boldsymbol{\phi}(t, 1, x). \end{aligned} \tag{13}$$

By (12) and (13), we conclude that $(\boldsymbol{\psi}(t), \mathbf{p}(t))$ satisfies the PIE (11) if and only if $(\boldsymbol{\phi}(t), \mathbf{p}(t))$ satisfies the PDE (10).

For the converse result, let $\boldsymbol{\phi}_0 \in Y_{\mathcal{T}_1\mathbf{v}(0)}$ and $\boldsymbol{\phi}(t) \in Y_{\mathcal{T}_1\mathbf{v}(t)}$, and define $\boldsymbol{\psi}_0 = \partial_s\partial_x^2\boldsymbol{\phi}_0$ and $\boldsymbol{\psi}(t) = \partial_s\partial_x^2\boldsymbol{\phi}(t)$. By Lemma 3, $\boldsymbol{\phi}(t) = \mathcal{T}_1\mathbf{v}(t) + \mathcal{T}_2\boldsymbol{\psi}(t)$ and $\boldsymbol{\phi}_0 = \mathcal{T}_1\mathbf{v}(0) + \mathcal{T}_2\boldsymbol{\psi}_0$. By the first implication, it follows that $(\boldsymbol{\phi}, \mathbf{p})$ is a solution to the PDE with initial state $\boldsymbol{\phi}_0$ if and only if $(\boldsymbol{\psi}, \mathbf{p})$ is a solution to the PIE with initial state $\boldsymbol{\psi}_0$. ∎

IV. FEEDBACK INTERCONNECTION OF PIES

Having constructed a PIE representation of both the 1D and 2D subsystems of the PDE (6), we now take the feedback interconnection of these PIE subsystems to obtain a PIE representation for the full delayed PDE. This PIE will have state $\mathbf{w}(t) = \begin{bmatrix} \mathbf{v}(t) \\ \boldsymbol{\psi}(t)\end{bmatrix} \in \mathrm{Z}^{(n,n)}[\Omega_{0a}^{1b}]$, where we define

$$\mathrm{Z}^{(n_1,n_2)}[\Omega_{0a}^{1b}] := L_2^{n_1}[\Omega_a^b] \times L_2^{n_2}[\Omega_{0a}^{1b}]. \tag{14}$$

We represent a generalized PIE on such a state space as

$$\begin{aligned} \mathcal{T}_p\mathbf{p}_t(t) + \mathcal{T}\mathbf{w}_t(t) &= \mathcal{A}\mathbf{w}(t) + \mathcal{B}\mathbf{p}(t), \mathbf{w}(t) \in \mathrm{Z}^{n_\mathrm{w}}[\Omega_{0a}^{1b}], \\ \mathbf{q}(t) &= \mathcal{C}\mathbf{w}(t) + \mathcal{D}\mathbf{p}(t), \end{aligned} \tag{15}$$

with input $\mathbf{p}(t) \in Z^{n_p}$ and output $\mathbf{q}(t) \in Z^{n_q}$ for $n_w, n_p, n_q \in \mathbb{N}^2$, and where $\mathcal{T}_p$ through $\mathcal{D}$ are all PI operators. We collect these PI operators as $\mathbf{G} := \{\mathcal{T}, \mathcal{T}_p, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$, writing $\mathbf{G} \in \Pi^{(n_v,n_q)\times(n_v,n_p)}$, or $\mathbf{G} := \{\mathcal{T}, \mathcal{A}\} \in \Pi^{n_v\times n_v}$ if $n_p = n_q = 0$.

Definition 3 (Solution to the PIE): For a given input signal $\mathbf{p}$ and initial state $\mathbf{w}_0 \in Z^{n_v}$, we say that $(\mathbf{w}, \mathbf{q})$ is a solution to the PIE defined by $\mathbf{G} := \{\mathcal{T}, \mathcal{T}_p, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ if $\mathbf{w}$ is Fréchet differentiable, $\mathbf{w}(0) = \mathbf{w}_0$, and for all $t \geq 0$, $(\mathbf{w}(t), \mathbf{q}(t), \mathbf{p}(t))$ satisfies Eqn. (15).

