Dual Representations and H_∞-Optimal Control of Partial Differential Equations

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***Abstract*—We consider H_∞-optimal state-feedback control of the class of linear Partial Differential Equations (PDEs), which admit a Partial Integral Equation (PIE) representation. While linear matrix inequalities are commonly used for optimal control of Ordinary Differential Equations (ODEs), the absence of a universal state-space representation and suitable dual form prevents such methods from being applied to optimal control of PDEs. Specifically, for ODEs, the controller synthesis problem is defined in state-space and duality is used to resolve the bilinearity in that synthesis problem. Recently, the PIE representation was proposed as a universal state-space representation for linear PDE systems. In this paper, we show that any PDE system represented by a PIE admits a dual PIE with identical stability and I/O properties. This result allows us to reformulate the stabilizing and optimal state-feedback control problems as convex optimization over the cone of positive Partial Integral (PI) operators. Operator inversion formulae then allow us to construct feedback gains for the original PDE system. The results are verified through application to several canonical problems in optimal control of PDEs.**

***Index Terms*—Partial Differential Equations, Optimal Control, Robust Control, Linear Matrix Inequalities**

I. Introduction

Partial Differential Equations (PDEs) are used to model spatially-distributed phenomena such as vibrations in beams [46], turbulent fluid flows [1], and reaction kinetics [6, 7]. In many systems governed by PDEs, optimal control can significantly improve safety and reduce operational costs. For example: controllers designed using Euler or Timoshenko beam models can suppress seismic and wind disturbances in buildings and bridges [23, 19, 14] (thereby reducing structural damage); controllers for fluid-flow models can reduce drag on aircraft wings [41] (thereby reducing fuel costs); and controllers for reaction-diffusion equations can improve homogeneity (or desired stratification) of concentration and temperature in chemical reactors [28, 6] (thereby optimizing reaction rates). See [38] for a survey on PDE models in optimal control.

While the problem of optimal feedback control of PDEs is underdeveloped, efficient algorithms exist for robust and optimal feedback control of linear state-space Ordinary Differential Equations (ODEs), with such controllers typically obtained by solving either Riccati Equations [27, 17] or Linear Matrix Inequalities (LMIs) [5]. Because efficient algorithms exist for controller synthesis of linear state-space systems, the most common approach to control of PDEs is to approximate the PDE model with a lumped state-space ODE model using methods such as projection [3, 2, 9] or finite-difference [8, 20, 21]. However, stability and performance gains of the closed-loop state-space lumped ODE do not necessarily translate to stability or performance of the optimal closed-loop PDE [25, 32] – PDEs that have a finite number of unstable modes [40] are an exception. The downsides of these *early-lumping* methods are: closed-loop stability is not guaranteed; large discretized state-spaces increase computational cost; and implementation requires mapping measurements of the physical system to states of the ODE approximation.

To avoid reducing the PDE model to a linear state-space ODE, one can formulate the optimal control problem in an abstract operator-theoretic state-space framework [10] to obtain an operator equivalent of the Riccati Equations for controller synthesis [26, 31]. Unfortunately, however, the operators in these Riccati equations are unbounded and cannot be easily parameterized. As a result, one needs to project the operator equations onto a finite-dimensional subspace – implying that the solution is only valid on the projected subspace. This approach is often referred to as *late-lumping* and has associated convergence proofs that show the error of approximation decreases with the increase in the number of bases for the projected subspace. The downsides of late-lumping are: the projection requires extensive ad-hoc analysis for any given PDE; the operator solutions are never obtained explicitly (only their projection onto a finite-dimensional subspace); and the closed-loop is not guaranteed to be stable for any given order of projection.

Related to the Riccati-based *late-lumping* approach is backstepping, which uses a feedback controller to transform the closed-loop dynamics into a form that is equivalent (through an invertible state transformation) to that of a nominal stable system [24, 29]. The resulting state transformation is an integral operator whose kernel is defined by a set of algebraic equations that must be solved numerically and, in certain cases, convergence proofs are available. The downsides of backstepping are: the kernel map must be re-derived for every PDE (recent work has focused on power series [47] or neural networks [4] to find the kernel); a parameterization of the kernels is required in order to numerically solve the resulting PDEs; and the controllers are stabilizing, not optimal.

To avoid lumping and numerical solution of kernels, recent work has focused on explicit parameterizations of positive Lyapunov functions, often using positive matrices and LMIs to enforce the positivity of these Lyapunov functions [15, 16]. Since the resulting conditions for stability or performance

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of the controller are formulated in terms of LMIs, one can use efficient interior-point solvers to solve these LMIs to obtain provable properties of the PDE. The downsides of this approach are: the assumption of specific structure on the Lyapunov function and controller adds conservatism to the problem; the use of ad-hoc steps such as Poincaré and Wirtinger inequalities to upper bound the derivative of the Lyapunov function; and the failure to resolve the bilinearity between the Lyapunov variable and the feedback gain variable often renders the problem non-convex or severely limits the structure of the Lyapunov function and/or controller.

We conclude, therefore, that existing methods for optimal feedback control of PDEs: lack provable properties, are ad-hoc, or are conservative. In this paper, then, we attempt to overcome some of these disadvantages by using a newly developed state-space representation of PDEs to obtain dual representations of the PDE. This dual representation is then used to propose a convex formulation of the H_∞-optimal state-feedback controller synthesis problem. This approach has advantages over prior work in that it applies to any suitably well-posed PDE, requires no ad-hoc steps or manipulation, and has few obvious sources of conservatism.

The results in this paper are based on the use of the recently developed Partial Integral Equation (PIE) state-space representation of PDE optimal control problems. Specifically, a PIE has the following state-space form

$$\begin{bmatrix}\mathcal{T}\dot{\mathbf{x}}(t)\\ z(t)\\ y(t)\end{bmatrix} = \begin{bmatrix}\mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2\\ \mathcal{C}_1 & D_{11} & D_{12}\\ \mathcal{C}_2 & D_{21} & D_{22}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ w(t)\\ u(t)\end{bmatrix} + \mathcal{B}_w\dot{w}(t) + \mathcal{B}_u\dot{u}(t)$$

where u is the control input, w is the exogeneous disturbance, y is the measured output, z is the regulated output, and $\mathbf{x}$ is the system state. The operators $\mathcal{T}, \mathcal{A}, \mathcal{B}_i, \mathcal{C}_i$ are all bounded integral operators and D_{ij} are matrices – for which analytic formulae have been obtained in prior work [44]. The major difference between the PIE and PDE representations is that the state of the PIE is the highest-order spatial derivative of the PDE state (e.g. $\mathbf{x} = \mathbf{v}_{ss}$), which is related to the PDE state through an integral transformation ($\mathbf{v} = \mathcal{T}\mathbf{x}$ – whose kernel can be thought of as an explicit polynomial parameterization of a Green's function). Since the state of the PIE is the highest-order spatial derivative of the PDE state, the PIE does not require boundary conditions. Instead, the boundary conditions are implicit in the map $\mathcal{T} : \mathbf{x} \mapsto \mathbf{v}$, and their effect on the dynamics is made explicit in the operators $\mathcal{A}$ and $\mathcal{B}_i$.

The motivation behind the PIE representation is that it retains many of the advantages of the state-space framework used for ODEs (e.g. $\dot{x} = Ax$). For ODEs, a necessary and sufficient condition for $V(x)$ to be a positive quadratic Lyapunov candidate on the solution space is that it has the form $V(x) = x^T P x$ for some positive definite matrix $P > 0$. More importantly, square matrices form a linear algebra, and hence the derivative of such Lyapunov candidates are also square matrices, i.e., $A^T P + PA$ is itself a matrix and hence a necessary and sufficient condition for $\dot{V}(x) = x^T(A^T P + PA)x \le 0$ is that $A^T P + PA \le 0$. Extending these concepts to PIEs, we find that some vector spaces of integral operators on $\mathbb{R}^m \times L_2^n$ also form a linear algebra. Specifically, we define the class of bounded linear Partial Integral (PI) operators (denoted $\mathcal{P} \in \mathbf{\Pi}_4$) to be those of the form

$$\left(\mathcal{P}\left[\begin{array}{c|c} P & Q_1\\ \hline Q_2 & \{R_i\}\end{array}\right]\begin{bmatrix}x\\ \mathbf{\Phi}\end{bmatrix}\right)(s) := \begin{bmatrix}Px + \int_{-1}^{0} Q_1(s)\mathbf{\Phi}(s)ds\\ Q_2(s)x + (\mathcal{P}\mathbf{\Phi})(s)\end{bmatrix}$$

where

$$(\mathcal{P}\mathbf{\Phi})(s) = R_0(s)\mathbf{\Phi}(s) + \int_{-1}^{s} R_1(s,\theta)\mathbf{\Phi}(\theta)d\theta + \int_{s}^{0} R_2(s,\theta)\mathbf{\Phi}(\theta)d\theta.$$

The square elements of this subspace $\mathbf{\Pi}_4$ form a linear composition algebra. As a result, many of the LMI methods used in the analysis and control of state-space ODEs can be generalized to optimization of positive PI operators. For example, if one considers a quadratic Lyapunov function for the PDE $V(\mathbf{v}) = \langle \mathbf{v}, \mathcal{P}\mathbf{v}\rangle$ defined on a PDE state $\mathbf{v} = \mathcal{T}\mathbf{x}$, where $\mathcal{P} \succ 0$ is a PI operator, then if $u = w = 0$, $\dot{V} = \langle \mathbf{x}, (\mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T})\mathbf{x}\rangle$ and hence negativity of the PI operator $\mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T}$ is necessary and sufficient for the stability of the PDE. Furthermore, the positivity of $\mathcal{P}$ and negativity of $\mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T}$ can be verified using convex optimization solvers embedded in software packages such as PIETOOLS [44]. This approach to a generalization of LMI methods to PIEs has previously been used to solve the problems of stability analysis, L_2-gain, and optimal estimator design for linear PDEs [36, 45, 12].

