A Computationally Tractable H_2-optimal State-Estimator Synthesis for PIEs

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***Abstract*— In this paper, we present a computationally tractable convex formulation for the H_2-optimal state estimation problem applicable to a general class of linear time-invariant systems modeled by Ordinary Differntial Equations (ODEs) coupled with Partial Differential Equations (PDEs), with one spatial dimension. These convex optimization problems are derived by using an analysis and control framework called the"Partial Integral Equation" (PIE) framework, which utilizes the PIE representation of infinite-dimensional systems. Since PIEs are parameterized by Partial Integral (PI) operators that form an algebra, H_2-optimal estimation and control problems for PIEs can be formulated as Linear PI Inequalities (LPIs). Furthermore, if a PDE admits a PIE representation, then the stability and H_2 performance of the PIE system implies that of the PDE system. Consequently, the optimal estimator obtained for a PIE using LPIs provide the same stability and performance when applied to the corresponding original system. LPI problems can be solved computationally once converted to semi-definite programming; positive polynomial matrices are imposed to parameterize a cone of positive PI operators, and the polynomial positivity constraints are tightened to sum-of-squares constraints, which can be converted to Linear Matrix Inequalities and solved by semi-definite programming solvers. The application of these methods is illustrated by synthesizing estimators for important subclasses of linear ODE coupled with PDE systems: time-delay systems and simple standard PDEs.**

I. Introduction

Partial Differential Equations are used to describe the evolution of some process whose state cannot be represented using a finite set of values, but which is rather distributed over a spatial domain. Examples of such processes include fluid flow [1], [2], vibroaccoustics [3], [4], chemical reaction networks, and time-delay systems [], among others. The states in these processes may include velocity profile, displacement, species concentration, and history. For such systems, it is often desirable to be able to track the evolution of the system using sensor measurements – either for the purpose of feedback control [5], [6], [7], [8], [9], [10] or for monitoring and fault detection [].

Unlike Ordinary Differential Equations and other such lumped-parameter systems, however, direct measurement of the system state of a PDE requires an uncountable number of sensors – a practical impossibility. Consequently, there has been significant interest in the development of observers wherein by tracking a finite set of measurements, we may infer real-time estimates of the entire distributed state. For ODEs, the problem of state estimation has been largely solved, with special cases including the Luenberger observer, the Kalman filter, and Linear Matrix Inequalities (LMIs) for H_∞-optimal observers and filters – methods which can be applied to state-estimation for any linear ODE with state-space representation. For PDEs, however, the need to integrate boundary conditions and a distributed system state precludes the existence of a convenient and universal state-space representation. This means that most efforts to design estimators for such systems are ad hoc – requiring significant modification for even minor changes in the model [11]. As a result, most approaches to the estimation of the PDE state entail a reduction of the PDE state to finite dimensions, either through early or late lumping.

Early-lumping [12], [13], [14] entails reduction of the PDE to a state-space ODE through discretize via Galerkin projection, spatial discretization or modal decomposition. Late-lumping [15], [16], [17], by contrast, formulates synthesis conditions using the distributed state, but then enforces those conditions on a finite number of test functions. In both cases, neither stability or performance of the estimator can be proven unless the truncation error can be bounded through some auxiliary, ad-hoc process.

Recently, efforts have been made to synthesize observers for PDE systems without lumping through the use of a more convenient state-space representation of PDEs. This method integrates the PDE evolution equation with the boundary conditions by defining the state as the highest spatial derivative of the distributed state and parameterizing the evolution of this state by means of integral operators with polynomial kernels. This method has the advantage that integration operators form an algebra of bounded linear operators, which can be represented using matrices and optimized using LMIs. The representation of a PDE using such operators is referred to as a Partial Integral Equation (PIE), and methods for the construction of PIE representations of a broad class of PDEs are well-established [18], [19], [20].

Observer designs for PDE systems that admit a PIE representation have previously been presented in [21], [22] and for time-delay systems in [23]. These results parameterize the observer dynamics using PIEs and pose conditions for stability and performance bounds as the solution of a convex optimization problem expressed in terms of partial integral (PI) operator variables and linear operator inequalities (LPIs), which can be enforced using recently developed Matlab toolboxes such as [24]. These results use a generalization of the KYP lemma to partial integral equations to ensure stability and bound the L_2-gain of a tracking error to disturbances

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such as sensor noise. The problem with such approaches, however, is that disturbances such as sensor noise are not typically characterized in terms of energy, but rather in terms of frequency content and power spectral density – such as is the case for the well-known Kalman filtering problem.

The goal of this paper, then, is to combine and extend the results in [21] and [23]. Specifically, this paper integrates [21] and [23] by considering a broader class of PIEs that represent PDEs coupled with ODEs. More significantly, this paper extends these results by formulating the problem of optimal observer synthesis, which minimizes a bound on H_2 performance. Unlike H_∞-optimal observer synthesis, wherein a proxy for H_∞ performance is L_2-gain, the main technical difficulty for H_2-optimal estimation is the identification of a time-domain proxy for H_2 performance. To address this difficulty, we rely on an initial condition to output L_2-gain characterization of the H_2 metric as proposed in [25]. This allows us to extend classical LMIs for H_2-performance to LPI-type conditions to performance bounds on the error dynamics of the PIE-based observer.

To paper is structured as follows.

II. Notation

We denote $L_2^p[a,b]$ the set of *Lesbegue* square-integrable functions defined in the spatial domain, a compact interval $[a,b]\subset\mathbb{R}$ for real a and b, and evaluated in $\mathbb{R}^p$, with natural p. Similarly, $L_2^p[0,\infty)$ is used for functions defined in the temporal domain $[0,\infty)$ and evaluated in $\mathbb{R}^p$. For brevity, $\mathbb{R}L_2^{m,p}[a,b]$ denotes the space $\mathbb{R}^m\times L_2^p[a,b]$, for natural m, endowed with the inner-product defined as follows:

$$\left\langle \begin{bmatrix} x_1\\ \mathbf{x}_2\end{bmatrix}, \begin{bmatrix} y_1\\ \mathbf{y}_2\end{bmatrix}\right\rangle_{\mathbb{R}L_2} := \langle x_1, y_1\rangle_2 + \langle \mathbf{x}_2, \mathbf{y}_2\rangle_{L_2},$$

where $\begin{bmatrix} x_1\\ \mathbf{x}_2\end{bmatrix}, \begin{bmatrix} y_1\\ \mathbf{y}_2\end{bmatrix} \in \mathbb{R}L_2^{m,p}$ and two different inner-products were used: the norm-2 induced inner-product of $\mathbb{R}^m$, $\langle x_1, y_1\rangle_2 = x_1^T y_1$ and the usual L_2 inner-product, $\int_{[a,b]} \mathbf{x}_2(\theta)^T\mathbf{y}_2(\theta)d\theta$. Since these norms make $\mathbb{R}^m$ and $L_2^p[a,b]$ both *Hilbert* spaces, it follows that $\mathbb{R}L_2^{m,p}[a,b]$ is also a *Hilbert* space. Occasionally, we omit the domain and simply write L_2^p or $\mathbb{R}L_2^{m,p}$. We also omit the inner-product subscripts whenever it is clear from the context and define the extended space $L_{2e}^p[0,\infty)$ as

$$L_{2e}^p[0,\infty) = \{x \quad : \quad x\in L_2^p[0,T] \quad \forall \quad T\geq 0\}.$$

We use the bold font to indicate scalar or vector-valued functions of the spatial variable, e.g. $\mathbf{x}(t)\in L_2^p[a,b]$. For these functions of time and space, $\partial_s\mathbf{x}$ denotes $\frac{\partial \mathbf{x}}{\partial s}$. Moreover, for simplicity of notation and in analogy with the usual state vector in ODE systems, $\mathbf{x}(t)$ is used to denote $\mathbf{x}$ evaluated in a given instant of time t .

Furthermore, we use the calligraphic font, e.g. $\mathcal{A}$, to represent bounded linear operators on *Hilbert* spaces, e.g. $\mathcal{A}\in B(X)$ where X is a *Hilbert* space with inner product $\langle\cdot,\cdot\rangle_X$. Finally, for any $\mathcal{A}\in B(X)$, $\mathcal{A}^*$ denotes the adjoint operator satisfying $\langle \mathbf{x}, \mathcal{A}\mathbf{y}\rangle_X = \langle \mathcal{A}^*\mathbf{x}, \mathbf{y}\rangle_X$ for all $\mathbf{x},\mathbf{y}\in X$, $\succ$ and $\succeq$ means positive definiteness and semidefiniteness of self-adjoint PI operators. The set of 4-PI operators is denoted $\mathbf{\Pi}_4$, and the cone of positive 4-PI operators, $\mathbf{\Pi}_4^+$.

In this section, we introduce the notation used in the paper and we provide the basis the

III. State Space and Convex Optimization: PIs, PIEs, and LPIs

PI operators is a spetial class of operators that generalize the algebraic structure of matrices to infinite-dimensional spaces. On the other hand, PIEs, which are parametrized by PI operators, generalize the state-space representation to linear time-invariant systems modeled by PDEs; it do so by removing continuity and boundary constraints associated with the PDE representation. Finally, the LPIs generalize the convex optimization problems of LMIs since its feasable spaces are the convex cone of positive PI operators.

These three ingredients are combined in this section, providing the basis for the PIE framework, used in the control problems addressed by this work. The usefulness of PIE framework is illustrated in the last section, where state-estimators are synthesized for PDEs with provable bounds and without the necessity of tailored discretization schemes.

A. An Algebraic set of Operators

We begin by defining the PI operators that parametrize PIE systems. After properly defining these operators, we show how they expand the computationally tractable parametrization in finite-dimensional systems, given by matrices, to infinite-dimensional systems; this motivates their use in PDE systems coupled with ODEs, where the state has both finite and infinite parts.