Using the composition and addition rules of PI operators, we now show that the feedback interconnection of two suitable PIEs as in (15) can be represented as a PIE as well.

Proposition 2 (Interconnection of PIEs): Let

$$\mathbf{G}_1 := \{\mathcal{T}_1, \mathcal{T}_p, \mathcal{A}_1, \mathcal{B}_p, \mathcal{C}_q, \mathcal{D}_{qp}\} \in \Pi^{(n_1,n_q)\times(n_1,n_p)},$$
$$\mathbf{G}_2 := \{\mathcal{T}_2, \mathcal{T}_q, \mathcal{A}_2, \mathcal{B}_q, \mathcal{C}_p, 0\} \in \Pi^{(n_2,n_p)\times(n_2,n_q)},$$

and define $\mathbf{G} := \{\mathcal{T}, \mathcal{A}\} \in \Pi^{n_v\times n_v}$ with $n_v = n_1 + n_2$ as

$$\mathcal{T} := \begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_p\mathcal{C}_p \\ \mathcal{T}_q\mathcal{C}_q & \mathcal{T}_2 + \mathcal{T}_q\mathcal{D}_{qp}\mathcal{C}_p \end{bmatrix}, \quad \mathcal{A} := \begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_p\mathcal{C}_p \\ \mathcal{B}_q\mathcal{C}_q & \mathcal{A}_2 + \mathcal{B}_q\mathcal{D}_{qp}\mathcal{C}_p \end{bmatrix}.$$

Then, $\left[\begin{smallmatrix} \mathbf{v} \\ \boldsymbol{\psi} \end{smallmatrix}\right]$ solves the PIE defined by $\mathbf{G}$ with initial state $\left[\begin{smallmatrix} \mathbf{v}_0 \\ \boldsymbol{\psi}_0 \end{smallmatrix}\right]$ if and only if $(\mathbf{v}, \mathbf{q})$ and $(\boldsymbol{\psi}, \mathbf{p})$ solve the PIEs defined by $\mathbf{G}_1$ and $\mathbf{G}_2$ with initial states $\mathbf{v}_0$ and $\boldsymbol{\psi}_0$ and inputs $\mathbf{p}$ and $\mathbf{q}$, respectively, where for all $t \geq 0$

$$\mathbf{p}(t) = \mathcal{C}_p\boldsymbol{\psi}(t), \qquad \mathbf{q}(t) = \mathcal{C}_q\mathbf{v}(t) + \mathcal{D}_{qp}\mathbf{p}(t). \tag{16}$$

Proof: Let $\mathbf{v}(t) \in Z^{n_1}$ and $\boldsymbol{\psi}(t) \in Z^{n_2}$ for $t \geq 0$. Then, $\mathbf{p}(t)$ and $\mathbf{q}(t)$ satisfy the PIEs defined by $\mathbf{G}_2$ and $\mathbf{G}_1$, respectively, if and only if they are as in (16). In that case,