Unlike analysis, however, controller synthesis is fundamentally a non-convex optimization problem, where the non-convexity arises because we are simultaneously searching for both a Lyapunov certificate of system properties and a controller that is being chosen to optimize those system properties. To understand what a dual system is and how it allows for convexification of the optimal control problem, let us first recall results developed for linear state-space ODEs. Specifically, for any state-space ODE system G, we may define a dual ODE state-space system G_d, where $G = \Sigma(A, B, C, D)$ (Primal ODE) and $G_d = \Sigma(A^T, C^T, B^T, D^T)$ (Dual ODE). Duality theory shows that the systems G and G_d have related properties – e.g. controllability of G implies observability of G_d and vice versa. Furthermore, and of particular importance to our work, duality theory shows that G and G_d have identical stability and input-output properties – e.g. $\|G\|_{\mathcal{L}(L_2)} = \|G_d\|_{\mathcal{L}(L_2)}$.

Equipped with this duality result, control of linear state-space ODEs is relatively simple. Specifically, if we want to design a controller $u(t) = Kx(t)$ then, by the KYP lemma applied to the primal closed-loop system, the H_∞-norm of the optimal closed-loop system is given by the smallest γ such that

$$\begin{bmatrix}P(A+BK) + (A+BK)^T P & PB & (C+DK)^T\\ B^T P & -\gamma I & D^T\\ C+DK & D & -\gamma I\end{bmatrix} \le 0$$

for some K and $P > 0$ – a condition which is *bilinear* in K and P. However, if we apply the KYP lemma to the dual closed-loop system, and define the new variable $Z = KP$, then the optimal closed-loop H_∞-norm is given by the smallest γ such that

$$\begin{bmatrix}(AP+BZ) + (AP+BZ)^T & (CP+DZ)^T & B\\ CP+DZ & -\gamma I & D\\ B^T & D^T & -\gamma I\end{bmatrix} \le 0$$

for some $P > 0$ and Z – a condition which is *linear* in P and Z, and where the optimal state-feedback controller gain is recovered as $K = ZP^{-1}$.

We conclude, therefore, that if we want to solve the problem of optimal control of PIEs (and hence PDEs), we need a theory of duality for this class of systems.

While dual representations of PDE systems have been studied in the context of the semigroup framework [13, 39], these methods require both extensive ad-hoc mathematical analysis and, when applied to controller synthesis, late lumping of the resulting operator equations [10].

The primary contribution of this paper, then, is, for a given PIE, $\mathcal{G}$, to provide a simple construction of associated dual PIE, $\mathcal{G}_d$, and to show that the stability and input-output properties of these two systems are equivalent. Specifically, for a given PIE (also referred to a *primal PIE*), $\mathcal{G} = \Sigma(\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, we define the *dual PIE* as $\mathcal{G}_d = \Sigma(\mathcal{T}^*, \mathcal{A}^*, \mathcal{C}^*, \mathcal{B}^*, \mathcal{D}^*)$. Then, in Theorem 5, we show that stability of $\mathcal{G}$ implies stability of $\mathcal{G}_d$ and vice-versa. Furthermore, in Theorem 8, we show that $\|\mathcal{G}\|_{\mathcal{L}(L_2)} = \|\mathcal{G}_d\|_{\mathcal{L}(L_2)}$.

Equipped with the duality results for PIEs (Thms. 5 and 8), generalization of the LMI for H_∞-optimal state-feedback control of PIEs becomes relatively simple – the resulting optimization problem is formulated in Theorem 13 and can be implemented using the PIETOOLS Matlab toolbox [44]. Finally, we note that, for actuation at the boundary, the controller synthesis conditions of this paper require filtering – See Subsec. III-B.

II. Notation

$L_2^n[a,b]$ denotes the set of $\mathbb{R}^n$-valued, Lesbegue square-integrable equivalence class of functions on spatial domain $[a,b] \subset \mathbb{R}$. $\mathbb{R}L_2^{m,n}[a,b]$ denotes the space $\mathbb{R}^m \times L_2^n[a,b]$ with inner-product

$$\left\langle \begin{bmatrix} x_1 \\ \mathbf{x}_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ \mathbf{y}_2 \end{bmatrix} \right\rangle_{\mathbb{R}L_2} := x_1^T y_1 + \langle \mathbf{x}_2, \mathbf{y}_2 \rangle_{L_2}.$$

When clear from context, we occasionally omit the subscript on the inner product. (i.e. $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\mathbb{R}L_2}$). Since the only domain considered in this paper is $[a,b]$, we typically omit the domain and simply write L_2^n or $\mathbb{R}L_2^{m,n}$ – further omitting the dimensions when clear from context. For normed space X and Banach space, Y, $\mathcal{L}(X,Y)$ denotes the Banach space of bounded linear operators from X to Y with induced norm $\|\mathcal{P}\|_{\mathcal{L}(X,Y)} = \sup_{\|\mathbf{x}\|_X = 1} \|\mathcal{P}\mathbf{x}\|_Y$ and where $\mathcal{L}(X) := \mathcal{L}(X,X)$. Our notational convention is to write functions in **bold** so that $\mathbf{x}$ indicates $\mathbf{x} \in \mathbb{R}L_2$ and operators in calligraphic capital so $\mathcal{P}$ indicates $\mathcal{P} \in \mathcal{L}(\mathbb{R}L_2)$. For Hilbert space, X and $\mathcal{A} \in \mathcal{L}(X)$, $\mathcal{A}^*$ denotes the adjoint operator satisfying $\langle \mathbf{x}, \mathcal{A}\mathbf{y} \rangle_X = \langle \mathcal{A}^*\mathbf{x}, \mathbf{y} \rangle_X$ for all $\mathbf{x}, \mathbf{y} \in X$.

$L_2^n[0,\infty)$ denotes $\mathbb{R}^n$-valued square-integrable signals, where $[0,\infty)$ indicates a temporal domain – so for $x \in L_2^n[0,\infty)$, we have $x(t) \in \mathbb{R}^n$. Similarly, $\mathbb{R}L_2^n[0,\infty)$ denotes $\mathbb{R}L_2$-valued square-integrable signals. For suitably differentiable $\mathbf{x} \in \mathbb{R}L_2^n[0,\infty)$, $\dot{\mathbf{x}}$ denotes the partial derivative $\frac{\partial \mathbf{x}}{\partial t}$.

III. A State-Space Framework for Optimal Control of PDEs

Creation of a general framework for the control of PDEs is complicated by the lack of a universal parameterization or representation of such problems. In this section, we examine the class of Partial Integral Equations (PIEs) and use this framework to propose a unified formulation of the stabilization and H_∞-optimal state-feedback controller synthesis problems.

PIEs are parameterized by PI operators, which are elements of the algebra, $\mathbf{\Pi}_4$, defined as follows.

Definition 1. *We say $\mathcal{P} \in \mathbf{\Pi}_4 \subset \mathcal{L}(\mathbb{R}L_2^{m_1,n_1}, \mathbb{R}L_2^{m_2,n_2})$ if there exists a matrix P and matrix polynomials $Q_1, Q_2, R_0, R_1,$ and R_2 (of compatible dimension) such that*

$$\mathcal{P} = \mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}(s) := \begin{bmatrix} Px + \int_a^b Q_1(s)\mathbf{x}(s)ds \\ Q_2(s)x + \mathcal{R}\mathbf{x}(s) \end{bmatrix},$$

$$(\mathcal{R}\mathbf{x})(s) = R_0(s)\mathbf{x}(s) + \int_a^s R_1(s,\theta)\mathbf{x}(\theta)d\theta + \int_s^b R_2(s,\theta)\mathbf{x}(\theta)d\theta.$$

If $m_1 = m_2$ and $n_1 = n_2$, then the PI operators, parameterized as above, are closed under composition, addition, and adjoint [45, 42].

Lemma 2. *The vector space $\mathbf{\Pi}_4 \subset \mathcal{L}(\mathbb{R}L_2^{m,n})$ is a *-algebra under composition.*

The notation $\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right]$ is widely used throughout this paper to indicate the PI operator associated with the matrix P and polynomial parameters Q_i, R_j. The dimensions (m_1, n_1, m_2, n_2) of the domain $(\mathbb{R}L_2^{m_1,n_1})$ and range $(\mathbb{R}L_2^{m_2,n_2})$ of these operators are inherited from the dimensions of the matrices $P \in \mathbb{R}^{n_2 \times n_1}$ and polynomials $R_0(s) \in \mathbb{R}^{m_2 \times m_1}$. When clear from context, we will omit the dimensions of the domain and range and simply use $\mathbb{R}L_2$. In the case where a dimension is zero, we use $\emptyset$ in place of the associated parameter with dimension zero, so that for example, if $m_1 = 0$, we have an operator of the form

$$\mathit{\Pi}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline Q_2 & \{R_i\} \end{array}\right].$$

The stabilization and optimal control problems, considered in this paper, can be formulated using a class of dynamics parameterized by PI operators as follows

$$\begin{bmatrix} \partial_t(\mathcal{T}\mathbf{x})(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C} & D_1 & D_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ w(t) \\ u(t) \end{bmatrix}, \tag{1}$$

where $\mathcal{T}$, $\mathcal{A}$, $\mathcal{B}_i$, $\mathcal{C}$ are PI operators and D_i are matrices. Formally, we say $\mathbf{x}$ satisfies the PIE defined by $\{\mathcal{T}, \mathcal{A}, \mathcal{B}_i, \mathcal{C}, D_i\}$ with initial condition $\mathbf{x}_0$ if Eqn. (1) is satisfied for all $t \geq 0$ and $\mathcal{T}\mathbf{x}(0) = \mathcal{T}\mathbf{x}_0$. For such systems, we can formulate the stabilization and H_∞-optimal state-feedback control problems as follows.

