H_∞-Optimal Estimator Synthesis for Linear 2D PDEs using Convex Optimization ⋆

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Abstract: Any suitably well-posed PDE in two spatial dimensions can be represented as a Partial Integral Equation (PIE) – with system dynamics parameterized using Partial Integral (PI) operators. Furthermore, L_2-gain analysis of PDEs with a PIE representation can be posed as a linear operator inequality, which can be solved using convex optimization. In this paper, these results are used to derive a convex-optimization-based test for constructing an H_∞-optimal estimator for 2D PDEs. In particular, a PIE representation is first derived for arbitrary well-posed 2D PDEs with sensor measurements along boundaries of the domain. An associated Luenberger-type estimator is then parameterized using a PI operator $\mathcal{L}$ as the observer gain. Next, it is shown that an upper bound on the H_∞-norm of the error dynamics for the estimator can be minimized by solving a linear operator inequality on PI operator variables. Finally, an analytical formula for inversion of a sub-class of 2D PI operators is derived and used to reconstruct the Luenberger gain $\mathcal{L}$. Results are implemented in the PIETOOLS software suite – applying the methodology and simulating the estimator for an unstable 2D heat equation.

Keywords: Distributed Parameter Systems, Observer Synthesis, PDEs, LMIs.

1. INTRODUCTION

Partial Differential Equations (PDEs) are frequently used to model physical systems, relating the temporal evolution of an internal state variable to its spatial distribution. For example, to model the density $\mathbf{u}(t,x,y)$ of an exponentially growing population on a 2D domain $(x,y)\in[0,1]^2$, we can use the following PDE (see e.g. Holmes et al. (1994))

$$\mathbf{u}_t(t) = \mathbf{u}_{xx}(t) + \mathbf{u}_{yy}(t) + r\mathbf{u}(t) + w(t), \qquad (1)$$
$$z(t) = \int_0^1\int_0^1 \mathbf{u}(t,x,y)dxdy,$$

wherein r is a parameter determining the population growth, $w(t)$ is an external disturbance, $z(t)$ denotes the total population size, and where the evolution of the state is further constrained by boundary conditions such as

$$\mathbf{u}(t,0,y) = \mathbf{u}_x(t,1,y) = \mathbf{u}(t,x,0) = \mathbf{u}_y(t,x,1) = 0. \qquad (2)$$

For state feedback control of such systems, we require real-time knowledge of the distributed internal state $\mathbf{u}(t)$. However, in practice, direct measurement of the distributed state would require a prohibitive number of sensors. To alleviate the sensing burden, therefore, we commonly make a smaller number of observations – typically on the boundary of the domain. For example, in the population model, we might only measure the population density on the upper boundaries ($x=1$, $y=1$), yielding observed outputs

$$\mathbf{q}_1(t,y) = \mathbf{u}(t,1,y) \qquad \text{and} \qquad \mathbf{q}_2(t,x) = \mathbf{u}(t,x,1). \qquad (3)$$

The role of an estimator, then, is to reconstruct the distributed state in the full domain from these limited observations. Unfortunately, designing an estimator is complicated by the infinite-dimensional nature of the system.

For comparison, consider a linear Ordinary Differential Equation (ODE), with a finite-dimensional state $u(t)\in\mathbb{R}^n$, sensed output $q(t)\in\mathbb{R}^m$, and regulated output $z(t)\in\mathbb{R}^p$, as

$$\dot{u}(t) = Au(t) + Bw(t), \qquad z(t) = Cu(t) + Dw(t),$$
$$q(t) = C_q u(t) + D_q w(t).$$

The most common approach for estimating the state, $u(t)$, is to construct a Luenberger-type observer, with state estimate $\hat{u}(t)$ and parameterized by a gain matrix L as

$$\dot{\hat{u}}(t) = A\hat{u}(t) + Bw(t) + L(C_q\hat{u}(t) - q(t)), \quad \hat{z}(t) = C\hat{u}(t),$$

where $\hat{z}(t)$ is the output estimate. Then, a matrix L which minimizes $\sup_{w\neq 0}\|\hat{z}-z\|_{L_2}/\|w\|_{L_2}$ (the H_∞-norm) may be found by solving the Linear Matrix Inequality (LMI)

$$\min_{\gamma>0,P,W} \gamma, \quad \text{s.t. } P\succ 0, \quad \begin{bmatrix} -\gamma I & -D & C \\ (\cdot)^* & -\gamma I & -PB^* - WD_q \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + PA + WC_q \end{bmatrix} \preceq 0,$$

and setting $L = P^{-1}W$ (see e.g. Duan and Yu (2013)).

However, consider now a 2D PDE such as in (1) with state $\mathbf{u}(t,x,y)$. Given observed outputs $\mathbf{q}(t) = (\mathbf{q}_1(t),\mathbf{q}_2(t))$ as in (3), it is relatively simple to define an equivalent of the Luenberger-type estimator for ODEs, where we have

$$\hat{\mathbf{u}}_t(t) = \hat{\mathbf{u}}_{xx}(t) + \hat{\mathbf{u}}_{yy}(t) + r\hat{\mathbf{u}}(t) + w(t) + \mathcal{L}(\hat{\mathbf{q}}(t) - \mathbf{q}(t)),$$
$$\hat{\mathbf{q}}_1(t,y) = \hat{\mathbf{u}}(t,1,y), \qquad \hat{\mathbf{q}}_2(t,x) = \hat{\mathbf{u}}(t,x,1),$$

with regulated output estimate $\hat{z}(t) = \int_0^1\int_0^1 \hat{\mathbf{u}}(t,x,y)dxdy$. However, finding an observer gain $\mathcal{L}$ that minimizes the H_∞-norm of the map from w to $\hat{z}-z$ is complicated by the fact that the state and sensed outputs are now infinite-dimensional, requiring us to parameterize infinite-dimensional operators and optimize performance of PDEs.

To avoid these challenges with estimator synthesis of PDE systems, a common approach is to project the PDE state onto a finite-dimensional subspace and synthesizing an estimator based on this finite-dimensional (ODE) approximation. Recent applications of this approach include: 1D systems with observer delay in Lhachemi and Prieur (2022), 1D stochastic systems in Wang and Fridman

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(2024), and 2D Navier-Stokes equations in Zayats et al. (2021), each deriving LMI conditions for verifying stability of the resulting error dynamics. However, parameterizing an estimator only by a finite-dimensional operator (a matrix) necessarily introduces conservatism. In addition, properties such as optimality of the estimator for the ODE do not a priori guarantee optimality or even convergence of the estimator for the PDE (see e.g. Zuazua (2005)), requiring conditions for convergence to be proven a posteriori.

Estimator synthesis of PDEs is also frequently performed using the backstepping method, in which a Luenberger-type estimator is parameterized by a multiplier operator, and convergence of the estimator is ensured by mapping the resulting error dynamics to a stable target system. Using this approach, estimators can be designed for a variety of PDEs, including e.g. 1D hyperbolic systems as in Yu et al. (2020) and PDEs in multiple spatial variables as in Jadachowski et al. (2015). However, the backstepping method also does not offer any guarantee of optimality of the obtained estimators, and introduces conservatism by parameterizing observer gains only by multiplier operators. In addition, each estimator is constructed only for a narrow class of systems – and extending the approach to new systems may require significant expertise.