$$\begin{aligned} \mathcal{T}\begin{bmatrix} \mathbf{v}_t(t) \\ \boldsymbol{\psi}_t(t) \end{bmatrix} - \mathcal{A}\begin{bmatrix} \mathbf{v}(t) \\ \boldsymbol{\psi}(t) \end{bmatrix} &= \begin{bmatrix} \mathcal{T}_1\mathbf{v}_t(t) + \mathcal{T}_p\mathcal{C}_p\boldsymbol{\psi}_t(t) \\ \mathcal{T}_q\mathcal{C}_q\mathbf{v}_t(t) + \mathcal{T}_2\boldsymbol{\psi}_t(t) + \mathcal{T}_q\mathcal{D}_{qp}\mathcal{C}_p\boldsymbol{\psi}_t(t) \end{bmatrix} \\ &\quad - \begin{bmatrix} \mathcal{A}_1\mathbf{v}(t) + \mathcal{B}_p\mathcal{C}_p\boldsymbol{\psi}(t) \\ \mathcal{B}_q\mathcal{C}_q\mathbf{v}(t) + \mathcal{A}_2\boldsymbol{\psi}(t) + \mathcal{B}_q\mathcal{D}_{qp}\mathcal{C}_p\boldsymbol{\psi}(t) \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{T}_p\mathbf{p}_t(t) + \mathcal{T}_1\mathbf{v}_t(t) - \mathcal{A}_1\mathbf{v}(t) - \mathcal{B}_p\mathbf{p}(t) \\ \mathcal{T}_q\mathbf{q}_t(t) + \mathcal{T}_2\boldsymbol{\psi}_t(t) - \mathcal{A}_2\boldsymbol{\psi}(t) - \mathcal{B}_q\mathbf{q}(t) \end{bmatrix}. \end{aligned}$$

From this expression, it follows that $\left[\begin{smallmatrix} \mathbf{v}(t) \\ \boldsymbol{\psi}(t) \end{smallmatrix}\right]$ satisfies the PIE defined by $\mathbf{G}$ if and only if $\mathbf{v}(t)$ and $\boldsymbol{\psi}(t)$ satisfy the PIEs defined by $\mathbf{G}_1$ and $\mathbf{G}_2$, respectively. ■

Using this result, we finally construct a PIE representation for the full DPDE in (6).

Corollary 1: Suppose that $A, A_d \in L_\infty^{n\times 3n}[\Omega_a^b]$, $\tau > 0$ and $B \in \mathbb{R}^{2n\times 4n}$ satisfies the conditions of Lemma 1. Define $\mathcal{T}$ and $\mathcal{A}$ as in Block 1. Then, $\left[\begin{smallmatrix} \mathbf{v} \\ \boldsymbol{\psi} \end{smallmatrix}\right]$ is a solution to the PIE defined by $\{\mathcal{T}, \mathcal{A}\}$ with initial state $\left[\begin{smallmatrix} \mathbf{v}_0 \\ \boldsymbol{\psi}_0 \end{smallmatrix}\right]$ if and only if $\left[\begin{smallmatrix} \mathbf{u} \\ \boldsymbol{\phi} \end{smallmatrix}\right] = \mathcal{T}\left[\begin{smallmatrix} \mathbf{v} \\ \boldsymbol{\psi} \end{smallmatrix}\right]$ is a solution to the DPDE defined by $\{A, A_d, B, \tau\}$ with initial state $\left[\begin{smallmatrix} \mathbf{u}_0 \\ \boldsymbol{\phi}_0 \end{smallmatrix}\right] = \mathcal{T}\left[\begin{smallmatrix} \mathbf{v}_0 \\ \boldsymbol{\psi}_0 \end{smallmatrix}\right]$. Conversely, $\left[\begin{smallmatrix} \mathbf{u} \\ \boldsymbol{\phi} \end{smallmatrix}\right]$ is a solution to the DPDE defined by $\{A, A_d, B, \tau\}$ with initial state $\left[\begin{smallmatrix} \mathbf{u}_0 \\ \boldsymbol{\phi}_0 \end{smallmatrix}\right]$ if and only if $\left[\begin{smallmatrix} \mathbf{v} \\ \boldsymbol{\psi} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \partial_x^2\mathbf{u} \\ \partial_s\partial_x^2\boldsymbol{\phi} \end{smallmatrix}\right]$ is a solution to the PIE defined by $\{\mathcal{T}, \mathcal{A}\}$ with initial state $\left[\begin{smallmatrix} \mathbf{v}_0 \\ \boldsymbol{\psi}_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \partial_x^2\mathbf{u}_0 \\ \partial_s\partial_x^2\boldsymbol{\phi}_0 \end{smallmatrix}\right]$.