Definition 1. *We say* $\mathcal{P} = \mathit{\Pi}\left[\begin{array}{c|c} P & Q_1\\ \hline Q_2 & \{R_i\}\end{array}\right] \in \mathbf{\Pi}_4 \subset B(\mathbb{R}L_2^{m_1,n_1}, \mathbb{R}L_2^{m_2,n_2})$ *if there exists a matrix* P *and polynomials* $Q_1, Q_2, R_0, R_1,$ *and* R_2 *such that*

$$\left(\mathcal{P}\begin{bmatrix} x\\ \mathbf{x}\end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_a^b Q_1(\theta)\mathbf{x}(\theta)d\theta\\ Q_2(s)x + \mathcal{R}\mathbf{y}(s)\end{bmatrix},$$

$$(\mathcal{R}\mathbf{x})(s) = R_0(s)\mathbf{x}(s) + \int_a^s R_1(s,\theta)\mathbf{x}(\theta)d\theta + \int_s^b R_2(s,\theta)\mathbf{x}(\theta)d\theta.$$

If $m_1 = m_2$ and $n_1 = n_2$, this set of PI operators is closed under composition, addition, and adjoint; explicit formulae for these operations can be obtained in terms of the polynomial matrices used to parametrize them. [26].

As shown in the definition, the notation $\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1\\ \hline Q_2 & \{R_i\}\end{array}\right]$ is used to indicate a generic 4-PI operators, a particular class of bounded linear operators, associated with the matrix P and polynomial parameters Q_i, R_j. The associated dimensions (m_1, n_1, m_2, n_2) are inherited from the dimensions of the constant matrix $P\in\mathbb{R}^{m_2\times m_1}$ and polynomial matrices $Q_1(s)\in\mathbb{R}^{m_2\times n_1}$, $Q_2(s)\in\mathbb{R}^{n_2\times m_1}$, and $R_0(s), R_1(s,\theta), R_2(s,\theta)\in\mathbb{R}^{n_2\times n_1}$.

In the case where a dimension is zero, we use $\emptyset$ in place of the associated parameter with zero dimension and may keep the 4-PI notation. For example, the particular case of

$p = q = 0$ makes an operator

$$P = \Pi\left[\begin{array}{c|c} P & \emptyset \\ \hline \emptyset & \{\emptyset\} \end{array}\right] : \mathbb{R}^{m_1} \to \mathbb{R}^{m_2},$$

a purely multiplicative operator, as used in ODE systems, where the state has only a finite-dimensional component. For simplicity, we don't use the 4-PI notation in this particular case; we use the matrix representation of multiplier operators instead. This is always the case for the map from finite-dimensional inputs to finite-dimensional outputs.

On the other hand, by making $m = n = 0$, the resultant operator has the form

$$\Pi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{R_i\} \end{array}\right] : L_2^{n_1} \to L_2^{n_2},$$

which is a particular class of 4-PI operators, called 3-PI, used in systems of PDEs where the state has only an infinite-dimensional component. It was shown in [18] that 3-PI operators form a $*$-algebra, and thus, the set is endowed with algebraic operations used in matrix algebra: addition, composition, adjoint, and inverse.

Other particular case used to represent finite dimensional inputs to state behavior is $n_1 = 0$, giving

$$\Pi\left[\begin{array}{c|c} P & \emptyset \\ \hline Q_2 & \{\emptyset\} \end{array}\right] : \mathbb{R}^{m_1} \to \mathbb{R}L^{m_2,n_2}.$$

Finally, to represent state to finite dimensional outputs, we use

$$\Pi\left[\begin{array}{c|c} P & Q_1 \\ \hline \emptyset & \{\emptyset\} \end{array}\right] : \mathbb{R}L^{m_1,n_1} \to \mathbb{R}^{m_2}.$$

This generalization, together with the fact that polynomial matrices are easily stored in computers, allows the numerical results that have been presented so far using the PIEs framework: an algebraic mimic of established results in convex optimization of ODE systems in terms of PI operators and search for optimal solutions using the developed computational toolbox.

B. Partial Integral Equations

It has been shown in, e.g. [26], that a large class of coupled ODE-PDE systems with sensed and regulated outputs, $y(t) \in \mathbb{R}^{n_y}$, $z(t) \in \mathbb{R}^{n_z}$, and in-domain disturbances, $w(t) \in \mathbb{R}^{n_w}$, may be equivalently represented using a partial integral equation (PIE) of the form

$$\begin{aligned} \partial_t(\mathcal{T}\mathbf{x}(t)) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t), \qquad \mathbf{x}(0) = 0 \in \mathbb{R}L_2, \\ z(t) &= \mathcal{C}_1\mathbf{x}(t), \qquad y(t) = \mathcal{C}_2\mathbf{x}(t) + \mathcal{D}_{21}w(t), \end{aligned} \quad (1)$$

where the solution of the PIE, $\mathbf{x}(t) \in \mathbb{R}\times L_2$ yields a solution to the PDE as $\mathcal{T}\mathbf{x}(t)$. The PIE state, $\mathbf{x}(t)$ combines ODE state with a spatial derivative of the PDE state and admits no boundary conditions or continuity contraints. The system parameters $\mathcal{A}, \mathcal{B}_1, \mathcal{C}_2$, et c. are all 4-PI operators.

The solution of this class of PIE is formally defined as follows, where $x \in L_{2e}^p[0,\infty)$ means $x(t) \in \mathbb{R}^p$ and $\int_0^T \|x(t)\|^2 dt$ if finite for all $T \geq 0$.

Definition 2 (PIE solution)**.** *Given PI operators $\mathcal{T}$, $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{D}_{21}$ we say $\{\mathbf{x}, z, y\}$ is a solution to the PIE system for given initial condition $\mathbf{x}(0) \in \mathbb{R}L_2^{m,n}[a,b]$ and input $w \in L_{2e}^{n_w}[0,\infty)$, if $\mathcal{T}\mathbf{x}(t)$ is Frechét differentiable for all $t \in [0,\infty)$, and if $\mathbf{x}(t) \in \mathbb{R}L_2^{m,n}[a,b]$, $z \in L_{2e}^{n_z}[0,\infty)$, and $y \in L_{2e}^{n_y}[0,\infty)$ satisfy Eq.* (1) *for all $t \in [0,\infty)$.*

C. Linear PI Operator Inequalities

As described in Subsection **??**, 4-PI operators of the form given in Defn. 1 constitute a composition algebra of bounded linear operators and are parameterized by polynomial matrices, which in turn can be parameterized by the coefficients of those polynomials. In this paper, we reformulate the problem of H_2-optional estimator synthesis as an optimization problem where the decision variables are themselves PI operators and are subject to inequality constraints which are affine in those decision variables – See, e.g. Eqn. (12) in Thm. 8. Optimization problems in this form may be solved by using matrices to parameterize the coefficients of the polynomials which define the PI operator variables. Inequalities are enforced by using positive matrices to parameterize positive PI operators, as described in [18]. For convenience, we denote the set of positive 4-PI operators which admit such a parameterization as Π_4^+.

Thus, affine inequality constraints of the form, e.g.

$$\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} \prec 0$$

can be enforced by using the equality constraint $\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} + \epsilon I = -\mathcal{Q}$ where $\mathcal{Q} \in \Pi_4^+$ and the equality constraint is enforced by equating the coefficients of the polynomials which define $\mathcal{Q} \in \Pi_4$ and $\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} + \epsilon I \in \Pi_4$.

IV. Problem Formulation

The purpose of this section is to introduce a suitable time-domain characterization of the H_2 norm and use this characterization to define the problems of H_2 norm bounding and H_2-optimal estimation for systems which admit a PIE representation.

A. The H_2 norm of a PIE

For this subsection, we restrict our consideration to characterization of the H_2 norm of a system represented by a PIE of the form

$$\begin{aligned} \partial_t(\mathcal{T}\mathbf{x}(t)) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t), \\ z(t) &= \mathcal{C}_1\mathbf{x}(t), \qquad \mathbf{x}(0) = 0, \end{aligned} \quad (2)$$

where $\mathbf{x}(t) \in \mathbb{R}L_2$ is the state, $w \in L_2[0\infty)$ is a disturbance, and z is the output. Specifically, in Defn. 3, we define the H_2 norm of this system as L_2-gain of initial condition to output of an auxiliary system with no disturbance. While non-standard, we will see that this characterization of H_2 performance is equivalent in a certain sense to the standard definition of H_2 norm.

Definition 3. *Consider solutions of the auxiliary PIE*

$$\begin{aligned} \partial_t(\mathcal{T}\mathbf{x}(t)) &= \mathcal{A}\mathbf{x}(t), \\ z(t) &= \mathcal{C}_1\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathcal{B}_1 x_0. \end{aligned} \quad (3)$$

We define the H_2 norm of System (2) *(denoted G) as*

$$\|G\|_{H_2} := \sup_{\substack{z,\mathbf{x} \text{ satisfy } (3) \\ \|x_0\|=1}} \|z\|_{L_2}$$

we have that $\|z\|_{L_2} \leq \gamma$.

To see the relationship between the definition of H_2 norm in Definition 3 and the standard definition, recall the usual state-space representation of an ODE.

$$\begin{bmatrix}\dot{x}(t)\\ z(t)\end{bmatrix}=\begin{bmatrix}A & B\\ C & 0\end{bmatrix}\begin{bmatrix}x(t)\\ w(t)\end{bmatrix}\qquad \forall t\in[0,\infty) \tag{4}$$

Then if A is Hurwitz, and we define the transfer function as $\hat{G}(s)=C(sI-A)^{-1}B$, the standard definition of H_2 norm is given as

$$\|\hat{G}^2\|_{H_2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\text{trace}\left(G^*(i\omega)G(i\omega)d\omega\right)$$
$$\text{trace}\left(B_1^T\int_0^{\infty}e^{A^T\tau}C_1^TC_1e^{A\tau}d\tau B_1\right)$$

where we have used the inverse Laplace transform to obtain the time-domain characterization [27].