In this paper we present an alternative, LMI-based method for constructing Luenberger-type estimators for 2nd order, 2D PDEs. In particular, we focus on systems of the form

$$\mathbf{u}_t(t) = \sum_{i,j=0}^{2} A_{ij}\partial_x^i\partial_y^j\mathbf{u}(t) + Bw(t), \qquad \mathbf{u}(t)\in X,$$
$$z(t) = \sum_{i,j=0}^{2} \int_{[0,1]^2}[C_{ij}]\partial_x^i\partial_y^j\mathbf{u}(t) + Dw(t),$$

where the set $X \subseteq L_2^n[[0,1]^2]$ is constrained by suitable linear boundary conditions. The value of the state is assumed to be observed along the boundary of the domain, yielding a sensed output along e.g. the boundary $y=1$ as

$$\mathbf{q}(t,x) = \sum_{i,j=0}^{1} F_{ij}(x)\partial_x^i\partial_y^j\mathbf{u}(t,x,1) + Gw(t).$$

Similar outputs on other boundaries are allowed as well. An estimator for the resulting system will then be constructed by adopting an approach similar to that presented for 1D PDEs with finite-dimensional sensed outputs $\mathbf{q}(t)\in\mathbb{R}^m$ in Das et al. (2019). In that paper, the 1D PDE was first converted to an equivalent Partial Integral Equation (PIE), expressing the dynamics of the system in terms of an associated *fundamental state* $\mathbf{v}(t)$ as

$$\mathcal{T}\mathbf{v}_t(t) = \mathcal{A}\mathbf{v}(t) + \mathcal{B}w(t), \qquad z(t) = \mathcal{C}\mathbf{v}(t) + \mathcal{D}w(t),$$
$$\mathbf{q}(t) = \mathcal{C}_q\mathbf{v}(t) + \mathcal{D}_q w(t),$$

where now the parameters ($\mathcal{T}$, $\mathcal{A}$, etc.) are all Partial Integral (PI) operators. Given this PIE representation of the system, the authors then proposed parameterizing a Luenberger-type estimator by a PI operator $\mathcal{L}$ as

$$\mathcal{T}\hat{\mathbf{v}}_t(t) = \mathcal{A}\hat{\mathbf{v}}(t) + \mathcal{B}w(t) + \mathcal{L}(\mathcal{C}_q\hat{\mathbf{v}} - \mathbf{q}(t)), \quad \hat{z}(t) = \mathcal{C}\hat{\mathbf{v}}(t).$$

Finally, the authors showed that if there exist PI operators $\mathcal{P}$ and $\mathcal{W}$ that solve the linear operator inequality

$$\min_{\gamma>0,\mathcal{P},\mathcal{W}} \gamma, \quad \text{s.t. } \mathcal{P}\succ 0, \qquad \begin{bmatrix} -\gamma I & -\mathcal{D} & \mathcal{C} \\ (\cdot)^* & -\gamma I & -[\mathcal{B}^*\mathcal{P} + \mathcal{D}_q^*\mathcal{W}^*]\mathcal{T} \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + \mathcal{T}^*[\mathcal{P}\mathcal{A} + \mathcal{W}\mathcal{C}_q] \end{bmatrix} \preceq 0, \tag{4}$$

then, using the estimator gain $\mathcal{L} = \mathcal{P}^{-1}\mathcal{W}$, the H_∞-norm of the associated map from w to $\hat{z}-z$ is upper-bounded by γ. This linear operator inequality can be efficiently solved using convex optimization methods with the PIETOOLS software suite (Shivakumar et al. (2021)).

Unfortunately, when using the approach presented in Das et al. (2019) to synthesize estimators for systems of 2D PDEs, we encounter several challenges. In particular, although it has been shown how a PIE representation can be constructed for a broad class of 2D PDEs with finite-dimensional inputs (see Jagt and Peet (2022)), a similar representation has not been derived for infinite-dimensional output signals, such as the sensed outputs $\mathbf{q}(t)\in L_2^m[0,1]$. Furthermore, since the sensed outputs $\mathbf{q}(t)$ are not finite-dimensional, this also requires a more complicated parameterization of the gain $\mathcal{L}: L_2^m[0,1]\to L_2^n[[0,1]^2]$ and hence the variable $\mathcal{W}$ in (4). Finally, although a Luenberger-type estimator for the 2D PIE representation may again be synthesized by solving the operator inequality in (4), computing the associated estimator gain $\mathcal{L}=\mathcal{P}^{-1}\mathcal{W}$ requires inverting the operator $\mathcal{P}$ – posing the challenge of computing the inverse of PI operators in 2D.

In the remainder of this paper, we address each of these challenges. First, in Subsec. 3.1, an equivalent PIE representation is derived for a broad class of 2D PDEs with infinite-dimensional (sensed) outputs. In Subsec. 3.2, $\mathcal{W}$ in (4) is then parameterized, and it is shown how optimal estimator synthesis for the PIE can be performed by solving the operator inequality. Finally, in Sec. 4, an explicit expression is derived for the inverse of a certain class of 2D PI operators, and the operator inequality in (4) is posed as an LMI. In Sec. 5, the methodology is implemented via the software suite PIETOOLS, and illustrated for an unstable heat equation using numerical simulation.

2. PRELIMINARIES

2.1 Notation

For a given domain $\Omega\subset\mathbb{R}^2$, let $L_2^n[\Omega]$ denote the set of $\mathbb{R}^n$-valued square-integrable functions on Ω, where we omit the domain when clear from context. Define $W_2^n[[0,1]^2]$ as a Sobolev subspace of $L_2^n[[0,1]^2]$, where

$$W_2^n[[0,1]^2] = \left\{\mathbf{v} \mid \partial_x^{\alpha_1}\partial_y^{\alpha_2}\mathbf{v}\in L_2^n[[0,1]^2],\ \forall\alpha\in\mathbb{N}^2: \alpha_1,\alpha_2\le 2\right\}.$$

For $\mathbf{v}\in W_2^n[[0,1]^2]$, denote the Dirac delta operators

$$(\Delta_x^0\mathbf{v})(y) := \mathbf{v}(0,y) \quad \text{and} \quad (\Delta_y^1\mathbf{v})(x) := \mathbf{v}(x,1).$$

For a function $N\in L_2^{m\times n}[[0,1]^2]$, define an associated multiplier operator $\mathrm{M}[N]:\mathbb{R}^n\to L_2^m[[0,1]^2]$ and integral operator $\int_{[0,1]^2}[N]: L_2^n[[0,1]^2]\to\mathbb{R}^m$ by

$$(\mathrm{M}[N]v)(x,y) := N(x,y)v, \qquad \forall v\in\mathbb{R}^n,$$
$$\left(\textstyle\int_{[0,1]^2}[N]\mathbf{v}\right) := \int_0^1\int_0^1 N(\theta,\eta)\mathbf{v}(\theta,\eta)d\eta d\theta, \quad \forall\mathbf{v}\in L_2^n[[0,1]^2].$$

2.2 Algebras of PI Operators on 2D Functions

Partial Integral (PI) operators are bounded, linear operators, parameterized by square-integrable functions. We briefly recall the definition and properties of a class of such operators on $L_2[[0,1]^2]$ here, referring to Jagt and Peet (2022) and the references therein for more details and proofs.

Definition 1. (2D PI Operators, $\mathbf{\Pi}_{2D}$). For given parameters $R := \left[\begin{smallmatrix} R_{00} & R_{01} & R_{02} \\ R_{10} & R_{11} & R_{12} \\ R_{20} & R_{21} & R_{22}\end{smallmatrix}\right]$ with $R_{00}\in L_2^{m\times n}[[0,1]^2]$, $R_{ij}\in L_2^{m\times n}[[0,1]^4]$, and $R_{i0},R_{0j}\in L_2^{m\times n}[[0,1]^3]$ for $i,j\in\{1,2\}$, define an associated operator $\Pi[R]: L_2^n[[0,1]^2]\to L_2^m[[0,1]^2]$ such that, for any $\mathbf{v}\in L_2^n[[0,1]^2]$,

$$
\begin{aligned}
&(\Pi[R]\mathbf{v})(x,y) := R_{00}(x,y)\mathbf{v}(x,y) \\
&+ \int_0^x R_{10}(x,y,\theta)\mathbf{v}(\theta,y)\,d\theta + \int_x^1 R_{20}(x,y,\theta)\mathbf{v}(\theta,y)\,d\theta \\
&+ \int_0^y R_{01}(x,y,\eta)\mathbf{v}(x,\eta)\,d\eta + \int_y^1 R_{02}(x,y,\eta)\mathbf{v}(x,\eta)\,d\eta \\
&+\int_0^y\!\left(\int_0^x R_{11}(x,y,\theta,\eta)\mathbf{v}(\theta,\eta)d\theta + \int_x^1 R_{21}(x,y,\theta,\eta)\mathbf{v}(\theta,\eta)d\theta\right)d\eta \\
&+\int_y^1\!\left(\int_0^x R_{12}(x,y,\theta,\eta)\mathbf{v}(\theta,\eta)d\theta + \int_x^1 R_{22}(x,y,\theta,\eta)\mathbf{v}(\theta,\eta)d\theta\right)d\eta.
\end{aligned}
$$

We refer to an operator $\mathcal{R} = \Pi[R]$ of this form as a 2D PI operator, writing $\mathcal{R} \in \mathbf{\Pi}_{2D}^{m\times n}$.