Proof: By definition of the operators $\mathcal{T}, \mathcal{A}$, and invoking Proposition 2, $\left[\begin{smallmatrix} \mathbf{v} \\ \boldsymbol{\psi} \end{smallmatrix}\right]$ is a solution to the PIE defined by $\{\mathcal{T}, \mathcal{A}\}$ if and only if $\mathbf{v}$ and $\boldsymbol{\psi}$ are solutions to the PIEs (9) and (11), respectively. By Lemma 2 and Lemma 4, it follows that $\left[\begin{smallmatrix} \mathbf{v} \\ \boldsymbol{\psi} \end{smallmatrix}\right]$ is a solution to the PIE defined by $\{\mathcal{T}, \mathcal{A}\}$ if and only if $\mathbf{u} = \mathcal{T}_1\mathbf{v}$ and $\boldsymbol{\phi} = \mathcal{T}_1\mathbf{v} + \mathcal{T}_2\boldsymbol{\psi}$ are solutions to the PDEs (8) and (10), respectively. Taking the interconnection of these PDEs, we finally conclude that $\left[\begin{smallmatrix} \mathbf{v} \\ \boldsymbol{\psi} \end{smallmatrix}\right]$ is a solution to the PIE defined by $\{\mathcal{T}, \mathcal{A}\}$ if and only if $\left[\begin{smallmatrix} \mathbf{u} \\ \boldsymbol{\phi} \end{smallmatrix}\right] = \mathcal{T}\left[\begin{smallmatrix} \mathbf{v} \\ \boldsymbol{\psi} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{T}_1 & 0 \\ \mathcal{T}_1 & \mathcal{T}_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathbf{v} \\ \boldsymbol{\psi} \end{smallmatrix}\right]$ is a solution to the DPDE (6), defined by $\{A, A_d, B, \tau\}$. The converse result follows by similar reasoning. ■

V. Testing Stability in the PIE Representation

Having established a bijective map between the solution of the DPDE (6) and that of an associated PIE, we now show how this PIE can be used to formulate a convex optimization problem to test stability of the DPDE. To derive this test, we use the following inner product on $\mathbf{w} = \left[\begin{smallmatrix} \mathbf{v} \\ \boldsymbol{\psi} \end{smallmatrix}\right] \in Z^n[\Omega]$,

$$\langle \mathbf{w}_1, \mathbf{w}_2\rangle_Z = \langle \mathbf{v}_1, \mathbf{v}_2\rangle_{L_2[\Omega_a^b]} + \langle \boldsymbol{\psi}_1, \boldsymbol{\psi}_2\rangle_{L_2[\Omega_{0a}^{1b}]}.$$

Theorem 1: Let $\{\mathcal{T}, \mathcal{A}\} \in \Pi^{n\times n}$, and suppose that there exist constants $\epsilon > 0$, $\alpha \geq 0$, and a PI operator $\mathcal{P} : Z^n \to Z^n$ such that $\mathcal{P} = \mathcal{P}^*$, $\mathcal{P} \succeq \epsilon^2 I$, and

$$\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} \preceq -2\alpha\mathcal{T}^*\mathcal{P}\mathcal{T}. \tag{17}$$

Then, any solution $\mathbf{w}$ to the PIE defined by $\{\mathcal{T}, \mathcal{A}\}$ satisfies

$\|\mathcal{T}\mathbf{w}(t)\|_Z \leq (\zeta/\epsilon)\|\mathcal{T}\mathbf{w}(0)\|_Z\, e^{-\alpha t}$, where $\zeta := \sqrt{\|\mathcal{P}\|_{\mathcal{L}_Z}}$.