Corollary 4. *Suppose A is Hurwitz and $\hat{G}(s)=C(sI-A)^{-1}B$. Consider solutions of the auxiliary ODE*

$$\begin{aligned}&\dot{x}(t)=Ax(t),\\ &z(t)=Cx(t),\qquad x(0)=Bx_0,\end{aligned} \tag{5}$$

Then

$$\sup_{\substack{z \text{ satisfies } (5)\\ \|x_0\|=1}}\|z\|_{L_2}\le\|G\|_{H_2}\le n_w\sup_{\substack{z \text{ satisfies } (5)\\ \|x_0\|=1}}\|z\|_{L_2}$$

Proof. Suppose $\{x,z\}$ satisfy 4 with initial condition $x(0)=Bx_0$. Then $x(t)=e^{At}Bx_0$ and hence if $\|x_0\|=1$, we have

$$\begin{aligned}\|z\|_{L_2}^2&=\int_0^{\infty}x(\tau)^TC^TCx(\tau)d\tau\\ &=\int_0^{\infty}x_0^TB^Te^{A^T\tau}C^TCe^{A\tau}Bx_0d\tau\\ &\le\bar{\sigma}\left(\int_0^{\infty}B^Te^{A^T\tau}C^TCe^{A\tau}Bd\tau\right)\\ &\le\text{trace}\left(\int_0^{\infty}B^Te^{A^T\tau}C^TCe^{A\tau}Bd\tau\right)=\|G\|_{H_2}^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|G\|_{H_2}^2&=\text{trace}\left(\int_0^{\infty}B^Te^{A^T\tau}C^TCe^{A\tau}Bd\tau\right)\\ &\le n_w\bar{\sigma}\left(\int_0^{\infty}B^Te^{A^T\tau}C^TCe^{A\tau}Bd\tau\right)\\ &=n_w\sup_{\|x_0\|=1}\int_0^{\infty}x_0^TB^Te^{A^T\tau}C^TCe^{A\tau}Bx_0d\tau\\ &=n_w\sup_{\|x_0\|=1}\|z\|_{L_2}^2.\end{aligned}$$

□

Clearly, if the PIE has a single input, the proposed definition of H_2 norm coincides with the typical definition. Alternatively, in the case of multiple inputs, our time-domain characterization of H_2 norm would coincide with an alternative definition of H_2 norm given by

$$\|\hat{G}^2\|_{H_2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\sigma}\left(G^*(i\omega)G(i\omega)d\omega\right)$$

Having defined the H_2-norm, we proceed to present the two main problems of this work: computing optimal upper bounds on the H_2-norm of a PIE and synthesizing an optimal state-estimator based on the H_2-norm. Our approach derives a convex problem; optimal solutions can be ultimately searched by usual semi-definite programming solvers.

B. H_2-Optimal Estimators

Recall that our goal is to design an observer which provides an estimate of the state of a ODE-PDE with a PIE representation of the form,

$$\begin{aligned}&\partial_t(\mathcal{T}\mathbf{x}(t))=\mathcal{A}\mathbf{x}(t)+\mathcal{B}_1w(t),\qquad \mathbf{x}(0)=0,\\ &\quad z(t)=\mathcal{C}_1\mathbf{x}(t),\qquad y(t)=\mathcal{C}_2\mathbf{x}(t)+\mathcal{D}_{21}w(t),\end{aligned} \tag{6}$$

where recall the state of the original ODE-PDE is obtained from the solution of the PIE as $\mathcal{T}\mathbf{x}(t)$. The signal $y(t)$ are measurements of the ODE-PDE and $z(t)$ represents those parts of the state by which we will measure the performance of our estimator. Our estimator dynamics are then assumed to have the Luenberger observer structure

$$\begin{aligned}&\partial_t\left(\mathcal{T}\tilde{\mathbf{x}}(t)\right)=\mathcal{A}\tilde{\mathbf{x}}(t)+\mathcal{L}\left(\mathcal{C}_2\tilde{\mathbf{x}}(t)-y(t)\right),\\ &\tilde{\mathbf{x}}(0)=0,\end{aligned} \tag{7}$$

which mirror the dynamics of the observed system, but without the disturbance, which is unknown. The term, $\mathcal{C}_2\tilde{\mathbf{x}}(t)-y(t)$, reflects the difference between the predicted and measured output from the ODE-PDE. This term is weighted by the observer gain, $\mathcal{L}:\mathbb{R}^{n_y}\to\mathbb{R}L_2$ which is taken to be a PI operator. By combining the observer in Eqn. (7) with the measured output of an ODE-PDE, real-time estimates of the ODE-PDE state can be obtained as $\mathcal{T}\hat{\mathbf{x}}(t)$ and used in conjunction with state-feedback controllers or fault detection algorithms.

The H_2-optimal estimation problem, then, is to choose $\mathcal{L}$ which minimizes the H_2-norm of the map from disturbance w to error in the regulated output, which we define as $e_z(t)=\mathcal{C}_1\hat{\mathbf{x}}(t)-z(t)$. This map can likewise be represented as a PIE with state $\mathbf{e}(t)=\hat{\mathbf{x}}(t)-\mathbf{x}(t)$, where $\hat{\mathbf{x}}$ satisfies Eqn. (7) and $\mathbf{x}$ satisfies Eqn. (6) so that

$$\begin{aligned}&\partial_t\left(\mathcal{T}\mathbf{e}(t)\right)=(\mathcal{A}+\mathcal{L}\mathcal{C}_2)\mathbf{e}(t)-\left(\mathcal{B}_1+\mathcal{L}\mathcal{D}_{21}\right)w(t),\\ &\quad e_z(t)=\mathcal{C}_1\mathbf{e}(t),\qquad \mathbf{e}(0)=0.\end{aligned} \tag{8}$$

We see that System (2) is of the form in Eqn. (2) with $\mathcal{A}\mapsto\mathcal{A}+\mathcal{L}\mathcal{C}_2$, $\mathcal{P}\mapsto-\left(\mathcal{B}_1+\mathcal{L}\mathcal{D}_{21}\right)$ and $\mathcal{C}\mapsto\mathcal{C}_1$. Thus we can formulate the H_2-optimal synthesis problem using the auxiliary PIE from Defn. 3

$$\begin{aligned}&\partial_t\left(\mathcal{T}\mathbf{e}(t)\right)=(\mathcal{A}+\mathcal{L}\mathcal{C}_2)\mathbf{e}(t),\\ &\quad e_z(t)=\mathcal{C}_1\mathbf{e}(t),\qquad \mathbf{e}(0)=-\left(\mathcal{B}_1+\mathcal{L}\mathcal{D}_{21}\right)x_0.\end{aligned} \tag{9}$$

as

$$\min_{\mathcal{L}\in\Pi}\sup_{\substack{z,\mathbf{e}\text{ satisfy }(9)\\ \|x_0\|=1}}\|e_z\|_{L_2}. \tag{10}$$

In Section VI, we will reformulate the H_2-optimal estimation problem as an LPI. First, however, we need to address the problem of how to use LPIs to compute the H_2 norm of a PIE.

V. An LPI for the H_2 norm

In this section, we show how to use LPIs to compute the H_2 norm of a PIE. We begin by reformulating the following result from [25].

Theorem 5. *Suppose $\mathcal{T},\mathcal{A},\mathcal{C} \in \Pi_4$. Suppose there exists some $\mathcal{P} \succ 0$ such that:*

$$trace(\mathcal{B}^*\mathcal{P}\mathcal{B}) < \gamma^2,$$
$$\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{C}^*\mathcal{C} \prec 0. \tag{11}$$

Then

$$\sup_{\substack{z,\mathbf{x} \text{ satisfy } (3)\\ \|x_0\|=1}} \|z\|_{L_2} < \gamma$$

We now use an extension of the Schur complement to obtain an LPI for bounding the H_2 norm which will be used for estimator design in Section VI. This reformulation, however, requires us to define vertical and horizontal concatenation of Π_4 operators such that the concatenated operator is in Π_4 (See Lemmas 39,40 from [26]). This definition separately concatenates the real and distributed portions of the operator so that if, e.g. $\mathcal{P} \in \mathcal{L}(\mathbb{R}^n \times L_2^m)$ and $\mathcal{Q} \in \mathcal{L}(\mathbb{R}^p \times L_2^q)$, then

$$\begin{bmatrix} P & 0\\ 0 & Q\end{bmatrix} \in \mathcal{L}(\mathbb{R}^{n+p} \times L_2^{m+q}).$$

Lemma 6 (Schur Complement)**.** *Suppose $\mathcal{P},\mathcal{Q},\mathcal{R} \in \Pi_4$. Then the following are equivalent.*

1) $\begin{bmatrix} P & Q^*\\ Q & R\end{bmatrix} \succ \epsilon I$.