The structure of 2D PI operators will also be used to represent maps between $L_2^m[[0,1]^2]$ and $\mathbb{R}^{n_1} \times L_2^{n_2}[0,1]$. For example, for $Q := \left[\begin{smallmatrix} Q_0 & Q_1 & Q_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ and $R := \left[\begin{smallmatrix} 0 & 0 & 0 \\ R_0 & R_1 & R_2 \\ R_0 & R_1 & R_2 \end{smallmatrix}\right]$ with $Q_0, R_0 \in L_2^{m\times n}[[0,1]^2]$ and $Q_i, R_i \in L_2^{m\times n}[[0,1]^3]$, the associated PI operators $\Pi[Q] : L_2^n[0,1] \to L_2^m[[0,1]^2]$ and $\Pi[R] : L_2^n[[0,1]^2] \to L_2^m[0,1]$ take the form

$$
\begin{aligned}
&\big(\Pi[Q]\mathbf{u}\big)(x,y) = Q_0(x,y)\mathbf{u}(x) \qquad\qquad \forall \mathbf{u} \in L_2^n[0,1], \\
&\quad + \int_0^x Q_1(x,y,\theta)\mathbf{u}(\theta)d\theta + \int_x^1 Q_2(x,y,\theta)\mathbf{u}(\theta)d\theta, \\
&\big(\Pi[R]\mathbf{v}\big)(x) = \int_0^1 \bigg[R_0(x,\eta)\mathbf{v}(x,\eta) \qquad \forall \mathbf{v} \in L_2^n[[0,1]^2], \\
&\quad + \int_0^x R_1(x,\theta,\eta)\mathbf{v}(\theta,\eta)d\theta + \int_x^1 R_2(x,\theta,\eta)\mathbf{v}(\theta,\eta)d\theta\bigg]d\eta.
\end{aligned}
$$

Write $\Pi[Q] \in \mathbf{\Pi}_{2D\leftarrow 1D}^{m\times n}$ and $\Pi[R] \in \mathbf{\Pi}_{1D\leftarrow 2D}^{m\times n}$. Similarly, for $K \in L_2^{m\times n}[[0,1]^2]$, let $\Pi\left[\begin{smallmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] = \mathrm{M}[K]$ and $\Pi\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & K & K \\ 0 & K & K \end{smallmatrix}\right] = \int_{[0,1]^2}[K]$, writing $\mathrm{M}[K] \in \mathbf{\Pi}_{2D\leftarrow 0}^{m\times n}$ and $\int_{[0,1]^2}[K] : \mathbf{\Pi}_{0\leftarrow 2D}^{m\times n}$.

In Jagt and Peet (2022), it was shown that the sum $\mathcal{Q}+\mathcal{R}$, composition $\mathcal{Q}\mathcal{R}$, and adjoint $\mathcal{Q}^*$ of 2D PI operators $\mathcal{Q}, \mathcal{R}$ of suitable dimensions are PI operators as well, presenting explicit parameter maps defining each operation. These operations have also been implemented in the PIETOOLS software suite (Shivakumar et al. (2021)), allowing the sum, composition, and adjoint of PI operator objects `Q`,`R` to be readily computed as `Q+R`, `Q*R` and `Q'`, respectively.

2.3 A PIE Representation of 2D Input-Output PDEs

A Partial Integral Equation (PIE) is a linear differential equation, parameterized by PI operators, defining the dynamics of a state $\mathbf{v}(t) \in L_2^{n_u}$. For a system with finite-dimensional disturbance $w(t) \in \mathbb{R}^{n_w}$ and (regulated) output $z(t) \in \mathbb{R}^{n_z}$, a PIE on $\mathbf{v}(t)$ takes the form

$$
\mathcal{T}\mathbf{v}_t(t) = \mathcal{A}\mathbf{v}(t) + \mathcal{B}w(t), \quad z(t) = \mathcal{C}\mathbf{v}(t) + \mathcal{D}w(t), \tag{5}
$$

where the parameters ($\mathcal{T}$, $\mathcal{A}$, etc.) are PI operators.

Definition 2. (Solution to the PIE). For a given input w and initial value $\mathbf{v}_0 \in L_2^{n_u}$, $(\mathbf{v}, z)$ is a solution to the PIE defined by $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ if $\mathbf{v}$ is Frechét differentiable, $\mathbf{v}(0) = \mathbf{v}_0$, and for all $t \geq 0$, $(\mathbf{v}(t), z(t), w(t))$ satisfies (5).

It has previously been shown how a PIE representation can be constructed for a broad class of linear 2D PDEs with finite-dimensional inputs and outputs w, z. In this paper, we focus on 2nd order, coupled, 2D PDEs of the form

$$
\begin{aligned}
\mathbf{u}_t(t) &= \textstyle\sum_{i,j=0}^2 \mathrm{M}[A_{ij}]\partial_x^i\partial_y^j\mathbf{u}(t) + \mathrm{M}[B]w(t), \quad \mathbf{u}(t) \in X, \\
z(t) &= \textstyle\sum_{i,j=0}^2 \int_{[0,1]^2}[C_{ij}]\partial_x^i\partial_y^j\mathbf{u}(t) + \mathrm{M}[D]w(t),
\end{aligned} \tag{6}
$$

parameterized by matrix-valued functions

$$
\begin{bmatrix} A_{ij} & B \\ C_{ij} & D \end{bmatrix} \in \begin{bmatrix} L_\infty^{n_u\times n_u}[[0,1]^2] & L_2^{n_u\times n_w}[[0,1]^2] \\ L_2^{n_z\times n_u}[[0,1]^2] & \mathbb{R}^{n_z\times n_w} \end{bmatrix},
$$

and where the domain $X \subseteq W_2^{n_u}[[0,1]^2]$ of the state $\mathbf{u}(t)$ is defined by a set of suitable linear boundary conditions. These boundary conditions may be expressed in terms of all admissible derivatives of the state along the boundary of the domain, collecting these derivatives using the operator $\Lambda_{\text{bf}} := \left[\begin{smallmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{smallmatrix}\right] : L_2^{n_u}[[0,1]^2] \to \mathbb{R}^{16n_u} \times L_2^{8n_u}[0,1]$, where

$$
\Lambda_1 := \begin{bmatrix} \Delta_1 \\ \Delta_1\partial_x \\ \Delta_1\partial_y \\ \Delta_1\partial_x\partial_y \end{bmatrix}, \quad \Lambda_2 := \begin{bmatrix} \Delta_2\partial_x^2 \\ \Delta_2\partial_x^2\partial_y \end{bmatrix}, \quad \Lambda_3 := \begin{bmatrix} \Delta_3\partial_y^2 \\ \Delta_3\partial_x\partial_y^2 \end{bmatrix}, \tag{7}
$$

$$
\text{with} \quad \Delta_1 := \begin{bmatrix} \Delta_3\Delta_y^0 \\ \Delta_3\Delta_y^1 \end{bmatrix}, \quad \Delta_2 := \begin{bmatrix} \Delta_y^0 \\ \Delta_y^1 \end{bmatrix}, \quad \Delta_3 := \begin{bmatrix} \Delta_x^0 \\ \Delta_x^1 \end{bmatrix}.
$$

A general class of linear boundary conditions can then be parameterized by a matrix $E \in \mathbb{R}^{8n_u\times 24n_u}$ as

$$
\mathbf{u}(t) \in X := \left\{\mathbf{u} \in W_2^{n_u}[[0,1]^2] \;\middle|\; \mathrm{M}[E]\Lambda_{\text{bf}}\mathbf{u} = 0\right\}. \tag{8}
$$

Most common boundary conditions can be represented in this format, expressing e.g. the conditions in (2) as $\mathbf{u}(0,0) = \mathbf{u}_x(1,0) = \mathbf{u}_y(0,1) = \mathbf{u}_{xy}(1,1) = 0$, $\mathbf{u}_{xx}(\cdot,0) = \mathbf{u}_{xxy}(\cdot,1) = 0$ and $\mathbf{u}_{yy}(0,\cdot) = \mathbf{u}_{xyy}(1,\cdot) = 0$. Although, to reduce notation, disturbances in the boundary conditions will not be allowed here, such disturbances can be included using the approach presented in Jagt and Peet (2022).