Proof: Consider the candidate Lyapunov functional $V(\mathbf{w}) = \langle \mathcal{T}\mathbf{w}, \mathcal{P}\mathcal{T}\mathbf{w}\rangle_Z$. Since $\mathcal{P} \succeq \epsilon^2 I$ and $\|\mathcal{P}\|_{\mathcal{L}_Z} = \zeta^2$, this function is bounded below as $V(\mathbf{w}) \geq \epsilon^2\|\mathcal{T}\mathbf{w}\|_Z^2$, and bounded above as $V(\mathbf{w}) \leq \zeta^2\|\mathcal{T}\mathbf{w}\|_Z^2$, for all $\mathbf{w} \in Z^n$. Now, let $\mathbf{w}$ be an arbitrary solution to the PIE defined by $\{\mathcal{T}, \mathcal{A}\}$. Then, the temporal derivative of V along $\mathbf{w}$ satisfies

$$\begin{aligned} \dot V(\mathbf{w}) &= \langle \mathcal{A}\mathbf{w}, \mathcal{P}\mathcal{T}\mathbf{w}\rangle_Z + \langle \mathcal{T}\mathbf{w}, \mathcal{P}\mathcal{A}\mathbf{w}\rangle_Z \\ &= \langle \mathbf{w}, (\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A})\mathbf{w}\rangle_Z \\ &\leq -2\alpha\langle \mathcal{T}\mathbf{w}, \mathcal{P}\mathcal{T}\mathbf{w}\rangle_Z = -2\alpha V(\mathbf{w}). \end{aligned}$$

Applying the Grönwall-Bellman inequality, it follows that $V(\mathbf{w}(t)) \leq V(\mathbf{w}(0))e^{-2\alpha t}$, and therefore $\|\mathcal{T}\mathbf{w}(t)\|_Z^2 \leq (\zeta/\epsilon)^2\|\mathcal{T}\mathbf{w}(0)\|_Z^2\, e^{-2\alpha t}$. ■

Theorem 1 shows that, for a PIE defined by $\{\mathcal{T}, \mathcal{A}\}$, feasibility of the Linear PI Inequality (LPI) (17) proves exponential stability of $\mathcal{T}\mathbf{w}$ for all solutions $\mathbf{w}$ to the PIE. Using equivalence of the DPDE (6) to the PIE defined in Corollary 1, we can then test stability of the DPDE as follows.

Corollary 2: Let $\{A, A_d, B, \tau\}$ define a DPDE system, and let $\{\mathcal{T}, \mathcal{A}\}$ be as in Corollary 1. If there exist $\epsilon, \zeta > 0$, $\alpha \geq 0$, and $\mathcal{P}$ satisfying the conditions of Theorem 1, then any solution $\left[\begin{smallmatrix} \mathbf{u} \\ \boldsymbol{\phi} \end{smallmatrix}\right]$ to the DPDE defined by $\{A, A_d, B, \tau\}$ satisfies

$$\left\|\left[\begin{smallmatrix} \mathbf{u}(t) \\ \boldsymbol{\phi}(t) \end{smallmatrix}\right]\right\|_Z \leq (\zeta/\epsilon)\left\|\left[\begin{smallmatrix} \mathbf{u}(0) \\ \boldsymbol{\phi}(0) \end{smallmatrix}\right]\right\|_Z e^{-\alpha t}.$$

Proof: Let $\left[\begin{smallmatrix} \mathbf{u} \\ \boldsymbol{\phi} \end{smallmatrix}\right]$ be a solution to the DPDE defined by $\{A, A_d, B, \tau\}$, and let $\left[\begin{smallmatrix} \mathbf{v} \\ \boldsymbol{\psi} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbf{u}_{xx} \\ \boldsymbol{\phi}_{sxx} \end{smallmatrix}\right]$. By Corollary 1, $\left[\begin{smallmatrix} \mathbf{v} \\ \boldsymbol{\psi} \end{smallmatrix}\right]$ solves the PIE defined by $\{\mathcal{T}, \mathcal{A}\}$, and $\left[\begin{smallmatrix} \mathbf{u} \\ \boldsymbol{\phi} \end{smallmatrix}\right] = \mathcal{T}\left[\begin{smallmatrix} \mathbf{v} \\ \boldsymbol{\psi} \end{smallmatrix}\right]$. Applying Theorem 1 with $\mathbf{w} = \left[\begin{smallmatrix} \mathbf{v} \\ \boldsymbol{\psi} \end{smallmatrix}\right]$, the result follows. ■