2) $R - Q^*P^{-1}Q \succ \epsilon I$ *and* $P \succ \epsilon I$

Proof. In this proof, there is no rearrangement of rows or columns. Now, mirroring the standard proof of the Schur complement, suppose that 1) is true. Then, we have

$$\langle \mathbf{x}, \mathcal{P}\mathbf{x}\rangle = \left\langle \begin{bmatrix}\mathbf{x}\\0\end{bmatrix}, \begin{bmatrix}\mathcal{P} & \mathcal{Q}^*\\ \mathcal{Q} & \mathcal{R}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\0\end{bmatrix}\right\rangle \geq \epsilon \|\mathbf{x}\|^2$$

which implies that $\mathcal{P}$ is invertible. Now note that

$$\begin{bmatrix}\mathcal{P} & 0\\ 0 & \mathcal{R} - \mathcal{Q}^*P^{-1}\mathcal{Q}\end{bmatrix} = \begin{bmatrix} I & -\mathcal{P}^{-1}\mathcal{Q}\\ 0 & I\end{bmatrix}^* \begin{bmatrix}\mathcal{P} & \mathcal{Q}^*\\ \mathcal{Q} & \mathcal{R}\end{bmatrix}\begin{bmatrix} I & -\mathcal{P}^{-1}\mathcal{Q}\\ 0 & I\end{bmatrix}$$

and hence

$$\left\langle \mathbf{x}, (\mathcal{R} - \mathcal{Q}^*P^{-1}\mathcal{Q})\mathbf{x}\right\rangle = \left\langle \begin{bmatrix}0\\ \mathbf{x}\end{bmatrix}, \begin{bmatrix}\mathcal{P} & 0\\ 0 & \mathcal{R} - \mathcal{Q}^*P^{-1}\mathcal{Q}\end{bmatrix}\begin{bmatrix}0\\ \mathbf{x}\end{bmatrix}\right\rangle$$
$$= \left\langle \begin{bmatrix}-\mathcal{P}^{-1}\mathcal{Q}\mathbf{x}\\ \mathbf{x}\end{bmatrix}, \begin{bmatrix}\mathcal{P} & \mathcal{Q}^*\\ \mathcal{Q} & \mathcal{R}\end{bmatrix}\begin{bmatrix}-\mathcal{P}^{-1}\mathcal{Q}\mathbf{x}\\ \mathbf{x}\end{bmatrix}\right\rangle$$
$$\geq \epsilon \left\|\begin{bmatrix}-\mathcal{P}^{-1}\mathcal{Q}\mathbf{x}\\ \mathbf{x}\end{bmatrix}\right\|^2 \geq \epsilon\|\mathbf{x}\|^2$$

For the converse, suppose 2) is true. Then

$$\begin{bmatrix}\mathcal{P} & \mathcal{Q}^*\\ \mathcal{Q} & \mathcal{R}\end{bmatrix} = \begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q}\\ 0 & I\end{bmatrix}^* \begin{bmatrix}\mathcal{P} & 0\\ 0 & \mathcal{R} - \mathcal{Q}^*P^{-1}\mathcal{Q}\end{bmatrix}\begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q}\\ 0 & I\end{bmatrix}$$

which implies

$$\left\langle \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}, \begin{bmatrix}\mathcal{P} & \mathcal{Q}^*\\ \mathcal{Q} & \mathcal{R}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}\right\rangle \geq \epsilon \left\|\begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q}\\ 0 & I\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}\right\|^2$$

Now, define $\left\|\begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q}\\ 0 & I\end{bmatrix}^{-1}\right\|_{\mathcal{L}(\mathbb{R}L_2)} = \delta$. Then

$$\left\|\begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q}\\ 0 & I\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}\right\|^2 \geq \delta \left\|\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}\right\|^2$$

and hence

$$\left\langle \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}, \begin{bmatrix}\mathcal{P} & \mathcal{Q}^*\\ \mathcal{Q} & \mathcal{R}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}\right\rangle \geq \epsilon\delta \left\|\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}\right\|^2$$

as desired. □

Theorem 7. *Suppose $\mathcal{T},\mathcal{A},\mathcal{C} \in \Pi_4$. Suppose there exists some matrix $W \geq 0$, a 4-PI operator $P \succ 0$, and $\epsilon > 0$ such that:*

$$\begin{bmatrix} -\gamma I & \mathcal{C}\\ \mathcal{C}^* & \mathcal{T}^*P\mathcal{A} + \mathcal{A}^*P\mathcal{T}\end{bmatrix} \prec -\epsilon I \tag{12}$$
$$\begin{bmatrix} W & \mathcal{B}^*\mathcal{P}\\ \mathcal{P}\mathcal{B} & \mathcal{P}\end{bmatrix} \succ 0 \tag{13}$$
$$trace(W) < \gamma, \tag{14}$$

Then

$$\sup_{\substack{z,\mathbf{x} \text{ satisfy } (3)\\ \|x_0\|=1}} \|z\|_{L_2} < \gamma$$

Proof. Suppose $\gamma,\mathcal{P},\mathcal{Z}$ are as stated above. Then, Inequality (12) combined with Lemma 6 implies

$$\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} + \frac{1}{\gamma}\mathcal{C}^*\mathcal{C} \prec -\epsilon I$$

Likewise, Inequality (13) combined with Lemma 6 implies

$$W - \mathcal{B}^*\mathcal{P}\mathcal{P}^{-1}\mathcal{P}\mathcal{B} = W - \mathcal{B}^*\mathcal{P}\mathcal{B} > 0.$$

Now W and $\mathcal{B}^*\mathcal{P}\mathcal{B}$ are matrices and hence $\mathrm{trace}(\mathcal{B}^*\mathcal{P}\mathcal{B}) < \mathrm{trace}\, W < \gamma$. Define $\hat{\mathcal{P}} = \gamma\mathcal{P}$ so that $\mathcal{P} = \frac{1}{\gamma}\hat{\mathcal{P}}$ and hence

$$\mathcal{A}^*\hat{\mathcal{P}}\mathcal{T} + \mathcal{T}^*\hat{\mathcal{P}}\mathcal{A} + \mathcal{C}^*\mathcal{C} \prec -\gamma\epsilon I \qquad \mathrm{trace}(\mathcal{B}^*\hat{\mathcal{P}}\mathcal{B}) < \gamma^2$$

which implies the conditions of Thm 5 are satisfied. □

In Section VI, we use this LPI for the H_2 norm to synthesize observers which minimize a bound on the H_2 norm of the error dynamics.

VI. An LPI for H_2-optimal Estimator

In this section, we consider the problem of designing the estimator gain $\mathcal{L} \in \Pi_4$ which minimizes a bound on the H_2 norm of the error dynamics defined in Subsection IV-B. Specifically, recall these error dynamics are given by

Theorem 8. *Suppose there exist $\epsilon > 0, \delta > 0$, $\gamma \in \mathbb{R}$, matrix W, and PI operators $\mathcal{P} \succeq \epsilon I$ and $\mathcal{Z}$, such that*

$$\begin{bmatrix} -\gamma I & \mathcal{C}_1\\ \mathcal{C}_1^* & \mathcal{T}^*P\mathcal{A} + \mathcal{A}^*P\mathcal{T} + \mathcal{T}^*\mathcal{Z}\mathcal{C}_2 + \mathcal{C}_2^*\mathcal{Z}^*\mathcal{T}\end{bmatrix} \preceq \epsilon I \tag{15}$$
$$\begin{bmatrix} W & \mathcal{B}_1^*\mathcal{P} + D_{21}^T\mathcal{Z}^*\\ \mathcal{P}\mathcal{B}_1 + \mathcal{Z}D_{21} & \mathcal{P}\end{bmatrix} \succeq 0$$
$$\mathrm{trace}(W) < \gamma, \tag{16}$$

Then, the H_2-norm of the system in Eq. (8) *is upper bounded by γ. Moreover, from Eq.* **??***, $\tilde{x}(t)$ exponentially approaches $x(t)$, $\tilde{\mathbf{x}}(t)$ exponentially approaches $\mathbf{x}(t)$, and $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$.*

Proof. Let $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$. Then

$$\begin{bmatrix} -\gamma I & \mathcal{C}_1\\ \mathcal{C}_1^* & \mathcal{T}^*\mathcal{P}\left(\mathcal{A} + \mathcal{L}\mathcal{C}_2\right) + \left(\mathcal{A} + \mathcal{L}\mathcal{C}_2\right)^* P\mathcal{T}\end{bmatrix}$$
$$= \begin{bmatrix} -\gamma I & \mathcal{C}_1\\ \mathcal{C}_1^* & \mathcal{T}^*\mathcal{P}\left(\mathcal{A} + \mathcal{P}^{-1}\mathcal{Z}\mathcal{C}_2\right) + \left(\mathcal{A} + \mathcal{P}^{-1}\mathcal{Z}\mathcal{C}_2\right)^* P\mathcal{T}\end{bmatrix}$$
$$= \begin{bmatrix} -\gamma I & \mathcal{C}_1\\ \mathcal{C}_1^* & \mathcal{T}^*P\mathcal{A} + \mathcal{A}^*P\mathcal{T} + \mathcal{T}^*\mathcal{Z}\mathcal{C}_2 + \mathcal{C}_2^*\mathcal{Z}^*\mathcal{T}\end{bmatrix} \prec 0$$

Likewise

$$\begin{bmatrix} W & -\left(\mathcal{B}_1+\mathcal{L}D_{21}\right)^*\mathcal{P} \\ -\mathcal{P}\left(\mathcal{B}_1+\mathcal{L}D_{21}\right) & \mathcal{P} \end{bmatrix}$$
$$= \begin{bmatrix} W & -\left(\mathcal{B}_1+\mathcal{P}^{-1}\mathcal{Z}D_{21}\right)^*\mathcal{P} \\ -\mathcal{P}\left(\mathcal{B}_1+\mathcal{P}^{-1}\mathcal{Z}D_{21}\right) & \mathcal{P} \end{bmatrix}$$
$$= \begin{bmatrix} W & \mathcal{B}_1^*\mathcal{P}+D_{21}^T\mathcal{Z}^* \\ \mathcal{P}\mathcal{B}_1+\mathcal{Z}D_{21} & \mathcal{P} \end{bmatrix} \succ 0$$

Finally, $\operatorname{trace}(W)<\gamma$. Now, by applying Theorem 7, the above equations imply that γ is an upper bound on the H_2-norm of the PIE system defined by $\{\mathcal{T},(\mathcal{A}+\mathcal{L}\mathcal{C}_2),(\mathcal{B}_1+\mathcal{L}D_{21}),\mathcal{C}_1\}$ as in Eq. 8 and that the error approaches zero exponentially. □

Theorem 8, provides an optimization problem with 4-PI variables and LPI constraints, Eq. (15). Since Proposition 10 provides a convex constraint by appropriately choosing $g(s)$ and restricting the polynomial degrees, these constraints are convex, as well as the objective function γ.