For any $\mathbf{u}(t) \in X$, we define the associated *fundamental state* as $\mathbf{v}(t) := \partial_x^2\partial_y^2\mathbf{u}(t) \in L_2^{n_u}$ – free of any boundary conditions and continuity constraints. The following result from Jagt and Peet (2022) (Lem. 14) shows that if the boundary conditions are suitably well-defined, then we can define associated PI operators $\{\mathcal{T}, \ldots, \mathcal{D}\}$ such that $\mathbf{u}(t) = \mathcal{T}\mathbf{v}(t)$, and $\mathbf{u}(t)$ solves the PDE (6) if and only if $\mathbf{v}(t)$ solves the PIE (5). The formulae for computing these PI operators have been incorporated into the PIETOOLS software suite, allowing a linear 1D or 2D PDE system to be declared as a structure `PDE`, and the associated PIE representation to be computed by calling `PIE=convert(PDE)`.

Lemma 3. Let $\{A_{ij}, B, C_{ij}, D, E\}$ define a well-posed PDE as in (6) and (8), and define associated PI operators $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ as in Lem. 14 in Jagt and Peet (2022). Then

$$
\mathbf{u} = \mathcal{T}\,\partial_x^2\partial_y^2\mathbf{u},\ \forall \mathbf{u} \in X \quad \text{and} \quad \mathbf{v} = \partial_x^2\partial_y^2\,\mathcal{T}\mathbf{v},\ \forall \mathbf{v} \in L_2^{n_u}.
$$

Moreover, for any $w(t) \in \mathbb{R}^{n_w}$ and $t \geq 0$, $(\mathbf{u}(t), z(t))$ satisfies the PDE (6) if and only if $(\partial_x^2\partial_y^2\mathbf{u}(t), z(t))$ satisfies the PIE (5), and $(\mathbf{v}(t), z(t))$ satisfies the PIE (5) if and only if $(\mathcal{T}\mathbf{v}(t), z(t))$ satisfies the PDE (6).

2.4 A Linear PI Inequality (LPI) for L_2-Gain Analysis

Linear PI Inequalities (LPIs) are convex optimization programs involving linear operator inequalities on PI operator variables. In the next section, we will use the following LPI for L_2-gain analysis of 2D PDEs, presented in Lem. 8 in Jagt and Peet (2022), to derive a similar LPI for H_∞-optimal estimator synthesis.

Lemma 4. Let $\gamma > 0$, and suppose there exists a PI operator $\mathcal{P} \in \mathbf{\Pi}_{2D}^{n_u\times n_u}$ such that $\mathcal{P} = \mathcal{P}^* \succ 0$ and

$$
\begin{bmatrix} -\gamma I & \mathcal{D} & \mathcal{C} \\ \mathcal{D}^* & -\gamma I & \mathcal{B}^*\mathcal{P}\mathcal{T} \\ \mathcal{C}^* & \mathcal{T}^*\mathcal{P}\mathcal{B} & \mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} \end{bmatrix} \preceq 0.
$$

Then, for any $w \in L_2^{n_w}[0,\infty)$, if (w,z) satisfies the PIE (5) with $\mathbf{v}(0) = \mathbf{0}$, then $z \in L_2^{n_z}[0,\infty)$ and $\|z\|_{L_2} \leq \gamma\|w\|_{L_2}$.

3. AN H_∞-OPTIMAL ESTIMATOR FOR 2D PDES

In this section, we provide the main technical result of this paper, proposing an LPI for H_∞-optimal estimator synthesis for a class of 2D PDEs as in (6). Suppose that we have three observed output signals, $q_1(t) \in \mathbb{R}^{n_{q_1}}$, $\mathbf{q}_2(t) \in L_2^{n_{q_2}}[0,1]$, and $\mathbf{q}_3(t) \in L_2^{n_{q_3}}[0,1]$, defined by

$$\mathbf{q}(t) := \begin{bmatrix} q_1(t) \\ \mathbf{q}_2(t) \\ \mathbf{q}_3(t) \end{bmatrix} = \begin{bmatrix} \mathrm{M}[C_1]\Lambda_1 \\ \mathrm{M}[C_2]\Lambda_2 \\ \mathrm{M}[C_3]\Lambda_3 \end{bmatrix} \mathbf{u}(t) + \begin{bmatrix} \mathrm{M}[D_1] \\ \mathrm{M}[D_2] \\ \mathrm{M}[D_3] \end{bmatrix} w(t), \qquad (9)$$

where

$$\begin{bmatrix} C_1 & C_2 & C_3 \\ D_1 & D_2 & D_3 \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^{n_{q_1} \times 16 n_u} & L_2^{n_{q_2} \times 4 n_u}[0,1] & L_2^{n_{q_3} \times 4 n_u}[0,1] \\ \mathbb{R}^{n_{q_1} \times n_w} & L_2^{n_{q_2} \times n_w}[0,1] & L_2^{n_{q_3} \times n_w}[0,1] \end{bmatrix},$$

and where the operators $\Lambda_1, \Lambda_2, \Lambda_3$ are as in (7), evaluating admissible derivatives of the state along the boundary.

Definition 5. (Solution to the PDE). For a given input signal w and initial state $\mathbf{u}_0 \in X$, $(\mathbf{u}, z, \mathbf{q})$ is a solution to the PDE defined by $\{A_{ij}, B, C_{ij}, D, C_k, D_k, E\}$ if $\mathbf{u}$ is Frechét differentiable, $\mathbf{u}(0) = \mathbf{u}_0$, and for all $t \geq 0$, $(\mathbf{u}(t), z(t), \mathbf{q}(t))$ satisfies (6) and (9).

Now, to construct an estimator for the PDE with the proposed sensed output, first note that by Lem. 3 the system dynamics can be equivalently represented in terms of the fundamental state $\mathbf{v}(t) = \partial_x^2 \partial_y^2 \mathbf{u}(t)$, as the PIE (5). Invoking the identity $\mathbf{u} = \mathcal{T}\mathbf{v}$, the output signals can be represented in terms of this fundamental state as

$$\mathbf{q}(t) = \mathcal{C}_q \mathbf{v}(t) + \mathcal{D}_q w(t), \qquad (10)$$

where we define the operators

$$\mathcal{C}_q := \begin{bmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \\ \mathcal{C}_3 \end{bmatrix} = \begin{bmatrix} \mathrm{M}[C_1] \circ \Lambda_1 \circ \mathcal{T} \\ \mathrm{M}[C_2] \circ \Lambda_2 \circ \mathcal{T} \\ \mathrm{M}[C_2] \circ \Lambda_2 \circ \mathcal{T} \end{bmatrix}, \quad \mathcal{D}_q := \begin{bmatrix} \mathrm{M}[D_1] \\ \mathrm{M}[D_2] \\ \mathrm{M}[D_3] \end{bmatrix}. \qquad (11)$$

Then, a Luenberger-type estimator for the PIE (5) can be parameterized by a PI operator $\mathcal{L}$ as

$$\mathcal{T}\hat{\mathbf{v}}_t(t) = \mathcal{A}\hat{\mathbf{v}}(t) + \mathcal{L}(\mathcal{C}_q \hat{\mathbf{v}}(t) - \mathbf{q}(t)), \quad \hat{z}(t) = \mathcal{C}\hat{\mathbf{v}}(t), \qquad (12)$$

returning an estimate of the PDE state $\mathbf{u}(t) = \mathcal{T}\mathbf{v}(t)$ as $\hat{\mathbf{u}}(t) = \mathcal{T}\hat{\mathbf{v}}(t)$. The goal, then, is to choose the gain $\mathcal{L}$ such as to minimize the H_∞-norm of the resulting error dynamics, i.e. to solve the optimization program

$$\min_{\mathcal{L},\gamma} \ \gamma \quad \text{s.t.} \quad \|\hat{z} - z\|_{L_2} \leq \gamma \|w\|_{L_2}, \ \forall w \in L_2[0,\infty) \setminus \{0\}.$$

The following result shows that a solution to this program can be computed by solving an LPI.