By Corollary 2, we can finally test stability of the DPDE (6), by solving the LPI (17). Note that stability is proven in the norm $\left\|\left[\begin{smallmatrix} \mathbf{u}(t) \\ \boldsymbol{\phi}(t) \end{smallmatrix}\right]\right\|_Z^2 = \|\mathbf{u}(t)\|_{L_2}^2 + \int_0^1 \|\boldsymbol{\phi}(t,s)\|_{L_2}^2 ds$, bounding both the PDE state $\mathbf{u}(t)$ and its history $\boldsymbol{\phi}(t,s) = \mathbf{u}(t - s\tau)$.

VI. Numerical Examples

In this section, we provide several numerical examples, illustrating how stability of different PDE systems with delay can be numerically tested by verifying feasibility of the LPI from Theorem 1. In each case, the PIETOOLS software package [12] is used to declare the delayed system as a coupled system of (ODEs and) PDEs, convert the system to an equivalent PIE, and declare and solve the stability LPI.

TABLE I
MAXIMAL DELAY τ_{LPI} FOR WHICH EXPONENTIAL STABILITY OF SYSTEM (18) WAS VERIFIED USING THEOREM 1 WITH $\epsilon = 10^{-2}$, $\alpha = 0$

r	0.1	0.5	0.8	1.2	1.5	1.9
$\bar{\tau}$	6.17258	2.41839	1.80870	1.39768	1.20920	1.03472
τ_{LPI}	6.17248	2.41837	1.80869	1.39767	1.20919	1.03470

TABLE II
DECAY RATES α FOR WHICH EXPONENTIAL STABILITY OF SYSTEM (19) WITH $\mu = 0.4$ WAS VERIFIED USING THEOREM 1 WITH $\epsilon = 10^{-3}$

τ	0.125	0.25	0.5	1.0	2.0	4.0	8.0
α	0.2023	0.1908	0.1513	0.1333	0.1060	0.0701	0.0135

A. Heat Equation With Delay in Dynamics

Consider the following PDE with delay from [1], [16]

$$\begin{aligned}&\mathbf{u}_t(t,x) = \mathbf{u}_{xx}(t,x) + r\mathbf{u}(t,x) - \mathbf{u}(t-\tau,x), \qquad x \in \Omega_0^{\pi},\\ &\mathbf{u}(t,0) = \mathbf{u}(t,\pi) = 0. \end{aligned} \tag{18}$$

Modeling the delay as a 2D transport equation, and using PIETOOLS, we obtain an equivalent PIE representation as

$$(\mathcal{T}\mathbf{v}_t)(t,x) = \mathbf{v}(t,x) + (r-1)(\mathcal{T}\mathbf{v})(t,x) - \int_0^1 (\mathcal{T}\boldsymbol{\psi})(t,s,x)ds,$$

$$(\mathcal{T}\mathbf{v}_t)(t,x) + \int_0^s (\mathcal{T}\boldsymbol{\psi}_t)(t,\nu,x)d\nu = -\frac{1}{\tau}(\mathcal{T}\boldsymbol{\psi})(t,s,x),$$

where $\mathbf{v}(t,x) = \partial_x^2\mathbf{u}(t,x)$, $\boldsymbol{\psi}(t,s,x) = \partial_s\partial_x^2\boldsymbol{\phi}(t,s,x)$, and

$$(\mathcal{T}\mathbf{v})(t,x) := \int_0^x \theta[x-1]\mathbf{v}(t,\theta)d\theta + \int_x^{\pi} x[\theta-1]\mathbf{v}(t,\theta)d\theta.$$