Stabilization: find $\mathcal{K} \in \mathbf{\Pi}_4$ s.t., (2)

$$\lim_{t \to \infty} \mathbf{x}(t) \rightharpoonup 0$$

for any $\mathbf{x}(t)$ satisfying Eq. (1) with $u(t) = \mathcal{K}\mathbf{x}(t)$

H_∞ **Optimal State Feedback:** $\min_{\mathcal{K} \in \mathbf{\Pi}_4,\ \gamma \in \mathbb{R}} \gamma$ s.t., (3)

$$\|z\|_{L_2} \leq \gamma \|w\|_{L_2} \qquad \text{for any } w \in L_2 \tag{4}$$

where z satisfies Eq. (1) for some $\mathbf{x}$ with $\mathcal{T}\mathbf{x}(0) = 0$ and $u(t) = \mathcal{K}\mathbf{x}(t)$.

A. Representation of the PDE Control Problem Using PIEs

A broad class of linear PDEs on a bounded domain with inputs and outputs have been shown to admit a PIE representation of the form Eq. (1). While we will refer to [42] for the full class of linear PDEs that admit such a representation, we include the following example for illustration.

Example 3. *Consider the vibration suppression problem for a cantilevered Euler-Bernoulli beam:* $\ddot{\mathbf{u}} = -.1\mathbf{u}_{ssss} + w(t) + u(t)$ *with* $\mathbf{u}(0) = \mathbf{u}_s(0) = \mathbf{u}_{ss}(1) = \mathbf{u}_{sss}(1)$ *where* $\mathbf{u}$ *is displacement,* w *is external disturbance, and* u *is control input. As discussed in [34], to put this PDE in first-order form, we may define* $\mathbf{v}_1 := \dot{\mathbf{u}}$ *and* $\mathbf{v}_2 := \mathbf{u}_{ss}$*, which yields*

$$\begin{bmatrix}\dot{\mathbf{v}}_1(t,s)\\ \dot{\mathbf{v}}_2(t,s)\end{bmatrix} = \begin{bmatrix}0 & -0.1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\partial_s^2\mathbf{v}_1(t,s)\\ \partial_s^2\mathbf{v}_2(t,s)\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}w(t) + \begin{bmatrix}1\\0\end{bmatrix}u(t),$$

$$\mathbf{v}_1(t,0) = \partial_s\mathbf{v}_1(t,0) = \mathbf{v}_2(t,1) = \partial_s\mathbf{v}_2(t,1) = 0.$$

The optimal control problem requires us to specify a regulated output, which we define to be

$$z(t) = \begin{bmatrix} u(t) \\ \int_0^1 \mathbf{u}(t,s)ds\end{bmatrix} = \begin{bmatrix} u(t) \\ \int_0^1 (\frac{1}{2} - s - \frac{1}{2}s^2)\mathbf{v}_2(t,s)ds\end{bmatrix}.$$

The parameters $\mathcal{T}, \mathcal{A}, \mathcal{B}_i, \mathcal{C}, D_i$ *may be obtained from the formulae in [42]. However, for illustration, we derive this representation directly. Specifically, from Cauchy's rule for repeated integration, we have*

$$\mathbf{v}(t,s) = \mathbf{v}(t,0) + s\partial_s\mathbf{v}(t,0) + \int_0^s (s-\theta)\partial_s^2\mathbf{v}(t,\theta)d\theta.$$

Denoting $\mathbf{x} := \partial_s^2\mathbf{v}$ *and substituting boundary conditions, we obtain the map from PIE state,* $\mathbf{x}$ *to PDE state,* $\mathbf{v} = \mathcal{T}\mathbf{x}$*, as*

$$\mathbf{v}(t,s) = \int_0^s \begin{bmatrix}(s-\theta) & 0\\ 0 & 0\end{bmatrix}\mathbf{x}(t,\theta)d\theta + \int_s^1 \begin{bmatrix}0 & 0\\ 0 & (\theta - s)\end{bmatrix}\mathbf{x}(t,\theta)d\theta.$$

Substituting this expression into the dynamics and output equation, we obtain the PIE representation

$$\partial_t\left(\mathcal{T}\mathbf{x}(t)\right)(s) = \overbrace{\begin{bmatrix}0 & -0.1\\ 1 & 0\end{bmatrix}}^{\mathcal{A}}\mathbf{x}(t,s) + \overbrace{\begin{bmatrix}1\\0\end{bmatrix}}^{\mathcal{B}_1}w(t) + \overbrace{\begin{bmatrix}1\\0\end{bmatrix}}^{\mathcal{B}_1}u(t)$$

$$z(t) = \underbrace{\int_0^1 \begin{bmatrix}0 & 0\\ 0 & \frac{s^2}{4} - \frac{s^3}{6} - 0.042s^4\end{bmatrix}}_{\mathcal{C}}\mathbf{x}(t,s)ds + \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{D_2}u(t).$$

To illustrate the $\mathbf{\Pi}_4$ *notation, we may also write the system parameters as*

$$\mathcal{T} = \mathit{\Pi}\left[\begin{array}{c|c}\emptyset & \emptyset\\ \hline \emptyset & \{0, R_1, R_2\}\end{array}\right], \mathcal{A} = \mathit{\Pi}\left[\begin{array}{c|c}\emptyset & \emptyset\\ \hline \emptyset & \{R_0, 0, 0\}\end{array}\right],$$

$$\mathcal{B}_i = \mathit{\Pi}\left[\begin{array}{c|c}\emptyset & \emptyset\\ \hline Q_2 & \{\emptyset\}\end{array}\right], \mathcal{C} = \mathit{\Pi}\left[\begin{array}{c|c}\emptyset & Q_1\\ \hline \emptyset & \{\emptyset\}\end{array}\right], D_2 = \begin{bmatrix}1\\0\end{bmatrix},$$

where

$$R_1(s,\theta) = \begin{bmatrix}s-\theta & 0\\ 0 & 0\end{bmatrix},\ R_2(s,\theta) = \begin{bmatrix}0 & 0\\ 0 & \theta - s\end{bmatrix},\ Q_2 = \begin{bmatrix}1\\0\end{bmatrix},$$

$$R_0(s) = \begin{bmatrix}0 & -0.1\\ 1 & 0\end{bmatrix},\ Q_1(s) = \begin{bmatrix}0 & 0\\ 0 & \frac{s^2}{4} - \frac{s^3}{6} - 0.042s^4\end{bmatrix}.$$

Note that the process of constructing the PIE representation has been automated in the PIETOOLS software package with a dedicated command line and GUI input formats to simplify this step of finding the PIE representation.

B. A Note on Inputs at the Boundary

When the control input enters the dynamics of a PDE through the boundary conditions (e.g. $\mathbf{v}(t,0) = u(t)$), novel questions arise that are not readily apparent in the PDE representation but are made explicit when using the PIE framework. These questions arise because PDEs with distributed states are partly rigid – i.e. they are constrained by the continuity properties of Sobolev space necessary for boundary values to be well defined. The simplest illustration of this is the heat equation with boundary conditions $\mathbf{v}(t,0) = u_1(t)$ and $\mathbf{v}_s(t,0) = u_2(t)$. In this case, we have the relationship

$$\mathbf{v}(t,s) = u_1(t) + su_2(t) + \int_0^s (s-\eta)\mathbf{v}_{ss}(t,\eta)d\eta$$

which implies that the effect of the input is felt *immediately* throughout the distributed state and is NOT filtered through the dynamics (as is the case in ODEs or in-domain control). If we integrate this type of *semi-algebraic* relationship into the dynamics, we obtain a unitary PIE representation of the heat equation, defined in terms of PIE state $\mathbf{x} = \mathbf{v}_{ss}$, as

$$\partial_t\left(\int_0^s (s-\eta)\mathbf{x}(t,\eta)d\eta\right) = \mathbf{x}(t,s) - \dot{u}_1(t) - s\dot{u}_2(t).$$

In this representation, the partially algebraic nature of the boundary conditions is made explicit in that the dependence is not on u_1, u_2, but on their time-derivatives. This type of dependence is allowed in the parameterization of PIE defined in [42] but is not included in the controller synthesis approach defined in this paper. One reason is that there is a valid argument to be made that such types of control are non-physical in that they do not account for the inertia of the distributed state, and hence, such inputs would be better modeled by filtering through an ODE which represents the dynamics of the actuator. The other reason is that if we are searching for an H_∞-optimal controller, then we are trying to minimize the gain from $\|w\|_{L_2}$ to $\|z\|_{L_2}$ and if we were to include the derivative $\dot{w}$, this implies that the natural norm for w is the Sobolev norm – an approach taken in [10, Thm. 3.3].

Therefore, to account for the case of inputs at the boundary, we will assume that the actual disturbance or input signal is not w or u, but rather $\dot{w}, \dot{u}$, which we can relabel as disturbances $\hat{w}, \hat{u}$. This approach allows us to take any PIE optimal control problem involving time derivatives of the inputs and reformulate it as a PIE free of such derivatives. Specifically, if we are given a PIE representation of the form

$$\begin{bmatrix}\partial_t(\mathcal{T}\mathbf{x}(t))\\ z(t)\end{bmatrix} = \begin{bmatrix}\mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2\\ \mathcal{C} & D_1 & D_2\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ w(t)\\ u(t)\end{bmatrix} + \mathcal{B}_{1d}\dot{w}(t) + \mathcal{B}_{2d}\dot{u}(t),$$

then we will augment the state $\hat{\mathbf{x}}(t) := \begin{bmatrix}w(t)\\ u(t)\\ \mathbf{x}(t)\end{bmatrix}$ and redefine the PIE system as

$$\begin{bmatrix}\partial_t\left(\overbrace{\begin{bmatrix}I & 0 & 0\\ 0 & I & 0\\ 0 & 0 & \mathcal{T}\end{bmatrix}}^{\hat{\mathcal{T}}}\hat{\mathbf{x}}(t)\right)\\ z(t)\end{bmatrix}$$

$$= \left[\begin{array}{ccc} \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{A} \end{bmatrix}}^{\hat{\mathcal{A}}} & \overbrace{\begin{bmatrix} I \\ 0 \\ \mathcal{B}_{1d} \end{bmatrix}}^{\hat{\mathcal{B}}_1} & \overbrace{\begin{bmatrix} 0 \\ I \\ \mathcal{B}_{2d} \end{bmatrix}}^{\hat{\mathcal{B}}_2} \\ \underbrace{\begin{bmatrix} D_1 & D_2 & \mathcal{C} \end{bmatrix}}_{\hat{\mathcal{C}}} & \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\hat{D}_1} & \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\hat{D}_2} \end{array}\right] \begin{bmatrix} \hat{\mathbf{x}}(t) \\ \hat{w}(t) \\ \hat{u}(t) \end{bmatrix}$$

which is now of the form in Eq. (1) using the parameters $\hat{\mathcal{T}}$, $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}_1$, etc. Numerical examples of such boundary control problems are included in Section VIII as Examples 22 and 23.