Theorem 6. For given $\mathrm{G}_{\text{pde}} := \{A_{ij}, B, C_{ij}, D, C_k, D_k, E\}$, define PI operators $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ as in Lem. 3, and let $\{\mathcal{C}_q, \mathcal{D}_q\}$ be as in (11). For any $\gamma > 0$, suppose there exist $\mathcal{P} \in \mathbf{\Pi}_{2D}^{n_u \times n_u}$ and $\mathcal{W} \in \mathbf{\Pi}_{2D \leftarrow 0}^{n_u \times n_{q_1}} \times \mathbf{\Pi}_{2D \leftarrow 1D}^{n_u \times n_{q_2} + n_{q_3}}$ such that

$$\mathcal{P} = \mathcal{P}^* \succ 0, \qquad \begin{bmatrix} -\gamma I & -\mathcal{D} & \mathcal{C} \\ (\cdot)^* & -\gamma I & -[\mathcal{B}^*\mathcal{P} + \mathcal{D}_q^* \mathcal{W}^*]\mathcal{T} \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + \mathcal{T}^*[\mathcal{P}\mathcal{A} + \mathcal{W}\mathcal{C}_q] \end{bmatrix} \preceq 0, \qquad (13)$$

and let $\mathcal{L} = \mathcal{P}^{-1}\mathcal{W}$. If $(\mathbf{u}, z, \mathbf{q})$ is a solution to the PDE defined by G_{pde} for some disturbance $w \in L_2^{n_w}[0,\infty)$ and initial state $\mathbf{u}_0 \in X$, and $(\hat{\mathbf{v}}, \hat{z})$ is a solution to the PIE (12) with input $\mathbf{q}$ and initial state $\hat{\mathbf{v}}_0 = \partial_x^2 \partial_y^2 \mathbf{u}_0$, then $\hat{z} - z \in L_2^{n_z}[0,\infty)$ and $\|\hat{z} - z\|_{L_2} \leq \gamma \|w\|_{L_2}$.

To prove this result, in Subsec. 3.1, it is first proven that the operator $\mathcal{C}_q$ in (10) is indeed a PI operator, thus yielding a PIE representation of the considered PDE. In Subsec. 3.2, it is then shown how an estimator for this PIE can be synthesized by solving the proposed LPI.

3.1 Representation of Infinite-Dimensional PDE Outputs

Consider the sensed output $\mathbf{q}(t)$ in (10), expressed in terms of the fundamental state $\mathbf{v}(t)$ using operators $\{\mathcal{C}_q, \mathcal{D}_q\}$ as in (11). In order to show that an estimator based on $\mathbf{q}(t)$ can be computed by solving the LPI (13), in this subsection, it is first shown that the operators $\{\mathcal{C}_q, \mathcal{D}_q\}$ in this LPI are indeed PI operators.

To begin, consider e.g. the first element of the composition $\Lambda_2 \circ \mathcal{T}$ in (11), given by $\Delta_y^0 \partial_x^2 \circ \mathcal{T}$. It has previously been shown that the composition of suitable differential and PI operators can again be expressed as PI operators, so that in particular, we can explicitly define $\mathcal{R}_{k\ell} \in \mathbf{\Pi}_{2D}^{n_u \times n_u}$ such that $\partial_x^k \partial_y^\ell \circ \mathcal{T} = \mathcal{R}_{k\ell}$, for every $0 \leq k, \ell \leq 2$ (see also the extended version of this paper, Jagt and Peet (2024)). By definition of the operators Λ_i in (7), then, it follows that

$$\Lambda_1 \mathcal{T} = \begin{bmatrix} \Delta_1 \mathcal{T} \\ \Delta_1 \mathcal{R}_{10} \\ \Delta_1 \mathcal{R}_{01} \\ \Delta_1 \mathcal{R}_{11} \end{bmatrix}, \ \Lambda_2 \mathcal{T} = \begin{bmatrix} \Delta_2 \mathcal{R}_{20} \\ \Delta_2 \mathcal{R}_{21} \end{bmatrix}, \ \Lambda_3 \mathcal{T} = \begin{bmatrix} \Delta_3 \mathcal{R}_{02} \\ \Delta_3 \mathcal{R}_{12} \end{bmatrix}, \quad (14)$$

where the Dirac operators Δ_1, Δ_2, and Δ_3 are as in (7). To show that also the compositions $\Delta_i \circ \mathcal{R}_{k\ell}$ can be expressed as PI operators, note that e.g. evaluating the partial integral $\int_0^x R(x,y,\theta)\mathbf{v}(\theta,y)d\theta$ at $x = 1$, the result is a full integral $\int_0^1 R(1,y,\theta)\mathbf{v}(\theta,y)d\theta$. Prop. 7 extends this idea for more general Dirac and 2D PI operators.

Proposition 7. For $i, j \in \{1,2\}$, let $R_{i0}, R_{0j} \in L_2^{m \times n}[[0,1]^3]$ and $R_{ij} \in L_2^{m \times n}[[0,1]^4]$, and define $F_{kj}, G_{\ell i} \in L_2^{m \times n}[[0,1]^3]$ and $F_{k0}, G_{\ell 0}, H_{k\ell} \in L_2^{m \times n}[[0,1]^2]$ for $k, \ell \in \{0,1\}$ by

$$F_{k0}(y,\theta) := R_{(2-k)0}(k,y,\theta), \quad F_{kj}(y,\theta,\eta) := R_{(2-k)j}(k,y,\theta,\eta),$$
$$G_{\ell 0}(x,\eta) := R_{0(2-\ell)}(x,\ell,\eta), \quad G_{\ell i}(x,\theta,\eta) := R_{i(2-\ell)}(x,\ell,\theta,\eta),$$

and $H_{k\ell}(\theta,\eta) := F_{k(2-\ell)}(\ell,\theta,\eta)$, for $x, y, \theta, \eta \in [0,1]$. Then

$$\Delta_x^k \circ \Pi \begin{bmatrix} 0 & 0 & 0 \\ R_{10} & R_{11} & R_{12} \\ R_{20} & R_{21} & R_{22} \end{bmatrix} = \Pi \begin{bmatrix} 0 & 0 & 0 \\ F_{k0} & F_{k1} & F_{k2} \\ F_{k0} & F_{k1} & F_{k2} \end{bmatrix} \in \mathbf{\Pi}_{1D \leftarrow 2D}^{m \times n},$$
$$\Delta_y^\ell \circ \Pi \begin{bmatrix} 0 & R_{01} & R_{02} \\ 0 & R_{11} & R_{12} \\ 0 & R_{21} & R_{22} \end{bmatrix} = \Pi \begin{bmatrix} 0 & G_{\ell 0} & G_{\ell 0} \\ 0 & G_{\ell 1} & G_{\ell 1} \\ 0 & G_{\ell 2} & G_{\ell 2} \end{bmatrix} \in \mathbf{\Pi}_{1D \leftarrow 2D}^{m \times n},$$
$$\Delta_x^k \circ \Delta_y^\ell \Pi \begin{bmatrix} 0 & 0 & 0 \\ 0 & R_{11} & R_{12} \\ 0 & R_{21} & R_{22} \end{bmatrix} = \textstyle\int_{[0,1]^2}[H_{k\ell}] \in \mathbf{\Pi}_{0 \leftarrow 2D}^{m \times n}.$$