For $0 < r < 2$, the DPDE (18) is stable if and only if $\tau < \bar{\tau} := \frac{\cos^{-1}(r-1)}{\sqrt{2r-r^2}}$ [16]. Performing bisection on the delay τ, stability can be numerically verified for delays up to τ_{LPI} as in Table I. For each test, $\mathcal{P}$ in the LPI (17) was parameterized by $P \in \mathbb{R}^{27\times 27}$ (compare to $P \in \mathbb{R}^{5\times 5}$ for the LMI in [1]).

B. Wave Equation With Delay in Boundary Conditions

The methodology proposed in this letter can also be adapted to cases with delay in, e.g., the boundary conditions. For example, consider the wave equation

$$\begin{aligned}&\mathbf{u}_{tt}(t,x) = \mathbf{u}_{xx}(t,x) \;\; x \in \Omega_0^1,\\ &\mathbf{u}(t,0) = 0, \mathbf{u}_x(t,1) = (1-\mu)\mathbf{u}_t(t,1) + \mu\mathbf{u}_t(t-\tau,1). \end{aligned} \tag{19}$$

As shown in [2], this system is stable independent of delay if $\mu < \frac{1}{2}$, and unstable independent of delay if $\mu > \frac{1}{2}$. We examine the ability of the proposed algorithm to expand upon this result by determining bounds on the rate of decay for several values of μ and τ. First, introducing $\mathbf{u}_1(t) = \mathbf{u}(t)$, $\mathbf{u}_2(t) = \mathbf{u}_t(t)$, $\boldsymbol{\phi}_j(t,x) = \mathbf{u}_j(t-\tau x,1)$ and $u_0(t) = (1-\mu)\mathbf{u}_1(t,1) + \mu\boldsymbol{\phi}_1(t,1)$, we represent the system as

$$\begin{aligned}\dot{u}_0(t) &= \partial_x\mathbf{u}_1(t,1)\\ \partial_t\mathbf{u}_1(t,x) &= \mathbf{u}_2(t,x), \partial_t\mathbf{u}_2(t,x) = \partial_x^2\mathbf{u}_1(t,x),\\ \partial_t\boldsymbol{\phi}_1(t,x) &= -\frac{1}{\tau}\partial_x\boldsymbol{\phi}_1(t,x), \partial_t\boldsymbol{\phi}_2(t,x) = -\frac{1}{\tau}\partial_x\boldsymbol{\phi}_2(t,x),\\ \mathbf{u}_1(t,0) &= 0, \mathbf{u}_2(t,0) = 0,\\ \boldsymbol{\phi}_1(t,0) &= \mathbf{u}_1(t,1), \boldsymbol{\phi}_2(t,0) = \mathbf{u}_2(t,1),\\ u_0(t) &= (1-\mu)\mathbf{u}_1(t,1) + \mu\boldsymbol{\phi}_1(t,1),\\ \partial_x\mathbf{u}_1(t,1) &= (1-\mu)\mathbf{u}_2(t,1) + \mu\boldsymbol{\phi}_2(t,1),\end{aligned}$$

which can be readily converted to a PIE. Fixing $\tau = 1$, stability can then be numerically verified for any $\mu \leq 0.5 - 10^{-3}$. Next, fixing $\mu = 0.4$ and bisecting on the value of α, exponential decay rates can be computed as in Table I. For each test, the operator $\mathcal{P}$ was parameterized by a matrix $P \in \mathbb{R}^{73\times 73}$.