C. A Three State Approach to Controller Synthesis

Given a PIE formulation of the stabilizing and H_∞-optimal state-feedback problems in Eqs. (2) and (3), we will solve these problems in three stages.

First, for a given PIE, we define a dual PIE and show that this PIE has identical internal stability and input-output properties – Section IV. Second, in Section V, we define a class of optimization problems defined by linear PI operator inequality (LPI) constraints for which we have efficient convex optimization algorithms. We use this operator inequality (LPI) framework to solve the problems of internal stability and the input-to-output (I/O) L_2-gain in Subsections V-A and V-B, respectively. Third, in Section VI, we apply the results of Section V to the dual of the closed-loop PIE. We then use variable substitution to formulate the stabilization and H_∞-optimal state-feedback control problems as LPI problems. A formula for inversion of PI operators is presented in Section VII and used to obtain the controller gains. Because solutions of the PIE and the PDE it represents are equivalent, these controller gains can then be applied directly to the PDE model – a process illustrated in the examples in Section VIII.

IV. DUALITY IN PIES

In this section, we show that for any PIE system of the form in Eq. (1), we may associate a dual PIE system, also of the same form.

Definition 4. *(Dual PIE) Given a PIE system of the form*

$$\begin{bmatrix} \partial_t(\mathcal{T}\mathbf{x}(t)) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & D \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ w(t) \end{bmatrix}, \quad \mathbf{x}(0) \in \mathbb{R}L_2^{m,n}, \tag{5}$$

defined by PI operators $\mathcal{T}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and matrix D, we define the 'dual PIE system' as

$$\begin{bmatrix} \mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) \\ \bar{z}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & D^T \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}(t) \\ \bar{w}(t) \end{bmatrix}, \quad \bar{\mathbf{x}}(0) \in \mathbb{R}L_2^{m,n}, \tag{6}$$

where * represents the adjoint of an operator with respect to the $\mathbb{R}L_2$ inner product.

Given polynomial parameters defining the operators $\mathcal{T}, \mathcal{A}$, $\mathcal{B}, \mathcal{C}$, the polynomials that parameterize the dual PIE operators are easily obtained from the formula

$$\Pi\left[\begin{array}{c|c} P & Q_1(s) \\ \hline Q_2(s) & \{R_0(s), R_1(s,\theta), R_2(s,\theta)\} \end{array}\right]^* = \Pi\left[\begin{array}{c|c} P^T & Q_2(s)^T \\ \hline Q_1(s)^T & \{R_0(s)^T, R_2(\theta,s)^T, R_1(\theta,s)^T\} \end{array}\right].$$

Note that while the primal PIE in Eq. (5) is defined using $\partial_t(\mathcal{T}\mathbf{x})$, its dual in Eq. (6) is defined in terms of $\mathcal{T}\dot{\bar{\mathbf{x}}}$. This asymmetry may be relaxed if the primal solutions are assumed to be sufficiently differentiable. Having defined the dual of a PIE representation, the following subsections prove primal-dual equivalence in terms of stability and L_2-gain.

A. Dual Stability Theorem

First, let us look at the internal stability of a PIE and its dual.

Theorem 5. *(Dual Stability) Suppose $\mathcal{T}, \mathcal{A} \in \mathcal{L}(\mathbb{R}L_2^{m,n})$ are PI operators. The following statements are equivalent.*

- *a)* $\lim_{t\to\infty} \mathcal{T}\mathbf{x}(t) \rightharpoonup 0$ *for any* $\mathbf{x}$ *that satisfies* $\partial_t(\mathcal{T}\mathbf{x}(t)) = \mathcal{A}\mathbf{x}(t)$ *with initial condition* $\mathbf{x}(0) \in \mathbb{R}L_2^{m,n}$.
- *b)* $\lim_{t\to\infty} \mathcal{T}^*\bar{\mathbf{x}}(t) \rightharpoonup 0$ *for any* $\bar{\mathbf{x}}$ *that satisfies* $\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) = \mathcal{A}^*\bar{\mathbf{x}}(t)$ *with initial condition* $\bar{\mathbf{x}}(0) \in \mathbb{R}L_2^{m,n}$.

Note that "$\rightharpoonup 0$" in Thm. 5 denotes weak convergence in the Hilbert space $\mathbb{R}L_2$.

Proof. To show sufficiency (i.e. a implies b), suppose $\mathbf{x}$ satisfies $\partial_t(\mathcal{T}\mathbf{x}(t)) = \mathcal{A}\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) \in \mathbb{R}L_2^{m,n}$ and $\lim_{t\to\infty} \mathcal{T}\mathbf{x}(t) \rightharpoonup 0$. Let $\bar{\mathbf{x}}$ satisfy $\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) = \mathcal{A}^*\bar{\mathbf{x}}(t)$ with initial condition $\bar{\mathbf{x}}(0) \in \mathbb{R}L_2^{m,n}$. Then for any finite $t > 0$, using integration-by-parts, we get

$$\begin{aligned} &\int_0^t \langle \bar{\mathbf{x}}(t-s), \partial_s(\mathcal{T}\mathbf{x}(s)) \rangle_{\mathbb{R}L_2} \, ds \\ &= \langle \bar{\mathbf{x}}(0), \mathcal{T}\mathbf{x}(t) \rangle - \langle \bar{\mathbf{x}}(t), \mathcal{T}\mathbf{x}(0) \rangle \\ &\quad - \int_0^t \langle \partial_s \bar{\mathbf{x}}(t-s), \mathcal{T}\mathbf{x}(s) \rangle \, ds. \end{aligned} \tag{7}$$

Then, we use a change of variable ($\theta = t - s$) on the last term in Eq. (7) to show

$$\begin{aligned} \int_0^t \langle \partial_s \bar{\mathbf{x}}(t-s), \mathcal{T}\mathbf{x}(s) \rangle \, ds &= \int_0^t \left\langle \dot{\bar{\mathbf{x}}}(\theta), \mathcal{T}\mathbf{x}(t-\theta) \right\rangle d\theta \\ &= \int_0^t \left\langle \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta), \mathbf{x}(t-\theta) \right\rangle d\theta. \end{aligned}$$

Furthermore, using the same variable change on the left-hand side of Eq. (7), we get

$$\begin{aligned} &\int_0^t \langle \bar{\mathbf{x}}(t-s), \partial_s(\mathcal{T}\mathbf{x}(s)) \rangle_{\mathbb{R}L_2} \, ds \\ &= \int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{A}\mathbf{x}(s) \rangle \, ds = \int_0^t \langle \mathcal{A}^*\bar{\mathbf{x}}(\theta), \mathbf{x}(t-\theta) \rangle \, d\theta. \end{aligned}$$

Substituting these two expressions into Eq. (7), we have

$$\begin{aligned} &\int_0^t \langle \mathcal{A}^*\bar{\mathbf{x}}(\theta), \mathbf{x}(t-\theta) \rangle \, d\theta \\ &= \langle \bar{\mathbf{x}}(0), \mathcal{T}\mathbf{x}(t) \rangle - \langle \bar{\mathbf{x}}(t), \mathcal{T}\mathbf{x}(0) \rangle \\ &\quad + \int_0^t \left\langle \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta), \mathbf{x}(t-\theta) \right\rangle d\theta. \end{aligned}$$

However, $\mathcal{A}^*\bar{\mathbf{x}}(\theta) = \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta)$ for all $\theta \in [0, t]$, and hence

$$\langle \bar{\mathbf{x}}(0), \mathcal{T}\mathbf{x}(t) \rangle = \langle \mathcal{T}^*\bar{\mathbf{x}}(t), \mathbf{x}(0) \rangle, \quad \text{for all } t > 0. \tag{8}$$

Since $\lim_{t\to\infty} \mathcal{T}\mathbf{x}(t) = 0$, we have

$$\lim_{t\to\infty} \langle \mathcal{T}^*\bar{\mathbf{x}}(t), \mathbf{x}(0) \rangle = \lim_{t\to\infty} \langle \bar{\mathbf{x}}(0), \mathcal{T}\mathbf{x}(t) \rangle = 0.$$

Since $\mathbf{x}(0) \in \mathbb{R}L_2$ may be chosen arbitrarily and $\mathbb{R}L_2$ is Hilbert, this implies $\lim_{t\to\infty} \mathcal{T}^*\bar{\mathbf{x}}(t) \rightharpoonup 0$. Thus we have sufficiency.

Since $\mathcal{T}^{**} = \mathcal{T}$, and existence of $\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t)$ guarantees existence of $\partial_t(\mathcal{T}^*\bar{\mathbf{x}}(t))$, sufficiency implies necessity. $\square$

From a computational perspective, testing the above notion of stability is difficult, and hence, we show that exponential stability of the primal and the dual PIE are equivalent when we define exponential stability in the following sense.