Corollary 9. *Given $\epsilon\geq 0$, if*

$$\begin{aligned} \gamma^{2*} := & \min_{\gamma^2\in\mathbb{R},\mathcal{P},\mathcal{Z}\in\mathbf{\Pi}_4, W\succeq 0_+} && \gamma && (17)\\ & \operatorname{trace}(W) && \leq\gamma^2, \\ & \mathcal{P}-\epsilon I && \in\mathbf{\Pi}_4^+, \\ & \begin{bmatrix} \mathcal{P} & -(\mathcal{P}\mathcal{B}_1+\mathcal{Z}D_{21}) \\ -(\mathcal{P}\mathcal{B}_1+\mathcal{Z}D_{21})^* & W \end{bmatrix} && \in\mathbf{\Pi}_4^+, \\ & -\left(\mathcal{T}^*\mathcal{P}\mathcal{A}+\mathcal{A}^*\mathcal{P}\mathcal{T}+\mathcal{T}^*\mathcal{Z}\mathcal{C}_2+\mathcal{C}_2^*\mathcal{Z}^*\mathcal{T}+\mathcal{C}_1^*\mathcal{C}_1\right) && \in\mathbf{\Pi}_4^+, \end{aligned}$$

and $\{e,\mathbf{e},e_z\}$ satisfies the PIE in Eq. (8) for some initial e_0 and $\mathbf{e}_0$, we have that $\|e_z\|_{L_2}^2\leq\gamma^2\|\mathbf{x}_0\|_2^2$ and thus, by definition, the H_2 norm of Eq. (8) is upper bounded by γ.

The above SDP may be parsed by PIETOOLS. [24], resulting in the operators $\mathcal{P}$ and $\mathcal{Z}$. The estimator gain is the 4-PI operator $\mathcal{L}=\mathcal{P}^{-1}\mathcal{Z}$, which optimizes the performance of the error system Eq. (8), in terms of minimizing the H_2 norm in Definition 3. The reconstruction of the gains from the SDP solution is discussed in the next section.

VII. Estimator Reconstruction

The observer designed in the last section thus is a map $\mathcal{L}:\mathbb{R}^{n_y}\to\mathbb{R}L_2[a,b]$. In this section, we partition the state into a finite-dimensional state $x_1(t)\in\mathbb{R}^m$, and and infinite-dimensional state $\mathbf{x}(t)\in L_2^n[a,b]$, i.e. $\mathbf{x}(t)=\begin{bmatrix}x_1(t)\\ \mathbf{x}_2(t)\end{bmatrix}$. Thus, the structure of the estimator is parametrized by the matrix L_1 and polynomial matrix $L_2(s)$ as:

$$\mathcal{L}y(t)=\begin{bmatrix}L_1y(t)\\ L_2(s)y(t)\end{bmatrix}.$$

Recall the sensed out put has the form $y(t)=\mathcal{C}_2\mathbf{x}(t)+D_{21}w(t)$ where $\mathcal{C}_2\in\Pi_4$ has the form

$$\mathcal{C}_2\begin{bmatrix}x_1(t)\\ \mathbf{x}_2(t)\end{bmatrix}=C_{2,1}x_1(t)+\int_a^b C_{2,2}(\theta)x_2(\theta,t)d\theta.$$

Thus, the correction term added to the estimator dynamics in Eq.(7) becomes

$$\mathcal{L}\left(C_2^1x_1(t)+\int_a^b C_2^2(\theta)x_2(\theta,t)d\theta-y(t)\right)$$
$$=\begin{bmatrix} L_1\left(C_2^1x_1(t)-y(t)+\int_a^b C_2^2(\theta)x_2(\theta,t)d\theta\right) \\ L_2(s)\left(C_2^1x_1(t)-y(t)+\int_a^b C_2^2(\theta)x_2(\theta,t)d\theta\right)\end{bmatrix}, \quad (18)$$

The gains L_1 and $L_2(s)$ are computed by inverting the self-adjoint operator $\mathcal{P}$. In PIETOOLS, this inversion is done by a numerical approximation in general; however, in the particular case where the integral operators have a separable kernel ($R_1=R_2$) in Definition 1, an analytical formula exists. Lemma 17 of [] proves that the inversion can be approximated arbitrarily well in the general case.

It can be formally proved that a general class of PDEs allows for an equivalent PIE representation as in Eq. (1) and the two representations have the same internal stability and input-output properties (see [18] for linear time-invariant 1D PDEs with polynomial coefficients, [28] for a broad class of time delay systems, and [26] for more general time-invariant PDEs. In the next section, numerical examples of the main results are presented. The original problem is given in terms of PDEs that are converted to PIEs.

VIII. Numerical Examples

This tedious process of constructing the PIE representation has been automated in the PIETOOLS software package [29], [30] with a dedicated command line and GUI input formats. Typically, given a coupled ODE-PDE with sufficient boundary conditions, one can find a PIE representation of the form Eq. (**??**) using Cauchy's rule for repeated integration [18], [28], [26]. Besides converting PDEs or time-delay systems to PIEs, PIETOOLS offers convenient Matlab functions to, declare PI decision variables, add LPI constraints, and solve the resulting optimization problem.

For the following examples, namely, an unstable reaction-diffusion PDE and a neutrally-stable Euler-Bernoulli beam, we obtain a PIE representation of the PDEs and apply the results from Cor. 9 to find the H_2-optimal observers. Utilizing PIETOOLS functions to invert positive PI operators, we construct the closed-loop observer systems, which are simulated using first-order backward difference integration scheme for certain initial conditions and disturbance. For each example, we also provide a numerical estimate of the H_2-norm of the error (i.e., $\|e_z\|_{L_2}$ for $x_0=1$) as observed in the simulations — i.e., by performing numerical integration of the simultation output $z(t)^2$ to obtain $\|e_z\|_{L_2}$.

A. Reaction-diffusion proccess

In this example, we consider the reaction-diffusion PDE given by

$$\begin{aligned} &\dot{\mathbf{x}}(t,s)=3\mathbf{x}(t,s)+(s^2+0.2)\partial_s^2\mathbf{x}(t,s)+\frac{s^2-2s}{2}w(t), \\ &z(t)=\begin{bmatrix}\int_0^1\mathbf{x}(t,s)ds\\ u(t)\end{bmatrix}, \quad y(t)=\mathbf{x}(t,1)+w(t), \\ &\mathbf{x}(t,0)=\partial_s\mathbf{x}(t,1)=0. \end{aligned} \quad (19)$$

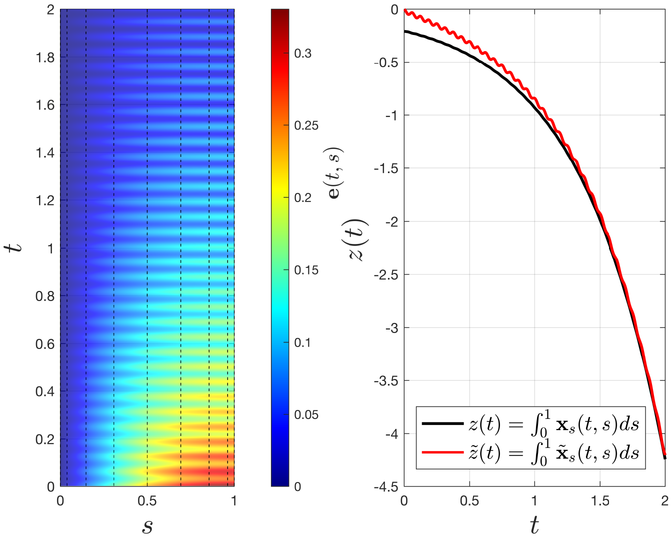

Fig. 1: This figure plots the error ($\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$) in the state estimate (on the left) and regulated error output (on the right) of the state observer for the system Eq. (19). The observer is initialized with zero initial conditions, whereas the PDE state starts with an initial condition $\mathbf{x}(0,s) = s$ and disturbance $w(t) = sin(100t)$. H_2-norm of the observer error system is 2.2949 (*numerical estimate* 0.23) where regulated and observed outputs are as defined in Subsec. VIII-A.

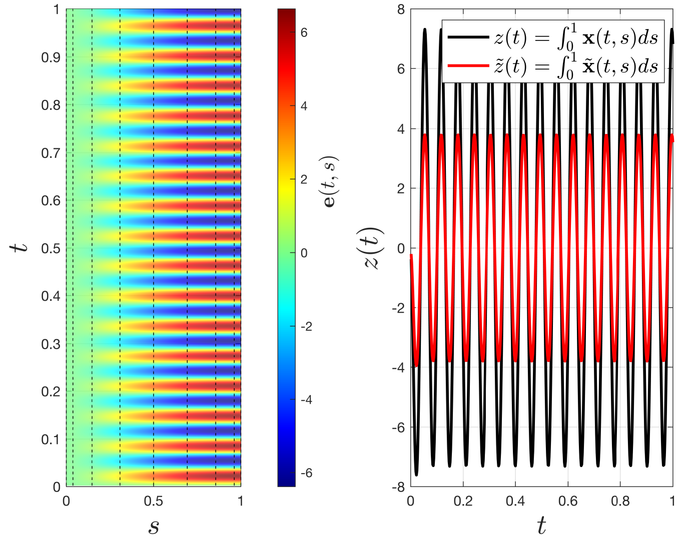

Fig. 2: This figure plots the error ($\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$) in the state estimate (on the left) and regulated outputs (on the right) of the state observer for the system Eq. (**??**). The observer is initialized with zero initial conditions, whereas the PDE state starts with zero initial conditions, under disturbance $w(t) = sin(100t), t \geq 0$. H_2-norm of the observer error system is 0.2615(Numerical estimate ?).