VII. CONCLUSION

In this letter, an LMI-based method for verifying stability of coupled, linear, delayed, PDE systems in a single spatial dimension was presented. In particular, it was shown that for any suitably well-posed PDE with delay, there exists an associated PIE with a corresponding bijective map from solution of the delayed PDE to that of the PIE. The PIE representation was then used to propose a stability test for the delayed PDE. This stability test was posed as a linear operator inequality on PI operator variables (an LPI). Finally, the PIETOOLS software package was used to convert the LPI to a semidefinite programming problem, and test stability of several examples of delayed PDEs. While these results only apply to fixed delays, an extension to time-varying delays may be possible using PDE representations such as in [4].

REFERENCES

[1] E. Fridman and Y. Orlov, "Exponential stability of linear distributed parameter systems with time-varying delays," *Automatica*, vol. 45, no. 1, pp. 194–201, 2009.
[2] G. Q. Xu, S. P. Yung, and L. K. Li, "Stabilization of wave systems with input delay in the boundary control," *ESAIM Control Optim. Calculus Variations*, vol. 12, no. 4, pp. 770–785, 2006.
[3] V. Kolmanovskii and A. Myshkis, *Applied Theory of Functional Differential Equations*, vol. 85. Dordrecht, The Netherlands: Springer, 2012.
[4] S. Nicaise, J. Valein, and E. Fridman, "Stability of the heat and of the wave equations with boundary time-varying delays," *Discrete Continuous Dyn. Syst.-Series S*, vol. 2, no. 3, pp. 559–581, 2009.
[5] K. Ammari, S. Nicaise, and C. Pignotti, "Feedback boundary stabilization of wave equations with interior delay," *Syst. Control Lett.*, vol. 59, no. 10, pp. 623–628, 2010.
[6] J.-W. Wang, C.-Y. Sun, X. Xin, and C.-X. Mu, "Sufficient conditions for exponential stabilization of linear distributed parameter systems with time delays," *IFAC Proc. Vol.*, vol. 47, no. 3, pp. 6062–6067, 2014.
[7] J.-W. Wang and C.-Y. Sun, "Delay-dependent exponential stabilization for linear distributed parameter systems with time-varying delay," *J. Dyn. Syst. Meas. Control*, vol. 140, no. 5, 2018, Art. no. 51003.
[8] O. Solomon and E. Fridman, "Stability and passivity analysis of semilinear diffusion PDEs with time-delays," *Int. J. Control*, vol. 88, no. 1, pp. 180–192, 2015.
[9] H. Lhachemi and C. Prieur, "Boundary output feedback stabilisation of a class of reaction-diffusion PDEs with delayed boundary measurement," *Int. J. Control*, vol. 96, pp. 2285–2295, Jun. 2022.
[10] M. M. Peet, "A partial integral equation (PIE) representation of coupled linear PDEs and scalable stability analysis using LMIs," *Automatica*, vol. 125, Mar. 2021, Art. no. 109473.
[11] D. S. Jagt and M. M. Peet, "A PIE representation of coupled 2D PDEs and stability analysis using LPIs," 2021, *arXiv:2109.06423*.
[12] S. Shivakumar, D. Jagt, D. Braghini, A. Das, and M. Peet, "PIETOOLS 2022: User manual," 2021, *arXiv:2101.02050*.
[13] M. Peet, "Representation of networks and systems with delay: DDEs, DDFs, ODE–PDEs and PIEs," *Automatica*, vol. 127, May 2021, Art. no. 109508.
[14] S. Shivakumar, A. Das, S. Weiland, and M. Peet, "Extension of the partial integral equation representation to GPDE input-output systems," 2022, *arxiv:2205.03735*.
[15] D. S. Jagt and M. M. Peet, "Representation of PDE systems with delay and stability analysis using convex optimization–extended version," 2022, *arXiv:2211.05326*.
[16] S. Y. Çalışkan and H. Özbay, "Stability analysis of the heat equation with time-delayed feedback," *IFAC Proc. Vol.*, vol. 42, no. 6, pp. 220–224, 2009.