Proof. The proof follows using the fact that, evaluating e.g. $\left(\Pi \begin{bmatrix} 0 & 0 & 0 \\ R_{10} & R_{11} & R_{12} \\ R_{20} & R_{21} & R_{22} \end{bmatrix} \mathbf{v}\right)(x,y)$ at $x = 0$, the integral terms $\int_0^x [R_{1j}]$ vanish, leaving only full integral terms $\int_0^1 [R_{2j}]$ as

$$\left(\Delta_x^0 \Pi \begin{bmatrix} 0 & 0 & 0 \\ R_{10} & R_{11} & R_{12} \\ R_{20} & R_{21} & R_{22} \end{bmatrix} \mathbf{v}\right)(y) = \int_0^1 \Big[R_{20}(0,y,\theta)\mathbf{v}(\theta,y)$$
$$+ \int_0^y R_{21}(0,y,\theta,\eta)\mathbf{v}(\theta,\eta)d\eta + \int_y^1 R_{22}(0,y,\theta,\eta)\mathbf{v}(\theta,\eta)d\eta\Big]d\theta$$
$$= \left(\Pi \begin{bmatrix} 0 & 0 & 0 \\ R_{20}(0,.) & R_{21}(0,.) & R_{22}(0,.) \\ R_{20}(0,.) & R_{21}(0,.) & R_{22}(0,.) \end{bmatrix} \mathbf{v}\right)(y) = \left(\Pi \begin{bmatrix} 0 & 0 & 0 \\ F_{00} & F_{01} & F_{02} \\ F_{00} & F_{01} & F_{02} \end{bmatrix} \mathbf{v}\right)(y).$$

A full proof is provided in the extended version of the paper, Jagt and Peet (2024).

By Prop. 7, the compositions $\Delta_i \circ \mathcal{R}_{k\ell}$ in (14) can again be expressed as PI operators if the operators $\mathcal{R}_{k\ell} := \partial_x^k \partial_y^\ell \circ \mathcal{T}$ for $0 \leq k, \ell \leq 2$ have suitable structures. As shown in the extended version of this paper (Jagt and Peet (2024), Lem. 7), the operators $\mathcal{R}_{k\ell}$ indeed satisfy the conditions of Prop. 7, and thus the compositions $\Lambda_i \circ \mathcal{T}$ and consequently $\mathrm{M}[C_i] \circ \Lambda_i \circ \mathcal{T}$ can be expressed as 2D PI operators, for each $i \in \{1,2,3\}$. It follows that the operator $\mathcal{C}_q$ in (11) is in fact a PI operator, thus yielding an equivalent PIE representation of the PDE with sensed outputs as follows.

Lemma 8. For given $\mathrm{G}_{\text{pde}} := \{A_{ij}, B, C_{ij}, D, C_k, D_k, E\}$, define associated PI operators $\mathrm{G}_{\text{pie}} := \{\mathcal{T}, \mathcal{A}, \mathcal{B}, \left[\begin{smallmatrix}\mathcal{C}\\ \mathcal{C}_q\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathcal{D}\\ \mathcal{D}_q\end{smallmatrix}\right]\}$ as in Lem. 3 and in (11). Then, for any input w, $(\mathbf{u}, z, \mathbf{q})$ is a solution to the PDE defined by G_{pde} with initial state $\mathbf{u}_0 \in X$ if and only if $(\mathbf{v}, \left[\begin{smallmatrix}z\\ \mathbf{q}\end{smallmatrix}\right])$ with $\mathbf{v} = \partial_x^2\partial_y^2\mathbf{u}$ is a solution to the PIE defined by G_{pie} with initial state $\mathbf{v}_0 = \partial_x^2\partial_y^2\mathbf{u}_0$.

Proof. Fix an arbitrary input w. Then, by Lem 3, $(\mathbf{u}, z, \mathbf{q})$ is a solution to the PDE defined by G_{pde} with initial state $\mathbf{u}_0 \in X$ if and only if $(\mathbf{v}, z)$ is a solution to the PIE defined by $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ with initial state $\mathbf{v}_0 = \partial_x^2\partial_y^2\mathbf{u}_0$, and $\mathbf{q}(t)$ satisfies (9) with $\mathbf{u}(t) = \mathcal{T}\mathbf{v}(t)$. Here, by definition of the operators $\{\mathcal{C}_q, \mathcal{D}_q\}$, $\mathbf{q}(t)$ satisfies (9) with $\mathbf{u}(t) = \mathcal{T}\mathbf{v}(t)$ if and only if $\mathbf{q}(t)$ satisfies (10), and hence $(\mathbf{v}, \left[\begin{smallmatrix}z\\ \mathbf{q}\end{smallmatrix}\right])$ is a solution to the PIE defined by G_{pie}.

3.2 An LPI for Optimal Estimation of PIEs

Having derived a PIE representation of the 2D PDE (6), consider now a Luenberger-type estimator for this PIE as in (12), parameterized by a PI operator $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) : \mathbf{\Pi}_{2D\leftarrow 0}^{n_u\times n_{q_1}} \times \mathbf{\Pi}_{2D\leftarrow 1D}^{n_u\times n_{q_2}} \times \mathbf{\Pi}_{2D\leftarrow 1D}^{n_u\times n_{q_3}}$. Then, for any solution $(\mathbf{v}, z)$ and $(\hat{\mathbf{v}}, \hat{z})$ to the PIEs (5) and (12), respectively, the errors $\mathbf{e}(t) := \hat{\mathbf{v}}(t) - \mathbf{v}(t)$ and $\tilde{z}(t) := \hat{z}(t) - z(t)$ will satisfy

$$\mathcal{T}\mathbf{e}_t(t) = \tilde{\mathcal{A}}\mathbf{e}(t) + \tilde{\mathcal{B}}w(t), \quad \tilde{z}(t) = \tilde{\mathcal{C}}\mathbf{e}(t) + \tilde{\mathcal{D}}w(t), \quad (15)$$

where we define the PI operators

$$\tilde{\mathcal{A}} := \mathcal{A} + \mathcal{L}\mathcal{C}_q, \quad \tilde{\mathcal{B}} := -(\mathcal{B} + \mathcal{L}\mathcal{D}_q), \quad \tilde{\mathcal{C}} := \mathcal{C}, \quad \tilde{\mathcal{D}} := -\mathcal{D}.$$

The H_∞-optimal estimator synthesis problem, then, is to find an operator $\mathcal{L}$ that minimizes the L_2-gain $\sup_{w\neq 0} \frac{\|\tilde{z}\|_{L_2}}{\|w\|_{L_2}}$ from disturbances w to the error $\tilde{z}$. To solve this problem, note that the challenge of verifying an upper bound γ on the L_2-gain of a PIE has already been posed as an LPI in Lem. 4, yielding the following corollary.

Corollary 9. Let $\gamma > 0$, and suppose there exist $\mathcal{P} \in \mathbf{\Pi}_{2D}^{n_u\times n_u}$ and $\mathcal{W} \in \mathbf{\Pi}_{2D\leftarrow 0}^{n_u\times n_{q_1}} \times \mathbf{\Pi}_{2D\leftarrow 1D}^{n_u\times n_{q_2}+n_{q_3}}$ that satisfy the LPI (13) in Thm. 6. Then, for any $w \in L_2^{n_w}[0,\infty)$, if $(w, \tilde{z})$ satisfies the PIE (15) with $\mathcal{L} = \mathcal{P}^{-1}\mathcal{W}$ and $\mathbf{e}(0) = \mathbf{0}$, then $\tilde{z} \in L_2^{n_z}[0,\infty)$ and $\|\tilde{z}\|_{L_2} \le \gamma\|w\|_{L_2}$.

Proof. Let the conditions of the corollary be satisfied for some γ, $\mathcal{P}$ and $\mathcal{W}$, and let $\mathcal{L} := \mathcal{P}^{-1}\mathcal{W}$. Then, by definition of the operators $\{\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}}\}$ defining the PIE (15),

$$[\mathcal{B}^*\mathcal{P} + \mathcal{D}_q^*\mathcal{W}^*] = -\tilde{\mathcal{B}}^*\mathcal{P} \quad \text{and} \quad [\mathcal{P}\mathcal{A} + \mathcal{W}\mathcal{C}_q] = \mathcal{P}\tilde{\mathcal{A}}.$$

By Lem. 4, the result follows.