For the above PDE, we use PIETOOLS toolbox to obtain a PIE representation and then solve the LPI optimization problems in Cor. 8 and **??** to obtain the gains corresponding to the H_2-optimal estimator. Then, the closed-loop PIE system is constructed and simulated using the PIESIM module of the PIETOOLS toolbox in MATLAB to find the estimation error under disturbance and initial conditions $\mathbf{x}(0,s) = \frac{s^2-2s}{2}$ (i.e., $u_0 = 1$); we initialize the observer state at zero. In Fig. 1, we show the response of the error system — i.e., we plot the error between state-estimate ($\hat{\mathbf{x}}$) and actual state ($\mathbf{x}$), given by $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$. Additionally, we also plot the regulated output of the error system given by $\hat{z}(t) - z(t) = \int_0^1 \mathbf{e}(t,s)ds$.

$\gamma^* = 2.0526$.

$Q_2(s) = 0.021514*s^10+2.9692*s^9-13.7173*s^8+24.6017*s^7-$ $-4.8494*s-$

B. 2D Blasius boundary layer

C. Eurler-Bernoulli beam

Consider an elastic beam distributed over the domain, fixed boundary at $s = 0$ and free boundary at $s = 1$ under the simplified hypothesis of the Euler-Bernoulli theory [31]:

$$\partial_t^2 \eta(s,t) = c\partial_s^4 \eta(s,t) - 0.5 * s^2 w(t), s \in (0,1), \forall t \in [0,\infty),$$
$$\eta(0,t) = \partial_s \eta(0,t) = 0,$$
$$\partial_s^2 \eta(1,t) = \partial_s^3 \eta(1,t) = 0,$$
$$y(t) = \partial_t \eta(s,1) + sin(100t)/100 \tag{20}$$

where $\eta(s,t)$ is the vertical displacement field of the beam, $w(t)$ is a disturbance in terms of acceleration, distributed

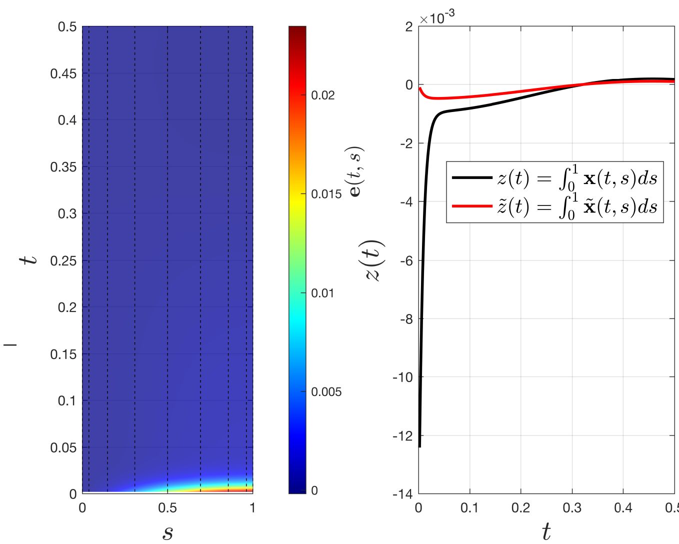

Fig. 3: This figure plots the error ($\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$) in the state estimate (on the left) and regulated outputs (on the right) of the state observer for the system Eq. (**??**). The observer is initialized with zero initial conditions, whereas the PDE state starts with zero initial conditions, under disturbance $w(t) = sin(100t), t \geq 0$. H_2-norm of the observer error system is 0.2615(Numerical estimate ?).

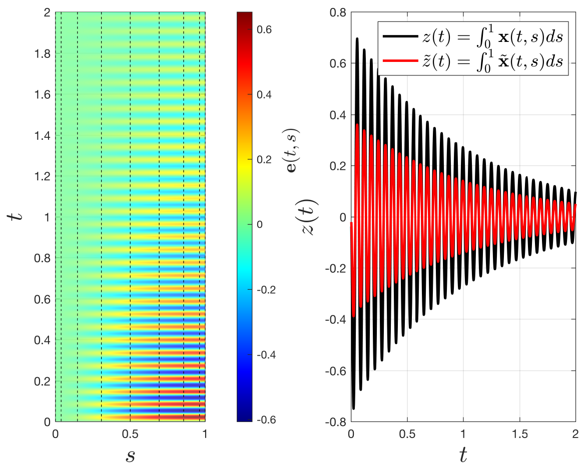

Fig. 4: This figure plots the error ($\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$) in the state estimate (on the left) and regulated outputs (on the right) of the state observer for the system Eq.?. The observer is initialized with zero initial conditions, whereas the PDE state starts with zero initial conditions, under disturbance $w(t) = 10^4 exp(-t) sin(100t), t \geq 0$. H_2-norm of the observer error system is 0.2615(Numerical estimate ?).

over the domain by the polynomial $-0.5 * s^2$.

$$\begin{aligned}
&\dot{\mathbf{x}}(t,s) = \begin{bmatrix} 0 & -0.1 \\ 1 & 0 \end{bmatrix} \partial_s^2 \mathbf{x}(t,s) + \begin{bmatrix} -0.5s^2 \\ 0 \end{bmatrix} w(t), \\
&\eta(0,t) = \partial_s \eta(0,t) = 0, \\
&\partial_s^2 \eta(1,t) = \partial_s^3 \eta(1,t) = 0, \\
&y(t) = \partial_t \eta(t,1) + w(t), \qquad (21)
\end{aligned}$$

IX. Conclusion

In this paper, we solved the H_2-optimal estimation and control problems for PDEs using the PIE framework developed for the analysis and control of PDE systems. Since formulating a PDE analysis/control problem using the PIE representation does not introduce any conservatism and leads to solvable convex optimization problems called Linear PI Inequalities (LPIs), we showed that H_2 analysis, estimation and control problems for PDEs can be solved using convex optimization without conservatism. For this purpose, we utilized an alternative definition of H_2-norm of a system that does not rely on the transfer function or impulse input; Instead, we characterized H_2-norm as the gain from an initial condition to the output of the system. Using this alternative, but equivalent, definition of H_2-norm, we showed that a PIE system and its corresponding dual PIE system have the same H_2-norm. Using this duality, we formulated two versions of LPIs problems (a primal and dual) to upper bound the H_2-norm of the system that were later used to formulate H_2-optimal state estimator and state-feedback control problems for PIEs as convex LPI optimization problems. By solving these LPI optimization problems, we demonstrated the application of this framework in estimator design and controller synthesis for PDE numerical examples.

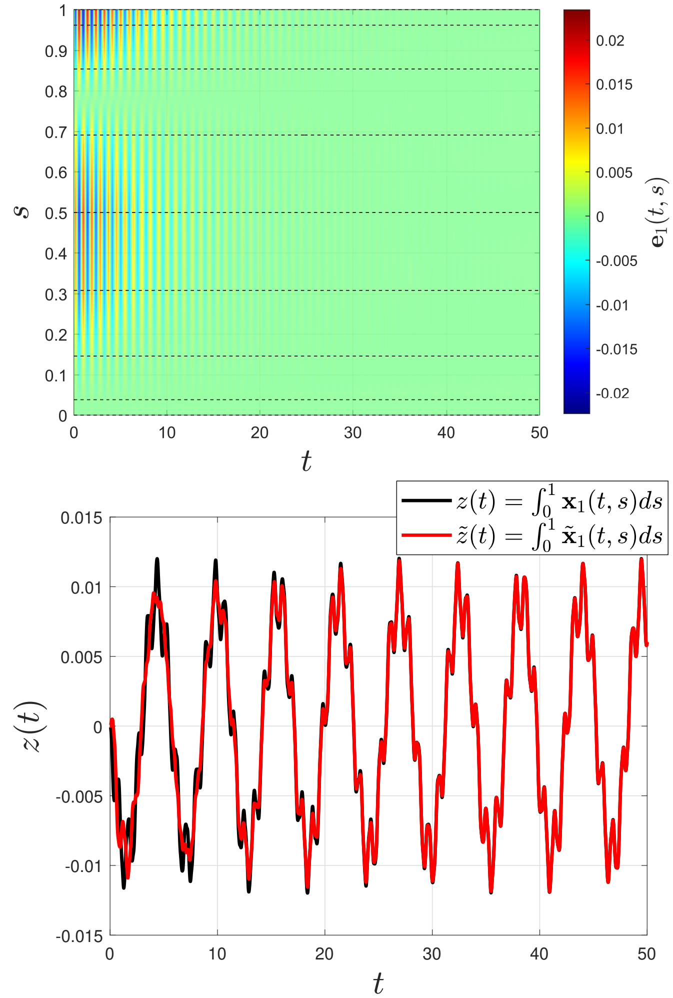

Fig. 5: The observer is initialized with zero initial conditions and disturbance $w(t) = exp(-10t) sin(10t)$.(a) Error ($\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$) in the state estimate. (b) Regulated error output of the state observer for the system Eq. (21). The H_2-norm of the observer error system is 0.3239 (Numerical estimate ?)