Definition 6 (Exponential Stability)**.** *We say that the PIE defined by $\{\mathcal{T},\mathcal{A}\}\subset \mathbf{\Pi}_4$ is **Exponentially Stable** with decay rate $\alpha>0$ if there exists some $M>0$ such that for any $\mathbf{x}_0\in\mathbb{R}L_2$, if $\mathcal{T}\mathbf{x}(0)=\mathcal{T}\mathbf{x}_0$ and $\partial_t(\mathcal{T}\mathbf{x}(t))=\mathcal{A}\mathbf{x}(t)$, then*

$$\|\mathcal{T}\mathbf{x}(t)\|_{\mathbb{R}L_2}\le M\|\mathbf{x}_0\|_{\mathbb{R}L_2}e^{-\alpha t}\quad \text{for all } t\ge 0.$$

Note that this definition implies that the PDE state ($\mathcal{T}\mathbf{x}$) decays exponentially in the $\mathbb{R}L_2$-norm but does not necessarily guarantee exponential stability of the PIE state ($\mathbf{x}$).

Corollary 7. *Suppose $\mathcal{T},\mathcal{A}\in\mathcal{L}(\mathbb{R}L_2^{m,n})$ are PI operators. The following statements are equivalent:*

a) $\|\mathcal{T}\mathbf{x}(t)\|\le M\|\mathbf{x}(0)\|e^{-\alpha t}$ for any $\mathbf{x}$ that satisfies $\partial_t(\mathcal{T}\mathbf{x})(t)=\mathcal{A}\mathbf{x}(t)$ with initial condition $\mathbf{x}(0)\in\mathbb{R}L_2^{m,n}$.

b) $\|\mathcal{T}^*\bar{\mathbf{x}}(t)\|\le M\|\bar{\mathbf{x}}(0)\|e^{-\alpha t}$ for any $\bar{\mathbf{x}}$ that satisfies $\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t)=\mathcal{A}^*\bar{\mathbf{x}}(t)$ with initial condition $\bar{\mathbf{x}}(0)\in\mathbb{R}L_2^{m,n}$.

Proof. To show sufficiency (i.e. a implies b), suppose $\bar{\mathbf{x}}$ satisfies $\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t)=\mathcal{A}^*\bar{\mathbf{x}}(t)$ for some initial condition $\bar{\mathbf{x}}(0)\in\mathbb{R}L_2^{m,n}$. Then for any $T>0$, let $\mathbf{x}$ satisfy $\partial_t(\mathcal{T}\mathbf{x}(t))=\mathcal{A}\mathbf{x}(t)$ with initial condition $\mathbf{x}(0)=\mathcal{T}^*\bar{\mathbf{x}}(T)$. Then, we have from Eq. (8) in the proof of Theorem 5,

$$\langle\bar{\mathbf{x}}(0),\mathcal{T}\mathbf{x}(T)\rangle=\langle\mathcal{T}^*\bar{\mathbf{x}}(T),\mathbf{x}(0)\rangle=\|\mathcal{T}^*\bar{\mathbf{x}}(T)\|^2.$$

Then, from the Cauchy-Schwarz inequality,

$$\begin{aligned}\|\mathcal{T}^*\bar{\mathbf{x}}(T)\|^2&=\langle\bar{\mathbf{x}}(0),\mathcal{T}\mathbf{x}(T)\rangle\le\|\mathcal{T}\mathbf{x}(T)\|\|\bar{\mathbf{x}}(0)\|\\&\le M\|\mathbf{x}(0)\|e^{-\alpha T}\|\bar{\mathbf{x}}(0)\|\\&=M\|\mathcal{T}^*\bar{\mathbf{x}}(T)\|e^{-\alpha T}\|\bar{\mathbf{x}}(0)\|,\end{aligned}$$

which implies

$$\|\mathcal{T}^*\bar{\mathbf{x}}(T)\|\le M\|\bar{\mathbf{x}}(0)\|e^{-\alpha T}.$$

Since T is arbitrary, we have sufficiency. Since $\mathcal{T}^{**}=\mathcal{T}$, sufficiency implies necessity. ☐

B. Dual L_2-Gain Theorem

Having proved that the internal stability of a PIE and its dual are equivalent, we now show that their I/O properties are identical (analogous to the dual KYP lemma for ODE systems).

Theorem 8. *(Dual L_2-gain) Suppose $\mathcal{T},\mathcal{A}\in\mathcal{L}(\mathbb{R}L_2^{m,n})$, $\mathcal{B}\in\mathcal{L}(\mathbb{R}^p,\mathbb{R}L_2^{m,n})$, and $\mathcal{C}\in\mathcal{L}(\mathbb{R}L_2^{m,n},\mathbb{R}^r)$ are PI operators and $D\in\mathbb{R}^{r\times p}$ is a matrix. The following statements are equivalent.*

a) For any $w\in L_2^p[0,\infty)$, $\mathbf{x}(t)\in\mathbb{R}L_2^{m,n}$ and $z(t)\in\mathbb{R}^r$ that satisfy $\mathbf{x}(0)=0$ and

$$\begin{bmatrix}\partial_t(\mathcal{T}\mathbf{x})(t)\\ z(t)\end{bmatrix}=\begin{bmatrix}\mathcal{A}&\mathcal{B}\\ \mathcal{C}&D\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ w(t)\end{bmatrix},\tag{9}$$

we have that $\|z\|_{L_2}\le\gamma\|w\|_{L_2}$.

b) For any $\bar{w}\in L_2^r[0,\infty)$, $\bar{\mathbf{x}}(t)\in\mathbb{R}L_2^{m,n}$ and $\bar{z}(t)\in\mathbb{R}^p$ that satisfy $\bar{\mathbf{x}}(0)=0$ and

$$\begin{bmatrix}\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t)\\ \bar{z}(t)\end{bmatrix}=\begin{bmatrix}\mathcal{A}^*&\mathcal{C}^*\\ \mathcal{B}^*&D^T\end{bmatrix}\begin{bmatrix}\bar{\mathbf{x}}(t)\\ \bar{w}(t)\end{bmatrix},\tag{10}$$

we have that $\|\bar{z}\|_{L_2}\le\gamma\|\bar{w}\|_{L_2}$.

Proof. To show sufficiency (i.e. a implies b), let $\mathbf{x}(t)\in\mathbb{R}L_2^{m,n}$ and $z(t)\in\mathbb{R}^r$ satisfy Eq. (9) for $\mathbf{x}(0)=0$ and some $w\in L_2^p[0,\infty)$. Then, $\|z\|_{L_2}\le\gamma\|w\|_{L_2}$. Let $\bar{\mathbf{x}}(t)\in\mathbb{R}L_2^{m,n}$ and $\bar{z}(t)\in\mathbb{R}^p$ satisfy Eq. (10) for $\bar{\mathbf{x}}(0)=0$ and some $\bar{w}\in L_2^r[0,\infty)$. Then, as in Eq. (7) in the proof of Theorem 5 and substituting initial conditions, we find

$$\int_0^t\langle\bar{\mathbf{x}}(t-s),\partial_s(\mathcal{T}\mathbf{x}(s))\rangle ds=\int_0^t\left\langle\mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta),\mathbf{x}(t-\theta)\right\rangle d\theta.\quad(\star)$$

Furthermore, by using the variable change $\theta=t-s$ on the left-hand side of the above equation,

$$\begin{aligned}&\int_0^t\langle\bar{\mathbf{x}}(t-s),\partial_s(\mathcal{T}\mathbf{x}(s))\rangle ds \qquad(\#)\\&=\int_0^t\langle\bar{\mathbf{x}}(t-s),\mathcal{A}\mathbf{x}(s)\rangle ds+\int_0^t\langle\bar{\mathbf{x}}(t-s),\mathcal{B}w(s)\rangle ds\\&=\int_0^t\langle\mathcal{A}^*\bar{\mathbf{x}}(\theta),\mathbf{x}(t-\theta)\rangle d\theta+\int_0^t(\mathcal{B}^*\bar{\mathbf{x}}(\theta))^Tw(t-\theta)d\theta.\end{aligned}$$

Combining the two Eqns. ($\star$) and ($\#$), we obtain

$$\begin{aligned}&\int_0^t\left\langle\mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta),\mathbf{x}(t-\theta)\right\rangle d\theta\\&=\int_0^t\langle\mathcal{A}^*\bar{\mathbf{x}}(\theta),\mathbf{x}(t-\theta)\rangle d\theta+\int_0^t(\mathcal{B}^*\bar{\mathbf{x}}(\theta))^Tw(t-\theta)d\theta.\end{aligned}$$

However, $\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t)-\mathcal{A}^*\bar{\mathbf{x}}(t)=\mathcal{C}^*\bar{w}(t)$, so

$$\begin{aligned}&\int_0^t\langle\mathcal{C}^*\bar{w}(\theta),\mathbf{x}(t-\theta)\rangle d\theta\\&=\int_0^t\left\langle\mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta)-\mathcal{A}^*\bar{\mathbf{x}}(\theta),\mathbf{x}(t-\theta)\right\rangle d\theta\\&=\int_0^t(\mathcal{B}^*\bar{\mathbf{x}}(\theta))^Tw(t-\theta)d\theta.\end{aligned}$$

Since $z=\mathcal{C}\mathbf{x}+Dw$, we obtain

$$\begin{aligned}&\int_0^t\bar{w}(\theta)^Tz(t-\theta)d\theta-\int_0^t\bar{w}(\theta)^T(Dw(t-\theta))d\theta\\&=\int_0^t\bar{w}(\theta)^T(\mathcal{C}\mathbf{x}(t-\theta))d\theta=\int_0^t\langle\mathcal{C}^*\bar{w}(\theta),\mathbf{x}(t-\theta)\rangle d\theta\\&=\int_0^t(\mathcal{B}^*\bar{\mathbf{x}}(\theta))^Tw(t-\theta)d\theta.\end{aligned}$$