Cor. 9 proves that, if the LPI (13) is feasible for some $(\gamma, \mathcal{P}, \mathcal{W})$, then, using the estimator defined by (12) with gain operator $\mathcal{L} := \mathcal{P}^{-1}\mathcal{W}$, the H_∞-norm $\sup_{w\neq 0} \frac{\|\hat{z}-z\|_{L_2}}{\|w\|_{L_2}}$ of the associated error dynamics is upper-bounded by γ. Using this result, we finally prove Thm. 6

Proof. [Proof of Thm. 6] Suppose that the conditions of the theorem are satisfied. Fix arbitrary $w \in L_2^{n_w}[0,\infty)$ and $\mathbf{u}_0 \in X$, and let $(\mathbf{u}, z, \mathbf{q})$ be an associated solution to the PDE defined by parameters G_{pde}. Then, by Lem 8, $(\mathbf{v}, \left[\begin{smallmatrix}z\\ \mathbf{q}\end{smallmatrix}\right])$ with $\mathbf{v} = \partial_x^2\partial_y^2\mathbf{u}$ is a solution to the PIE defined by $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \left[\begin{smallmatrix}\mathcal{C}\\ \mathcal{C}_q\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathcal{D}\\ \mathcal{D}_q\end{smallmatrix}\right]\}$, with $\mathbf{v}_0 = \partial_x^2\partial_y^2\mathbf{u}_0$. Let $(\hat{\mathbf{v}}, \hat{z})$ be a solution to the PIE (12) with input $\mathbf{q}(t)$ and initial state $\hat{\mathbf{v}}_0 = \partial_x^2\partial_y^2\mathbf{u}_0$, and define $\mathbf{e} = \hat{\mathbf{v}} - \mathbf{v}$ and $\tilde{z} = \hat{z} - z$. Then, $\mathbf{e}(0) = \mathbf{0}$, and $(w(t), \tilde{z}(t))$ satisfies (15) for $t \ge 0$. By Cor. 9 it follows that $\tilde{z} \in L_2^{n_z}[0,\infty)$ and $\|\tilde{z}\|_{L_2} \le \gamma\|w\|_{L_2}$.

4. ESTIMATOR SYNTHESIS USING AN LMI

Having shown how an H_∞-optimal estimator for 2D PDEs can be synthesized by solving the LPI (13), in this section, we show how this LPI can be numerically solved by parameterizing the PI operator variables $\mathcal{P}$ and $\mathcal{W}$ by matrices. In particular, for some $p, r, m \in \mathbb{N}$, fix $\mathcal{Z}_1 \in \mathbf{\Pi}_{2D}^{p\times n_u}$, $\mathcal{Z}_2 \in \mathbf{\Pi}_{2D}^{r\times n_u+n_z+n_w}$, and $\mathcal{Z}_3 \in \mathbf{\Pi}_{2D\leftarrow 0}^{m\times n_{q_1}} \times \mathbf{\Pi}_{2D\leftarrow 1D}^{m\times n_{q_2}+n_{q_3}}$ to be defined by monomials of degrees at most $d_1, d_2, d_3 \in \mathbb{N}$, in the variables $x, y, \theta, \eta \in [0,1]$. Then, parameterizing

$$\mathcal{P} = \mathcal{Z}_1^*\mathrm{M}[P]\mathcal{Z}_1, \quad \mathcal{Q} = \mathcal{Z}_2^*\mathrm{M}[Q]\mathcal{Z}_2, \text{ and } \mathcal{W} = \mathrm{M}[W]\mathcal{Z}_3,$$

by matrices $P \in \mathbb{R}^{p\times p}$, $Q \in \mathbb{R}^{r\times r}$, and $W \in \mathbb{R}^{n_u\times m}$, it can be shown that $P \succeq 0$ and $Q \preceq 0$ imply $\mathcal{P} \succeq 0$ and $\mathcal{Q} \preceq 0$, respectively (see e.g. Jagt and Peet (2022)). In this manner, the LPI conditions in Thm. 6 can be enforced as LMI conditions, allowing an H_∞-optimal estimator for a 2D PDE to be synthesized as in Algorithm 1.

Algorithm 1: H_∞-Optimal Estimator Synthesis

Data: PDE G_{pde}, PI operators $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$, scalar $\epsilon > 0$.
1. Compute $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{C}_q, \mathcal{D}_q\}$ as per Thm. 6;
2. Solve the semidefinite program

$$\min_{\gamma>0,P,Q,W} \gamma \quad \begin{array}{l} P \succeq 0, \\ Q \preceq 0, \end{array} \quad \begin{bmatrix} -\gamma I & -\mathcal{D} & \mathcal{C} \\ (\cdot)^* & -\gamma I & -[\mathcal{B}^*\mathcal{P} + \mathcal{D}_q^*\mathcal{W}^*]\mathcal{T} \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + \mathcal{T}^*[\mathcal{P}\mathcal{A} + \mathcal{W}\mathcal{C}_q] \end{bmatrix} = \mathcal{Q}, \quad (16)$$

where $\mathcal{P} = \mathcal{Z}_1^*\mathrm{M}[P]\mathcal{Z}_1 + \epsilon\mathrm{M}[I_{n_u}]$, $\mathcal{Q} = \mathcal{Z}_2^*\mathrm{M}[Q]\mathcal{Z}_2$ and $\mathcal{W} = \mathrm{M}[W]\mathcal{Z}_3$;
3. Compute the Luenberger gain $\mathcal{L} = \mathcal{P}^{-1}\mathcal{W}$;

Note that, since the operator $\mathcal{P}$ in Algorithm 1 is bounded, linear, and coercive, the inverse $\mathcal{P}^{-1}$ in $\mathcal{L} = \mathcal{P}^{-1}\mathcal{W}$ is well-defined. In order to actually compute this inverse, we recall that an explicit expression for the operator inverse has already been derived in Miao et al. (2019) for a class of *separable* 1D PI operators, taking the form $(\mathcal{R}\mathbf{v})(x) = R_0(x)\mathbf{v}(x) + Z(x)^T\int_0^1 HZ(\theta)\mathbf{v}(\theta)d\theta$. The following proposition replicates this result for separable 2D PI operators, taking the form $\mathcal{R} = \mathrm{M}[R_0] + \mathrm{M}[Z^T]H\int_{[0,1]^2}[Z]$.

Proposition 10. For $p, n \in \mathbb{N}$, let $R_0 \in L_2^{n\times n}[[0,1]^2]$ be invertible, and let $R_1(x,y,\theta,\eta) = Z(x,y)^T HZ(\theta,\eta)$ for some $Z \in L_2^{p\times n}[[0,1]^2]$ and $H \in \mathbb{R}^{p\times p}$. Let $Q_0 := R_0^{-1}$ and

$$Q_1(x,y,\theta,\eta) := Q_0(x,y)^T Z(x,y)^T \hat{H} Z(\theta,\eta) Q_0(\theta,\eta),$$

for $x, y, \theta, \eta \in [0,1]$, where $\hat{H} = -H(I_p + KH)^{-1} \in \mathbb{R}^{p\times p}$ with $K := \int_0^1\int_0^1 Z(\nu,\mu)Q_0(\nu,\mu)Z(\nu,\mu)^T d\mu d\nu \in \mathbb{R}^{p\times p}$. If $R := \left[\begin{smallmatrix} R_0 & 0 & 0 \\ 0 & R_1 & R_1 \\ 0 & R_1 & R_1 \end{smallmatrix}\right]$ and $Q := \left[\begin{smallmatrix} Q_0 & 0 & 0 \\ 0 & Q_1 & Q_1 \\ 0 & Q_1 & Q_1 \end{smallmatrix}\right]$, then

$$\Pi[Q] \circ \Pi[R] = \Pi[R] \circ \Pi[Q] = \mathrm{M}[I_n].$$

Proof. The result follows by the composition rules of 2D PI operators. A full proof is given in the extended version of this paper, Jagt and Peet (2024).

Using Prop. 10, the inverse of the operator $\mathcal{P} = \mathcal{Z}_1^*\mathrm{M}[P]\mathcal{Z}_1$ in Alg. 1 can be computed analytically if the operator $\mathcal{Z}_1$ is chosen to have a suitable, separable structure. Although such a restriction necessarily introduces conservatism, the fact that both $\mathcal{P}$ and $\mathcal{W}$ in (16) may be defined by (partial) integral operators still allows significantly more freedom than parameterizing the Luenberger gain by merely a multiplier operator, as is commonly done in practice.