Appendix

A. LPI to LMI

Proposition 10. *Take $Z_1 : [a,b] \to \mathbb{R}^{d_1 \times n}$, $Z_2 : [a,b] \times [a,b] \to \mathbb{R}^{d_2 \times n}$ as matrices whose rows are vector monomial basis for polynomials with bounded degrees d_1 and d_2, respectively. Given $g(s) \geq 0$ for all $s \in [a,b]$ and*

$$\begin{aligned}
P &= T_{11} \int_a^b g(s) ds, \\
Q(\eta) &= g(\eta) T_{12} Z_1(\eta) + \int_\eta^b g(s) T_{13} Z_2(s,\eta) ds + \int_a^\eta g(s) T_{14} Z_2(s,\eta) ds, \\
R_1(s,\eta) &= g(s) Z_1(s)^\top T_{23} Z_2(s,\eta) + g(\eta) Z_2(\eta,s)^\top T_{42} Z_1(\eta) + \\
&\quad + \int_s^b g(\theta) Z_2(\theta,s)^\top T_{33} Z_2(\theta,\eta) d\theta + \int_\eta^s g(\theta) Z_2(\theta,s)^\top T_{43} Z_2(\theta,\eta) d\theta + \\
&\quad \int_a^\eta g(\theta) Z_2(\theta,s)^\top T_{44} Z_2(\theta,\eta) d\theta, \\
R_2(s,\eta) &= g(s) Z_1(s)^\top T_{32} Z_2(s,\eta) + g(\eta) Z_2(\eta,s)^\top T_{24} Z_1(\eta) + \\
&\quad \int_\eta^b g(\theta) Z_2(\theta,s)^\top T_{33} Z_2(\theta,\eta) d\theta + \int_s^\eta g(\theta) Z_2(\theta,s)^\top T_{34} Z_2(\theta,\eta) d\theta + \\
&\quad + \int_a^s g(\theta) Z_2(\theta,s)^\top T_{44} Z_2(\theta,\eta) d\theta, \\
R_0(s) &= g(s) Z_1(s)^\top T_{22} Z_1(s). \qquad (22)
\end{aligned}$$

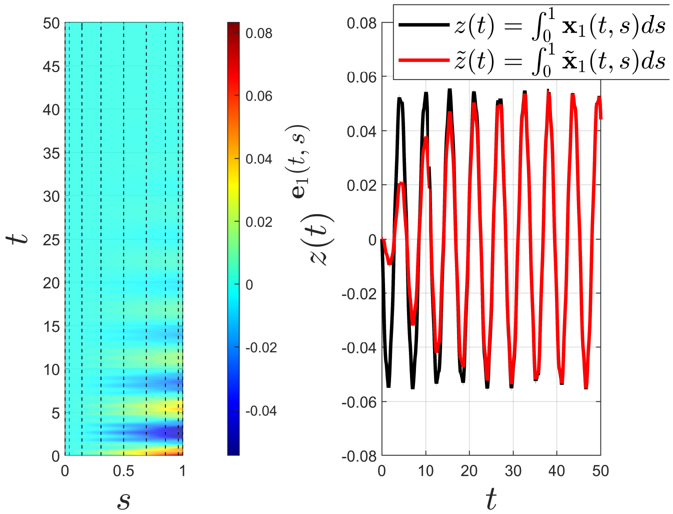

Fig. 6: This figure plots the error ($\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$) in the state estimate (on the left) and regulated error output (on the right) of the state observer for the system Eq. (21). The observer is initialized with zero initial conditions, whereas the PDE state starts with an initial condition $\mathbf{x}_1(0,s) = \partial_t \eta = -s^2/2$ without disturbance. The H_2-norm of the observer error system is 0.3241 (Numerical estimate).

where

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \succcurlyeq 0,$$

then the operator $\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right]$ *as defined in section III-A is positive semidefinite, i.e.* $\left\langle \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}, \mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] \right\rangle \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix} \geq$ 0 *for all* $\begin{bmatrix} x \\ \mathbf{x} \end{bmatrix} \in \mathbb{R}^m \times L_2^n[a,b]$.

B. Equivalence between PIEs and PDEs

To represent the class of problems covered by this work, we need to define the partitioned state of the original PDE system

$$\hat{\mathbf{x}}(t) = (\hat{\mathbf{x}}_0(t), \cdots, \hat{\mathbf{x}}_d(t)),$$

and the fundamental state that will appear in the PIE formulation,

$$\mathbf{x}(t) = (\hat{\mathbf{x}}_0(t), \cdots, \hat{\mathbf{x}}_d(t), \partial_s \hat{\mathbf{x}}_1(t), \cdots, \partial_s \hat{\mathbf{x}}_d(t), \cdots \partial_s^d \hat{\mathbf{x}}_d(t)),$$

which includes the PDE state and an ordered list with all its possible derivatives in s according to the continuity contraints [26]. In the above definitions, $\hat{\mathbf{x}}_i(t)$ is the part of the PDE state that admits up to the i-th derivative in s. Moreover, the differential operator $\mathbf{F}^d \hat{\mathbf{x}}(t) = \mathbf{x}(t)$, and $\Delta_{a,b}^d \mathbf{x}(t) = \begin{bmatrix} \Delta_a \\ \Delta_b \end{bmatrix} \mathbf{F}^d \hat{\mathbf{x}}(t)$, where $\Delta_a \mathbf{x}(t) = \mathbf{x}(t,a)$ is the *Dirac* operator.

The class of PDEs that may admit a PIE representation includes, but is not limited to, PDEs that have: ODE coupling, d-th order spatial derivatives, for some natural d, boundary terms, finite-dimensional inputs, and outputs. Using the 4-PI operators defined in the last section, the differential operator $\mathbf{F}^d \hat{\mathbf{x}}(t)$ and the operator $\Delta_{a,b}^d \hat{\mathbf{x}}(t)$, we can represent such PDEs as

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \\ \dot{\hat{\mathbf{x}}}(t) \end{bmatrix} = \mathit{\Pi}\left[\begin{array}{c|c} \begin{bmatrix} A & B_1 & B_{bx} \\ C_1 & D_{11} & C_{b1} \\ C_2 & D_{21} & C_{b2} \end{bmatrix} & \begin{bmatrix} B_{x,\mathbf{x}} \\ C_{\mathbf{x}1} \\ C_{\mathbf{x}2} \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{0} & \mathbf{B}_1 & \mathbf{0} \end{bmatrix} & \{A_i\} \end{array}\right] \begin{bmatrix} x(t) \\ w(t) \\ \Delta_{a,b}^{d-1} \hat{\mathbf{x}}(t) \\ \mathbf{F}^d \hat{\mathbf{x}}(t) \end{bmatrix},$$

$$\mathit{\Pi}\left[\begin{array}{c|c} \begin{bmatrix} B_x & B_w & B_b \end{bmatrix} & B_I \\ \hline \emptyset & \{\emptyset\} \end{array}\right] \begin{bmatrix} x(t) \\ w(t) \\ \Delta_{a,b}^{d-1} \hat{\mathbf{x}}(t) \\ \mathbf{F}^d \hat{\mathbf{x}}(t) \end{bmatrix} = 0,$$

$$\begin{bmatrix} x(0) \\ \hat{\mathbf{x}}(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \hat{\mathbf{x}}_0 \end{bmatrix} \in \mathbb{R}^m \times X \subset \mathbb{R}L_2^{m,n}[a,b], \tag{23}$$

where $x(t) \in R^{n_x}$ is the ODE state and $\hat{\mathbf{x}} \in X \subseteq L_2^n[a,b]$ is the variable in the PDEs, where X is the domain of the derivative operator $\mathbf{F}^d$, i.e. the space of functions satisfying the appropriate continuity constraints and boundary conditions [18]. The above parameterization extends the usual parametrization of input-output systems and allows:

- d-th-order spatial derivatives through the operator $\mathbf{F}^d$ and boundary valued terms through the operator $\Delta_{a,b}^{d-1}$, applied to the PDE part of the state, both of which can impact the dynamics (via A_i, $B_{x\mathbf{x}}$ and $B_{b\mathbf{x}}$), the outputs (via $C_{\mathbf{x}1}$, and $C_{\mathbf{x}2}$, C_{b1}, and C_{b2}), and the boundary conditions (via B_I and B_b);
- coupling with ODE via $B_{x,\mathbf{x}}$, $B_{b\mathbf{x}}$;
- linear boundary conditions, with possible effects from the ODE part of the state (via B_x), from the input (via B_w), and integral terms over the PDE part of the state and its existent derivatives (via B_I).

The problems with the PDE representation, from a computational point of view, can now be highlighted: the Derivative and Dirac operators cannot be appropriately represented in computers and, unlike 4-PIs, do not have algebraic properties similar to matrices [26]. In fact, $\mathbf{F}^d(t)$ is generally unbounded, and its domain imposes additional continuity constraints on $\hat{\mathbf{x}}(t)$, which is hard to represent in the optimization problems we wish to derive. On the other hand, $\Delta_{a,b}^{d-1}$ imposes singularities. In contrast, the PIE representation is fully parametrized by 4-PI operators.

In the PIE representation given in Eq.(1), the transformation given by $\mathcal{T} : RL^{m,n}[a,b] \to \mathbb{R}^m \times X$ is defined by successive applications of the Fundamental Theorem of Calculus, substituting the available boundary conditions. This transformation, when well defined according to the requisites detailed in [18] for simple PDEs and more generally in [26], gives an invertible map that formally states the equivalence between the two formats. In this paper, rather than restate the existent results, we show the equivalence for the particular class of systems defined above.

Theorem 11 (Equivalence between solutions)**.** *Given admissible polynomial matrices in Eq.* (23)*, according to the requisites in [26]. Let* $\mathcal{T}, \mathcal{A}, \mathcal{B}_1, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_{11}, \mathcal{D}_{12}$ *be 4-PI operators parametrizing Eq.* **??**. *Then*

$$\mathcal{A} = \mathit{\Pi}\left[\begin{array}{c|c} A - B_{bx} B_x & B_{x,\mathbf{x}} - B_{bx} B_I \\ \hline 0 & \{A_i\} \end{array}\right],$$

$$\mathcal{B} = \mathit{\Pi}\left[\begin{array}{c|c} B_1 - B_{bx} B_w & \emptyset \\ \hline \mathbf{B}_1 & \{A_i\} \end{array}\right]$$

$$\begin{bmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{bmatrix} = \mathit{\Pi}\left[\begin{array}{c|c} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} C_{b1} \\ C_{b2} \end{bmatrix} B_x & \begin{bmatrix} C_{x1} \\ C_{x2} \end{bmatrix} \\ \hline \emptyset & \{\emptyset\} \end{array}\right],$$

$$\begin{bmatrix} \mathcal{D}_{11} \\ \mathcal{D}_{12} \end{bmatrix} = \mathit{\Pi}\left[\begin{array}{c|c} \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} + \begin{bmatrix} C_{b1} \\ C_{b2} \end{bmatrix} B_w & \emptyset \\ \hline \emptyset & \{\emptyset\} \end{array}\right].$$

Furthermore, the following are equivalent

- $\{x, \hat{\mathbf{x}}\, z, y\}$ *satisfies the PDE for input* w *and initial condition*

$x_0, \hat{\mathbf{x}}_0$.