Likewise, we know $\bar{z}=\mathcal{B}^*\bar{\mathbf{x}}+D^T\bar{w}$. Hence

$$\begin{aligned}&\int_0^t\bar{z}(\theta)^Tw(t-\theta)d\theta-\int_0^tD^T\bar{w}(\theta)^Tw(t-\theta)d\theta\\&=\int_0^t(\mathcal{B}^*\bar{\mathbf{x}}(\theta))^Tw(t-\theta)d\theta\\&=\int_0^t\bar{w}(\theta)^Tz(t-\theta)d\theta-\int_0^t\bar{w}(\theta)^T(Dw(t-\theta))d\theta.\end{aligned}$$

We conclude that for any $t>0$, if z and w satisfy the primal PIE and $\bar{z}$ and $\bar{w}$ satisfy the dual PIE, then

$$\int_0^t\bar{z}(\theta)^Tw(t-\theta)d\theta=\int_0^t\bar{w}(\theta)^Tz(t-\theta)d\theta.\tag{11}$$

Now, for any $\bar{w}\in L_2$, let $\bar{z}$ solve the dual PIE for some $\bar{\mathbf{x}}$. For any fixed $T>0$, define $w(t)=\bar{z}(T-t)$ for $t\le T$ and $w(t)=0$ for $t>T$. Then $w\in L_2$ and for this input, let z solve the primal PIE for some $\mathbf{x}$. Then, if we define the truncation operator P_T, we have

$$\begin{aligned}\|P_T\bar{z}\|_{L_2}^2&=\int_0^T\bar{z}(s)^T\bar{z}(s)ds=\int_0^T\bar{z}(s)^Tw(T-s)ds\\&=\int_0^T\bar{w}(s)^Tz(T-s)ds\le\|P_T\bar{w}\|_{L_2}\|P_Tz\|_{L_2}\\&\le\|P_T\bar{w}\|_{L_2}\|z\|_{L_2}\le\gamma\|P_T\bar{w}\|_{L_2}\|w\|_{L_2}\end{aligned}$$

$$= \gamma \|P_T\bar{w}\|_{L_2} \|P_T w\|_{L_2} = \gamma \|P_T\bar{w}\|_{L_2} \|P_T\bar{z}\|_{L_2}.$$

Therefore, we have that $\|P_T\bar{z}\|_{L_2} \leq \gamma \|P_T\bar{w}\|_{L_2}$ for all $T \geq 0$. We conclude that $\|\bar{z}\|_{L_2} \leq \gamma \|\bar{w}\|_{L_2}$. Since $\mathcal{T}^{**} = \mathcal{T}$ and

$$\begin{bmatrix} (\mathcal{A}^*)^* & (\mathcal{B}^*)^* \\ (\mathcal{C}^*)^* & (D^T)^T \end{bmatrix} = \begin{bmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & D^T \end{bmatrix}^* = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & D \end{bmatrix}^{**} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & D \end{bmatrix}$$

we have that sufficiency implies necessity. □

Remark 1. *Note the relationship between primal and dual mappings $w \mapsto z$ and $\bar{w} \mapsto \bar{z}$ as given in Eq.* (11) *of the proof of Theorem 8:*

$$\int_0^t \bar{z}(\theta)^T w(t-\theta) d\theta = \int_0^t \bar{w}(\theta)^T z(t-\theta) d\theta.$$

If one were to define a Laplace transform for these inputs ($\hat{w}, \hat{z}, \hat{\bar{w}}, \hat{\bar{z}}$) and transfer function for the systems ($\hat{z}(s) = G(s)\hat{w}(s)$ and $\hat{\bar{z}}(s) = G_d(s)\hat{\bar{w}}(s)$), then this equation would imply $\hat{\bar{z}}(s)^T\hat{w}(s) = \hat{\bar{w}}(s)^T\hat{z}(s)$ or $\hat{\bar{w}}(s)^T G_d(s)^T \hat{w}(s) = \hat{\bar{w}}(s)^T G(s)\hat{w}(s)$ so that $G_d(s)^T = G(s)$ — which is precisely the standard interpretation of the dual transfer function for ODEs. In addition, we note that while Theorem 8 is on input-output stability of the primal and dual, the relationship in Eq. (11) *holds for any finite time, t, and hence does not require the primal or dual to be input-output stable.*

Remark 2. *Finally, we remark that the significance of Theorems 5 and 8 is not simply that a dual representation exists but that it has the same parameterization as the primal (making the primal and dual interchangeable). In addition, the proofs of Theorems 5 and 8 do not utilize the algebraic structure of the PI algebra – implying that the duality result (and intertwining relationship) holds for any class of well-posed systems parameterized by a set of bounded operators on a reflexive Hilbert space which is closed under adjoint.*

V. Primal-Dual LPIs for Stability and H_∞-Norm

In this section, we propose equivalent primal and dual conditions for both exponential stability and H_∞-norm (defined as maximum L_2-gain). These conditions are generalizations of the primal-dual Lyapunov and KYP LMIs used for stability and H_∞-norm analysis of state-space ODEs – e.g. $A^T P + PA < 0$ and $AP + PA^T < 0$ for stability. In Section VI, the dual conditions will be used to synthesize stabilizing and optimal controllers.

The primal-dual conditions in this section and Section VI are formulated using Linear PI operator inequality (LPI) conditions. Such LPI constraints are a generalization of LMI constraints – e.g. $\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} \prec 0$ and $\mathcal{A}\mathcal{P}\mathcal{T}^* + \mathcal{T}\mathcal{P}\mathcal{A}^* \prec 0$ for stability. Such conditions are constructed using a generalization of the quadratic storage functions used for state-space ODEs – i.e. $V(x) = x^T P x$ becomes $V(\mathbf{x}) = \langle \mathcal{T}\mathbf{x}, \mathcal{P}\mathcal{T}\mathbf{x}\rangle$. Note, however, that the duality results presented here are not quite as direct as in the state-space ODE case – where if $V = x^T P x$ proves the stability of $\dot{x} = Ax$, then $V(x) = x^T P^{-1} x$ proves the stability of the dual.

Because the conditions presented here are in the form of LPIs, we implicitly assume that these conditions can be solved efficiently using convex optimization. Fortunately, computational methods for solving LPIs are well-developed and have been presented in other work [45]. In brief, these methods construct a positive PI operator using a quadratic form involving a positive matrix and n^{th}*-order basis* of PI operators, $\mathcal{Z}_n$. For example, $\mathcal{P} \succeq 0$ if there exists some matrix $Q \geq 0$ such that $\mathcal{P} = \mathcal{Z}_n^* Q \mathcal{Z}_n = \mathcal{Z}_n^* Q^{\frac{1}{2}} Q^{\frac{1}{2}} \mathcal{Z}_n \succeq 0$, where the basis $\mathcal{Z}_n$ is constructed using the vector of monomials, Z_n, of degree n or less as

$$\mathcal{Z}_n \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}(s) := \begin{bmatrix} x \\ Z_n(s)\mathbf{x}(s) \\ \int_a^s (Z_n(s) \otimes Z_n(\theta))\mathbf{x}(\theta) d\theta \\ \int_s^b (Z_n(s) \otimes Z_n(\theta))\mathbf{x}(\theta) d\theta \end{bmatrix}. \tag{12}$$

The highest order of these monomials, n, can be used as a measure for the complexity of the LPIs, which will later be used in Section VIII to numerically verify the accuracy and convergence of the conditions. For this numerical verification, the associated LPI conditions will be unforced using the Matlab toolbox, PIETOOLS. This toolbox offers convenient functions to convert PDEs to PIE, declare PI decision variables, add LPI constraints, and solve the resulting optimization problem. We refer to the PIETOOLS User Manual [43] and [11] for details.

A. Primal and Dual LPIs for Stability of PDEs

In the following theorem, we propose primal and dual LPI tests for exponential stability and use Cor. 7 to show that feasibility of either implies exponential stability of both the primal and dual systems.

Theorem 9. *Suppose that* ***either*** *of the two statements hold for some $\alpha > 0$ and $\mathcal{P} \in \mathbf{\Pi}_4$ such that $\mathcal{P} = \mathcal{P}^* \succeq \eta I$ for some $\eta > 0$.*

a) $\mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} \preceq -2\alpha \mathcal{T}^*\mathcal{P}\mathcal{T}$
b) $\mathcal{T}\mathcal{P}\mathcal{A}^* + \mathcal{A}\mathcal{P}\mathcal{T}^* \preceq -2\alpha \mathcal{T}\mathcal{P}\mathcal{T}^*$

Then the PIEs defined by $\{\mathcal{T}, \mathcal{A}\} \subset \mathbf{\Pi}_4$ and $\{\mathcal{T}^*, \mathcal{A}^*\} \subset \mathbf{\Pi}_4$ are ***Exponentially Stable*** *with decay rate α.*

Proof. Suppose a) holds. Define $V(\mathbf{x}) = \langle \mathcal{T}\mathbf{x}, \mathcal{P}\mathcal{T}\mathbf{x}\rangle_{\mathbb{R}L_2}$. Since any $\mathcal{P} \in \mathbf{\Pi}_4$ is bounded in $\mathcal{L}(\mathbb{R}L_2)$,

$$\eta \|\mathcal{T}\mathbf{x}\|^2 \leq V(\mathbf{x}) \leq \|\mathcal{P}\|_{\mathcal{L}(\mathbb{R}L_2)} \|\mathcal{T}\mathbf{x}\|^2.$$

Suppose $\mathbf{x}(t)$ satisfies $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathcal{T}\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t)$. Differentiating $V(\mathbf{x}(t))$ with respect to time, t, we obtain

$$\begin{aligned} \dot{V}(\mathbf{x}(t)) &= \langle \mathcal{T}\mathbf{x}(t), \mathcal{P}\mathcal{A}\mathbf{x}(t)\rangle + \langle \mathcal{A}\mathbf{x}(t), \mathcal{P}\mathcal{T}\mathbf{x}(t)\rangle \\ &= \langle \mathbf{x}(t), (\mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T})\mathbf{x}(t)\rangle \leq -2\alpha V(\mathbf{x}(t)). \end{aligned}$$