5. A NUMERICAL EXAMPLE

The presented methodology for estimator synthesis of 2D PDEs has been fully incorporated into the PIETOOLS software suite (see Shivakumar et al. (2021)). Using this software, an H_∞-optimal estimator for a PDE can be synthesized by first declaring the PDE as a structure `PDE` with the user interface, converting it to a PIE using `PIE=convert(PDE)`, and finally solving the LMI (16) as `[Lop,gam]=lpisolve(PIE,'estimator')`, returning objects `Lop` and `gam` representing the optimal gain $\mathcal{L}$ and associated minimal value of $\gamma > 0$, respectively.

In this section, the PIETOOLS software is used to construct an optimal estimator for an unstable 2D heat equation, with sensing along the upper boundary of the domain. Performance of the estimator is tested by simulating the error $\mathbf{e}(t)$ in the fundamental state $\mathbf{v}(t)$, based on the PIE (15). We refer to the extended version of the paper (Jagt and Peet (2024)) for more information on the applied simulation scheme.

5.1 Estimator for a 2D Reaction-Diffusion Equation

Consider the following 2D reaction-diffusion equation,

$$\mathbf{u}_t(t) = \mathbf{u}_{xx}(t) + \mathbf{u}_{yy}(t) + r\mathbf{u}(t) + (x^2-1)(y^2-1)w(t),$$
$$z(t) = \int_0^1\int_0^1 \mathbf{u}(t,x,y)dxdy,$$
$$\mathbf{q}_1(t) = \mathbf{u}(t,1,\cdot) + \eta_1(t), \qquad \mathbf{q}_2(t) = \mathbf{u}(t,\cdot,1) + \eta_2(t),$$
$$\mathbf{u}(t,0,\cdot) = \mathbf{u}_x(t,1,\cdot) \equiv 0, \qquad \mathbf{u}(t,\cdot,0) = \mathbf{u}_y(t,\cdot,1) \equiv 0,$$

where $\mathbf{u}(t) \in L_2[[0,1]^2]$, $w(t), z(t), \eta_1(t), \eta_2(t) \in \mathbb{R}$, and $\mathbf{q}(t) = \left[\begin{smallmatrix}\mathbf{q}_1(t)\\ \mathbf{q}_2(t)\end{smallmatrix}\right] \in L_2^2[0,1]$. Using PIETOOLS, we construct an associated PIE representation and synthesize an estimator for parameter values $r = 4$ and $r = 8$, for which the system is stable and unstable, respectively. Solving the LMI (16), we obtain gain operators $\mathcal{L} = \mathcal{P}^{-1}\mathcal{W}$ that achieve bounds on the H_∞-norm of the associated error dynamics as $\gamma = 0.0476$ ($r = 4$) and $\gamma = 0.1403$ ($r = 8$).

The error dynamics corresponding to the obtained estimators were simulated with disturbance $w(t) = 5e^{-t/2}\sin(\pi t)$ and initial state $\mathbf{u}(0,x,y) = 5((x-1)^4-1)\sin(0.5\pi y)$, starting with an initial estimate $\hat{\mathbf{u}}(0) = 0$. Sensor noise $\eta_1(t)$ and $\eta_2(t)$ at each time step was generated from a Gaussian distribution with mean 0 and variance 0.04. Fig. 1 shows the norm of the error in the PDE state and output for $t \in [0,5]$, as well as the value of the disturbance. Both for the stable and unstable PDE, the errors in the state and output estimates rapidly converge to zero.

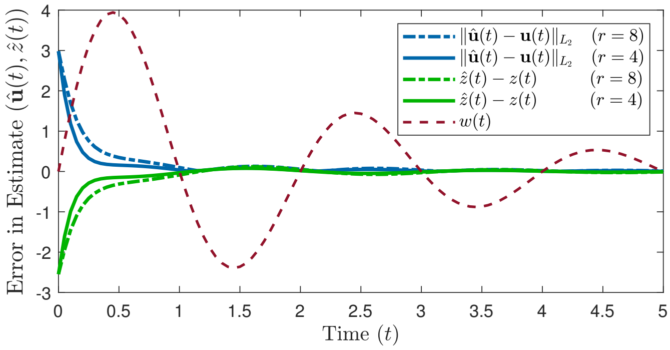

Fig. 1. Error in the estimates of the PDE state $\mathbf{u}(t)$ and the output $z(t)$ for the PDE in Subsec. 5.1 with $r \in \{4, 8\}$, for the initial error $\mathbf{u}(0,x,y) - \hat{\mathbf{u}}(0,x,y) = 5((x-1)^4-1)\sin(0.5\pi y)$ and disturbance $w(t) = 5e^{-t/2}\sin(\pi t)$. The estimators correspond to bounds on the H_∞-norm of $\gamma = 0.0476$ ($r = 4$) and $\gamma = 0.1403$ ($r = 8$).

6. CONCLUSION

In this paper, a new convex-optimization-based method was presented for estimator synthesis of linear, 2nd order, 2D PDEs with state observations along the boundary. To this end, it was proved that any sufficiently well-posed such PDE can be equivalently represented as a PIE, specifically proving that the value of the state $\mathbf{u}$ along the boundary can be expressed in terms of a PI operator acting on the fundamental state $\partial_x^2\partial_y^2\mathbf{u}$. Parameterizing a Luenberger-type estimator for the PIE by a PI operator, it was then shown that a value of this operator with guaranteed bound on the H_∞-norm of the estimator error dynamics can be computed by solving an LPI, which in turn could be solved as an LMI. The proposed methodology has been incorporated in the PIETOOLS software suite, and applied to construct an estimator for an unstable heat equation, using simulation to show convergence of the estimated state to the true value.

REFERENCES

Das, A., Shivakumar, S., Weiland, S., and Peet, M.M. (2019). $\mathcal{H}_\infty$ optimal estimation for linear coupled PDE systems. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, 262–267. IEEE.

Duan, G.R. and Yu, H.H. (2013). *LMIs in control systems: analysis, design and applications*. CRC press.

Holmes, E.E., Lewis, M., Banks, J., and Veit, R.R. (1994). Partial differential equations in ecology: Spatial interactions and population dynamics. *Ecology*, 75, 17–29.

Jadachowski, L., Meurer, T., and Kugi, A. (2015). Backstepping observers for linear PDEs on higher-dimensional spatial domains. *Automatica*, 51, 85–97.

Jagt, D.S. and Peet, M.M. (2022). L_2-gain analysis of coupled linear 2D PDEs using linear PI inequalities. In *2022 IEEE 61st Conference on Decision and Control (CDC)*, 6097–6104.

Jagt, D.S. and Peet, M.M. (2024). H_∞-optimal estimator synthesis for coupled linear 2D PDEs using convex optimization. *arXiv eprint:2402.05061*.

Lhachemi, H. and Prieur, C. (2022). Boundary output feedback stabilization of a class of reaction-diffusion PDEs with delayed boundary measurement. *International Journal of Control*, 1–11.

Miao, G., Peet, M.M., and Gu, K. (2019). Inversion of separable kernel operator and its application in control synthesis. *Delays and Interconnections: Methodology, Algorithms and Applications*, 265–280.

Shivakumar, S., Das, A., and Peet, M. (2021). PIETOOLS 2020a: User manual. *arXiv preprint arXiv:2101.02050*.

Wang, P. and Fridman, E. (2024). Sampled-data finite-dimensional observer-based control of 1D stochastic parabolic PDEs. *SIAM Journal on Control and Optimization*, 62(1), 297–325.

Yu, H., Gan, Q., Bayen, A., and Krstic, M. (2020). PDE traffic observer validated on freeway data. *IEEE Transactions on Control Systems Technology*, 29(3), 1048–1060.

Zayats, M., Fridman, E., and Zhuk, S. (2021). Global observer design for Navier-Stokes equations in 2D. In *2021 60th IEEE Conference on Decision and Control (CDC)*, 1862–1867.

Zuazua, E. (2005). Propagation, observation, and control of waves approximated by finite difference methods. *SIAM review*, 47(2), 197–243.