- $\{x, \mathbf{F}^d\hat{\mathbf{x}}\, z, y\}$ *satisfies the PIE for input* w *and initial condition* $x_0, \mathbf{F}^d\hat{\mathbf{x}}_0$.

Alternatively, we may say

- $\{x, \mathbf{x}, z, y\}$ *satisfies the PIE for input* w *and initial condition* $x_0, \mathbf{x}_0$.
- $\{x, \mathcal{T}\mathbf{x}, z, y\}$ *satisfies the PDE for input* w *and initial condition* x_0, x_0.

Proof. Eq. (23) may be rewritten as

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\hat{\mathbf{x}}}(t)\end{bmatrix} = \mathit{\Pi}\left[\begin{array}{c|c} A & B_{x,\mathbf{x}} \\ \hline 0 & \{A_i\}\end{array}\right]\begin{bmatrix}x(t)\\ \mathbf{F}^d\hat{\mathbf{x}}(t)\end{bmatrix} + \mathit{\Pi}\left[\begin{array}{c|c} B_1 & \emptyset \\ \hline \mathbf{B}_1 & \{\emptyset\}\end{array}\right] w(t) + \mathit{\Pi}\left[\begin{array}{c|c} B_{bx} & \emptyset \\ \hline \emptyset & \{\emptyset\}\end{array}\right]\Delta_{a,b}^{d-1}\hat{\mathbf{x}}(t) \tag{24}$$

$$\begin{bmatrix}z(t)\\ y(t)\end{bmatrix} = \mathit{\Pi}\left[\begin{array}{c|c} \begin{bmatrix}C_1\\ C_2\end{bmatrix} & \begin{bmatrix}C_{x1}\\ C_{x2}\end{bmatrix} \\ \hline \emptyset & \{\emptyset\}\end{array}\right]\begin{bmatrix}x(t)\\ \mathbf{F}^d\hat{\mathbf{x}}(t)\end{bmatrix} + \mathit{\Pi}\left[\begin{array}{c|c} \begin{bmatrix}D_{11}\\ D_{21}\end{bmatrix} & \emptyset \\ \hline \emptyset & \{\emptyset\}\end{array}\right] w(t) + \mathit{\Pi}\left[\begin{array}{c|c} \begin{bmatrix}C_{b1}\\ C_{b2}\end{bmatrix} & \emptyset \\ \hline \emptyset & \{\emptyset\}\end{array}\right]\Delta_{a,b}^{d-1}\hat{\mathbf{x}}(t) \tag{25}$$

.

Note that, from the boundary conditions equation:

$$\Delta_{a,b}^{d-1}\hat{\mathbf{x}}(t) = \mathit{\Pi}\left[\begin{array}{c|c} B_x & B_I \\ \hline \emptyset & \{\emptyset\}\end{array}\right]\begin{bmatrix}x(t)\\ \mathbf{x}(t)\end{bmatrix} + \mathit{\Pi}\left[\begin{array}{c|c} B_w & \emptyset \\ \hline \emptyset & \{\emptyset\}\end{array}\right] w(t) \tag{26}$$

To finish the proof, one just needs to substitute the state variable by the fundamental state in Eq. 24 and use the formulas for addition and composition of 4-PI operators [26]. □

References

[1] N. Fabbiane, O. Semeraro, S. Bagheri, and D. S. Henningson, "Adaptive and model-based control theory applied to convectively unstable flows," *Applied Mechanics Reviews*, vol. 66, no. 6, p. 060801, 2014.

[2] K. K. Chen and C. W. Rowley, "H_2 optimal actuator and sensor placement in the linearised complex Ginzburg–Landau system," *Journal of Fluid Mechanics*, vol. 681, pp. 241–260, 2011.

[3] W. Liu, Z. Hou, and M. A. Demetriou, "A computational scheme for the optimal sensor/actuator placement of flexible structures using spatial H_2 measures," *Mechanical systems and signal processing*, vol. 20, no. 4, pp. 881–895, 2006.

[4] P. Ambrosio, F. Resta, and F. Ripamonti, "An H_2 norm approach for the actuator and sensor placement in vibration control of a smart structure," *Smart Materials and Structures*, vol. 21, no. 12, p. 125016, 2012.

[5] A. Gahlawat and M. Peet, "A convex sum-of-squares approach to analysis, state feedback and output feedback control of parabolic PDEs," *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 1636–1651, 2017.

[6] S. Wu, C.-C. Hua, and M. Peet, "Estimator-based output-feedback stabilization of linear multi-delay systems using SOS," in *Proceedings of the IEEE Conference on Decision and Control*, 2019.

[7] A. Gahlawat and M. Peet, "Optimal state feedback boundary control of parabolic PDEs using SOS polynomials," in *Proceedings of the American Control Conference*, 2016.

[8] M. Peet, "Full-state feedback of delayed systems using SOS: A new theory of duality," in *Proceedings of the 11th IFAC Workshop on Time-Delay Systems*, 2013.

[9] A. Smyshlyaev and M. Krstic, "Closed-form boundary state feedbacks for a class of 1-d partial integro-differential equations," *IEEE Transactions on Automatic Control*, vol. 49, no. 12, pp. 2185–2202, 2004.

[10] P. Christofides and P. Daoutidis, "Feedback control of hyperbolic PDE systems," *AIChE Journal*, vol. 42, no. 11, pp. 3063–3086, 1996.

[11] A. Smyshlyaev and M. Krstic, "Backstepping observers for a class of parabolic pdes," *Systems & Control Letters*, vol. 54, no. 7, pp. 613–625, 2005.

[12] S. Mechhoud, E. Witrant, L. Dugard, and D. Moreau, "Estimation of heat source term and thermal diffusion in tokamak plasmas using a Kalman filtering method in the early lumping approach," *IEEE Transactions on Control Systems Technology*, vol. 23, no. 2, pp. 449–463, 2014.

[13] M. Balas, "Active control of flexible systems," *Journal of Optimization theory and Applications*, vol. 25, no. 3, pp. 415–436, 1978.

[14] F. Felici, M. De Baar, and M. Steinbuch, "A dynamic state observer for real-time reconstruction of the tokamak plasma profile state and disturbances," in *2014 American Control Conference*. IEEE, 2014, pp. 4816–4823.

[15] A. Schmidt and B. Haasdonk, "Reduced basis method for H_2 optimal feedback control problems," *IFAC-PapersOnLine*, vol. 49, no. 8, pp. 327–332, 2016.

[16] P. Benner and H. Mena, "Numerical solution of the infinite-dimensional lqr problem and the associated riccati differential equations," *Journal of Numerical Mathematics*, vol. 26, no. 1, pp. 1–20, 2018.

[17] A. A. Moghadam, I. Aksikas, S. Dubljevic, and J. F. Forbes, "Boundary optimal (LQ) control of coupled hyperbolic PDEs and ODEs," *Automatica*, vol. 49, no. 2, pp. 526–533, 2013.

[18] M. Peet, "A partial integral equation (PIE) representation of coupled linear PDEs and scalable stability analysis using LMIs," *Automatica*, vol. 125, p. 109473, 2021.

[19] S. Shivakumar, A. Das, S. Weiland, and M. Peet, "A generalized lmi formulation for input-output analysis of linear systems of odes coupled with pdes," in *Proceedings of the IEEE Conference on Decision and Control*, 2019.

[20] D. Jagt and M. Peet, "l_2-gain analysis of coupled linear 2D PDEs using Linear PI Inequalities," in *Proceedings of the IEEE Conference on Decision and Control*, 2022, pp. 6097–6104.

[21] A. Das, S. Shivakumar, S. Weiland, and M. Peet, "H_∞optimal estimation for linear coupled PDE systems," in *Proceedings of the IEEE Conference on Decision and Control*, 2019.

[22] D. S. Jagt and M. M. Peet, "H_∞-optimal estimator synthesis for coupled linear 2d pdes using convex optimization," *arXiv*, vol. preprint arXiv:2402.05061, 2024.

[23] S. Wu, M. M. Peet, F. Sun, and C. Hua, "H_∞-optimal observer design for systems with multiple delays in states, inputs and outputs: A pie approach," *International Journal of Robust and Nonlinear Control*, vol. 33, no. 8, pp. 4523–4540, 2023.

[24] S. Shivakumar, D. Jagt, D. Braghini, A. Das, and M. Peet, "Pietools 2022: User manual," *arXiv preprint arXiv:2101.02050*, 2021.

[25] D. Braghini and M. Peet, "Computing optimal upper bounds on the H_2-norm of ODE-PDE systems using linear Partial Inequalities," *IFAC-PapersOnLine*, vol. 56, no. 2, pp. 6994–6999, 2023.

[26] S. Shivakumar, A. Das, S. Weiland, and M. Peet, "Extension of the Partial Integral Equation representation to GPDE input-output systems," *arXiv preprint arXiv:2205.03735*, 2022.

[27] G. E. Dullerud and F. Paganini, *A course in robust control theory: a convex approach*. Springer Science & Business Media, 2013, vol. 36.

[28] M. M. Peet, "Representation of networks and systems with delay: Ddes, ddfs, ode–pdes and pies," *Automatica*, vol. 127, p. 109508, 2021.

[29] S. Shivakumar, D. Jagt, and M. Peet. (2019) PIETOOLS. https://github.com/CyberneticSCL/PIETOOLS.

[30] S. Shivakumar, D. Jagt, D. Braghini, A. Das, and M. Peet, "PIETOOLS 2022: User manual," *arXiv e-prints*, 2021.

[31] P. L. Gould and Y. Feng, *Introduction to linear elasticity*. Springer, 1994, vol. 2.