Therefore, $\dot{V}(\mathbf{x}(t)) \leq -2\alpha V(\mathbf{x}(t))$ for all t and, from the Gronwall-Bellman inequality, $V(\mathbf{x}(t)) \leq V(\mathbf{x}(0))e^{-2\alpha t}$. Let $\beta = \|\mathcal{T}\|_{\mathcal{L}(\mathbb{R}L_2)}$ and $\zeta = \|\mathcal{P}\|_{\mathcal{L}(\mathbb{R}L_2)}$. Then

$$\begin{aligned} \|\mathcal{T}\mathbf{x}(t)\|^2 &\leq \frac{1}{\eta} V(\mathbf{x}(t)) \leq \frac{1}{\eta} V(\mathbf{x}(0)) e^{-2\alpha t} \\ &\leq \frac{1}{\eta} \zeta \|\mathcal{T}\mathbf{x}(0)\|^2 e^{-2\alpha t} \leq \frac{\zeta\beta^2}{\eta} \|\mathbf{x}(0)\|^2 e^{-2\alpha t}. \end{aligned}$$

By taking square root on both sides,

$$\|\mathcal{T}\mathbf{x}(t)\| \leq M \|\mathbf{x}(0)\| e^{-\alpha t}$$

where $M = \sqrt{\frac{\zeta}{\eta}}\beta$. This implies the PIE defined by $\{\mathcal{T}, \mathcal{A}\} \subset \mathbf{\Pi}_4$ is **Exponentially Stable** with decay rate α. Then, from Corollary 7, the PIE defined by $\{\mathcal{T}^*, \mathcal{A}^*\} \subset \mathbf{\Pi}_4$ is **Exponentially Stable** with decay rate α.

The proof similarly establishes exponential stability for b) by swapping $\mathcal{T} \mapsto \mathcal{T}^*$ and $\mathcal{A} \mapsto \mathcal{A}^*$ and proving stability of the dual system. $\square$

Both a) and b) in Thm. 9 imply exponential stability of both primal and dual using the definition of exponential stability in Defn. 6, $\|\mathcal{T}\mathbf{x}(t)\|_{\mathbb{R}L_2} \leq M\|\mathbf{x}_0\|_{\mathbb{R}L_2} e^{-\alpha t}$ where the upper bound is defined using the L_2-norm of the PIE state (which is equivalent to the Sobolev norm of the PDE state). This slightly stronger norm is needed to preserve the symmetry of the primal and dual. However, we also note that from the proof of Thm. 9, a) implies exponential stability of the primal and b) implies exponential stability of the dual using an upper bound of the form $\|\mathcal{T}\mathbf{x}(t)\|_{\mathbb{R}L_2} \leq M\|\mathcal{T}\mathbf{x}_0\|_{\mathbb{R}L_2} e^{-\alpha t}$. Practically, however, there is no difference between these definitions of exponential stability since we always assume that $\mathbf{x}_0 \in \mathbb{R}L_2$.

B. Primal and Dual KYP Lemma for PDEs

In the following theorem, we propose LPI generalizations of the primal and dual versions of the KYP Lemma and use Theorem 8 to show that the solution of either proves a bound on the L_2-gain of both the primal and dual systems.

Note that the LPI conditions in Theorem 10 are expressed using an extension of block matrices to block PI operators – The formal definition of concatenation of PI operators can be found in [42, Lemmas 40 and 41]. However, because the domain and range of PI operators of the form given in Defn. 1 are an ordered concatenation of $\mathbb{R}$ and L_2, the arrangement of the blocks of the operators in the proposed LPI conditions are slightly different from that in the tradition formulations of the KYP Lemma for state-space ODEs.

Theorem 10. *Suppose that* ***either*** *of the two statements hold for some $\gamma > 0$ and $\mathcal{P} \in \mathbf{\Pi}_4$ such that $\mathcal{P} = \mathcal{P}^* \succeq 0$.*

a) $\begin{bmatrix} -\gamma I & D & \mathcal{C} \\ D^T & -\gamma I & \mathcal{B}^*\mathcal{P}\mathcal{T} \\ \mathcal{C}^* & \mathcal{T}^*\mathcal{P}\mathcal{B} & \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} \end{bmatrix} \preceq 0$

b) $\begin{bmatrix} -\gamma I & D^T & \mathcal{B}^* \\ D & -\gamma I & \mathcal{C}\mathcal{P}\mathcal{T}^* \\ \mathcal{B} & \mathcal{T}\mathcal{P}\mathcal{C}^* & \mathcal{T}\mathcal{P}\mathcal{A}^* + \mathcal{A}\mathcal{P}\mathcal{T}^* \end{bmatrix} \preceq 0$

Then, for any $w \in L_2$, if z satisfies either

$$\begin{bmatrix} \partial_t(\mathcal{T}\mathbf{x}(t)) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & D \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ w(t) \end{bmatrix}, \tag{13}$$

or

$$\begin{bmatrix} \mathcal{T}^*\dot{\mathbf{x}}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & D^T \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ w(t) \end{bmatrix}, \tag{14}$$

for some $\mathbf{x}(t)$ with $\mathcal{T}\mathbf{x}(0) = 0$, then $\|z\|_{L_2} \leq \gamma \|w\|_{L_2}$.

Proof. Suppose a) holds. Define $V(\mathbf{x}) = \langle \mathcal{T}\mathbf{x}, \mathcal{P}\mathcal{T}\mathbf{x}\rangle_{\mathbb{R}L_2}$. For any $w \in L_2$, suppose z satisfies Eq. (13) for some $\mathbf{x}$ with $\mathbf{x}(0) = 0$. Differentiating $V(\mathbf{x}(t))$ with respect to time, t, we obtain

$$\begin{aligned} \dot{V}(\mathbf{x}(t)) &= \langle \mathcal{T}\mathbf{x}(t), \mathcal{P}(\mathcal{A}\mathbf{x}(t) + \mathcal{B}w(t))\rangle \\ &\quad + \langle (\mathcal{A}\mathbf{x}(t) + \mathcal{B}w(t)), \mathcal{P}\mathcal{T}\mathbf{x}(t)\rangle \\ &= \left\langle \begin{bmatrix} w(t) \\ \mathbf{x}(t) \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{B}^*\mathcal{P}\mathcal{T} \\ \mathcal{T}^*\mathcal{P}\mathcal{B} & \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} \end{bmatrix} \begin{bmatrix} w(t) \\ \mathbf{x}(t) \end{bmatrix} \right\rangle. \end{aligned}$$

Now let $v(t) = \frac{1}{\gamma} z(t)$. Then we have

$$\begin{aligned} &\dot{V}(\mathbf{x}(t)) - \gamma \|w(t)\|_{\mathbb{R}}^2 + \frac{1}{\gamma}\|z(t)\|_{\mathbb{R}}^2 \\ &= \dot{V}(\mathbf{x}(t)) - \gamma \|w(t)\|^2 - \frac{1}{\gamma}\|z(t)\|^2 + \frac{2}{\gamma}\|z(t)\|^2 \\ &= \dot{V}(\mathbf{x}(t)) - \gamma \|w(t)\|^2 - \gamma\|v(t)\|^2 + v(t)^T z(t) + z(t)^T v(t) \\ &= \left\langle \begin{bmatrix} v(t) \\ w(t) \\ \mathbf{x}(t) \end{bmatrix}, \begin{bmatrix} -\gamma I & D & \mathcal{C} \\ D^T & -\gamma I & \mathcal{B}^*\mathcal{P}\mathcal{T} \\ \mathcal{C}^* & \mathcal{T}^*\mathcal{P}\mathcal{B} & \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \\ \mathbf{x}(t) \end{bmatrix} \right\rangle \leq 0. \end{aligned}$$

Integrating this inequality in time, we obtain

$$V(\mathbf{x}(T)) - V(\mathbf{x}(0)) \leq \gamma \int_0^T \|w(t)\|^2 dt - \frac{1}{\gamma}\int_0^T \|z(t)\|^2 dt.$$

Now, since $\mathcal{T}\mathbf{x}(0) = 0$ and $V(\mathbf{x}(T)) \geq 0$ for all $T \geq 0$, we obtain $\|z\|_{L_2}^2 \leq \gamma^2 \|w\|_{L_2}^2$. Furthermore, Theorem 8 implies the same bound hold if z and $\mathbf{x}$ satisfy Eq. (14).

Since $\mathcal{T}^{**} = \mathcal{T}$ and

$$\begin{bmatrix} (\mathcal{A}^*)^* & (\mathcal{B}^*)^* \\ (\mathcal{C}^*)^* & (D^T)^T \end{bmatrix} = \begin{bmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & D^T \end{bmatrix}^* = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & D \end{bmatrix}^{**} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & D \end{bmatrix}$$

we have that b) likewise implies the same bounds. $\square$

Before applying the results of Theorem 10 to controller synthesis, we note that while the operator variable $\mathcal{P}$, which represents the storage function $V(\mathbf{x}) = \langle \mathcal{T}\mathbf{x}, \mathcal{P}\mathcal{T}\mathbf{x}\rangle$ in Theorem 10, is not required to be *strictly* positive – thus allowing for the use of non-coercive storage functions (See [22]). When we turn to the problem of optimal controller synthesis in Subsection VI-B, however, we will require strict positivity of $\mathcal{P}$ so that we may reconstruct the controller gains as $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$.

C. Duality using Extended Lyapunov Functions

The LPI conditions for exponential stability and L_2-gain in Thms. 9 and 10 were obtained by parameterizing candidate Lyapunov/storage functions of the form $V(\mathbf{x}) = \langle \mathcal{T}\mathbf{x}, \mathcal{P}\mathcal{T}\mathbf{x}\rangle_{L_2}$ where $\mathbf{x}$ is the PIE state. This parametrization was chosen to ensure that the function V is both lower and upper bounded with respect to the original PDE state, $\mathbf{v} := \mathcal{T}\mathbf{x}$ – i.e. $V = \langle \mathcal{T}\mathbf{x}, \mathcal{P}\mathcal{T}\mathbf{x}\rangle_{L_2} = \langle \mathbf{v}, \mathcal{P}\mathbf{v}\rangle_{L_2}$ and $\mathcal{P} \succ 0$ implies $\alpha\|\mathbf{v}\|^2 \leq V(\mathbf{v}) \leq \beta\|\mathbf{v}\|^2$ for some $\alpha, \beta > 0$. However, these upper and lower bounds are unnecessary when computing L_2-gain. Therefore, in this subsection we propose an extension of Thm. 10 which allows V to be defined by a mix of PIE and PDE states as $V(\mathbf{x}) = \langle \mathcal{Q}\mathbf{x}, \mathcal{T}\mathbf{x}\rangle_{L_2}$ for some PI operator Q. By including the PIE state, $\mathbf{x}$, in addition to the PDE state, $\mathcal{T}\mathbf{x}$, this *extended* functional form allows for partial derivatives of the PDE state to appear.

To illustrate, consider the heat equation $\partial_t \mathbf{v}(t) = \partial_s^2 \mathbf{v}(t)$ with boundary conditions $\mathbf{u}(0) = \partial_s \mathbf{u}(1) = 0$. The extended parametrization now allows us to use storage functions defined in terms of partial derivatives – such as $V = \|\mathbf{v}_s\|^2$. Specifically, if we choose $\mathcal{Q} = -I$, and defining the PIE state as $\mathbf{x} := \mathbf{v}_{ss}$ (where $\mathbf{v} = \mathcal{T}\mathbf{x}$ for some $\mathcal{T}$), we have

$$V(\mathbf{x}) = \langle \mathcal{Q}\mathbf{x}, \mathcal{T}\mathbf{x}\rangle = -\langle \mathbf{v}_{ss}, \mathbf{v}\rangle = \|\mathbf{v}_s\|^2.$$

We note that this extended parametrization of storage functions includes, as a subset, those functions used in Thm. 10. Specifically, for any given $\mathcal{P}$, if we choose $\mathcal{Q}$ as $\mathcal{Q} = \mathcal{P}\mathcal{T}$, then the extended storage function has the form $V(\mathbf{x}) = \langle \mathcal{Q}\mathbf{x}, \mathcal{T}\mathbf{x}\rangle_{L_2} = \langle \mathcal{T}\mathbf{x}, \mathcal{P}\mathcal{T}\mathbf{x}\rangle_{L_2}$, as was used in Thm. 10. The drawback of such extended Lyapunov function candidates, however, is that they are not upper-bounded with respect to the PDE state since the L_2 norms and Sobolev norms are not equivalent. This means that we cannot extend the exponential stability criterion in Thm. 9 without redefining our notion of exponential stability.

Theorem 11. *Suppose that* ***either*** *of the two statements hold for some $\gamma > 0$ and $\mathcal{R}, \mathcal{Q} \in \mathbf{\Pi}_4$ such that $\mathcal{R} = \mathcal{R}^* \succeq 0$.*

a) $\mathcal{T}^*\mathcal{Q} = \mathcal{Q}^*\mathcal{T} = \mathcal{R}$ and

$$\begin{bmatrix} -\gamma I & D & \mathcal{C} \\ D^T & -\gamma I & \mathcal{B}^*\mathcal{Q} \\ \mathcal{C}^* & \mathcal{Q}^*\mathcal{B} & \mathcal{Q}^*\mathcal{A} + \mathcal{A}^*\mathcal{Q} \end{bmatrix} \preceq 0$$

b) $\mathcal{T}\mathcal{Q} = \mathcal{Q}^*\mathcal{T}^* = \mathcal{R}$ and

$$\begin{bmatrix} -\gamma I & D^T & \mathcal{B}^* \\ D & -\gamma I & \mathcal{C}\mathcal{Q}^* \\ \mathcal{B} & \mathcal{Q}\mathcal{C}^* & \mathcal{Q}\mathcal{A}^* + \mathcal{A}\mathcal{Q}^* \end{bmatrix} \preceq 0$$

Then, for any $w \in L_2$, if z satisfies either

$$\begin{bmatrix} \partial_t(\mathcal{T}\mathbf{x}(t)) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & D \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ w(t) \end{bmatrix}, \tag{15}$$

or

$$\begin{bmatrix} \mathcal{T}^*\dot{\mathbf{x}}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & D^T \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ w(t) \end{bmatrix}, \tag{16}$$

for some $\mathbf{x}(t)$ with $\mathbf{x}(0) = 0$, then $\|z\|_{L_2} \leq \gamma \|w\|_{L_2}$.

Proof. Suppose a) holds. Define $V(\mathbf{x}) = \langle \mathbf{x}, \mathcal{R}\mathbf{x} \rangle \geq 0$. Since $\mathcal{Q}^*\mathcal{T} = \mathcal{T}^*\mathcal{Q}$, we have $V(\mathbf{x}) = \langle \mathcal{T}\mathbf{x}, \mathcal{Q}\mathbf{x} \rangle = \langle \mathcal{Q}\mathbf{x}, \mathcal{T}\mathbf{x} \rangle$. Now, for any $w \in L_2$, suppose z satisfies Eq. (13) for some $\mathbf{x}$ with $\mathbf{x}(0) = 0$. Differentiating $V(\mathbf{x}(t))$ with respect to time, t, we obtain

$$\begin{aligned} &\dot{V}(\mathbf{x}(t)) = \langle \partial_t(\mathcal{T}\mathbf{x}(t)), \mathcal{Q}\mathbf{x}(t) \rangle_{\mathbb{R}L_2} + \langle \mathcal{Q}\mathbf{x}(t), \partial_t(\mathcal{T}\mathbf{x}(t)) \rangle_{\mathbb{R}L_2} \\ &= \langle \mathcal{Q}\mathbf{x}(t), \mathcal{A}\mathbf{x}(t) + \mathcal{B}w(t) \rangle + \langle \mathcal{A}\mathbf{x}(t) + \mathcal{B}w(t), \mathcal{Q}\mathbf{x}(t) \rangle \\ &= \left\langle \begin{bmatrix} w(t) \\ \mathbf{x}(t) \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{B}^*Q \\ \mathcal{Q}^*\mathcal{B} & \mathcal{Q}^*\mathcal{A} + \mathcal{A}^*\mathcal{Q} \end{bmatrix} \begin{bmatrix} w(t) \\ \mathbf{x}(t) \end{bmatrix} \right\rangle. \end{aligned}$$

Now let $v(t) = \frac{1}{\gamma} z(t)$. Then we have

$$\begin{aligned} &\dot{V}(\mathbf{x}(t)) - \gamma \|w(t)\|_{\mathbb{R}}^2 + \frac{1}{\gamma} \|z(t)\|_{\mathbb{R}}^2 \\ &= \dot{V}(\mathbf{x}(t)) - \gamma \|w(t)\|^2 - \frac{1}{\gamma} \|z(t)\|^2 + \frac{2}{\gamma} \|z(t)\|^2 \\ &= \dot{V}(\mathbf{x}(t)) - \gamma \|w(t)\|^2 - \gamma \|v(t)\|^2 + v(t)^T z(t) + z(t)^T v(t) \\ &= \left\langle \begin{bmatrix} v(t) \\ w(t) \\ \mathbf{x}(t) \end{bmatrix}, \begin{bmatrix} -\gamma I & D & \mathcal{C} \\ D^T & -\gamma I & \mathcal{B}^*\mathcal{Q} \\ \mathcal{C}^* & \mathcal{Q}^*\mathcal{B} & \mathcal{Q}^*\mathcal{A} + \mathcal{A}^*\mathcal{Q} \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \\ \mathbf{x}(t) \end{bmatrix} \right\rangle \leq 0. \end{aligned}$$

The rest of the proof is exactly as in the proof of Thm. 10. ☐

Finally, we note that feasibility of the conditions in Thm. 10 imply the conditions of Thm. 11 are satisfied with $\mathcal{Q} = \mathcal{P}\mathcal{T}$ for a) and $\mathcal{Q} = \mathcal{P}\mathcal{T}^*$ for b). In addition, if it is known that if $\{\mathcal{T}, \mathcal{A}\}$ or $\{\mathcal{T}^*, \mathcal{A}^*\}$ is asymptotically stable then the non-negativity constraints on $\mathcal{P}$ in Thm. 10 and on $\mathcal{Q}$ in Thm. 11 may be removed entirely.

VI. LPIs for Stabilizing and H_∞-Optimal State-Feedback Controller Synthesis

In this section, we return to the state-feedback controller synthesis problems defined in Section III (Eqs. (2) and (3)). Specifically, given a PIE system

$$\begin{bmatrix} \partial_t(\mathcal{T}\mathbf{x}(t)) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C} & D_1 & D_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ w(t) \\ u(t) \end{bmatrix},$$

our goal is to synthesize state-feedback controllers of the form $u(t) = \mathcal{K}\mathbf{x}(t)$, where $\mathbf{x}$ is the state of the PIE and the controller gain, $\mathcal{K}$, is a PI operator. To do this, we apply Corollary 7 and Theorem 10 to the closed-loop system

$$\begin{bmatrix} \partial_t(\mathcal{T}\mathbf{x}(t)) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} + \mathcal{B}_2\mathcal{K} & \mathcal{B}_1 \\ \mathcal{C} + D_2\mathcal{K} & D_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ w(t) \end{bmatrix}. \tag{17}$$

The resulting operator inequality then includes the term $\mathcal{K}\mathcal{P}$ which is bilinear in the decision variables $\mathcal{K}$ and $\mathcal{P}$. However, as described in the introduction, and following the approach used for state-space ODEs, we then construct an equivalent LPI by making the invertible variable substitution $\mathcal{K}\mathcal{P} \to \mathcal{Z}$. An iterative algorithm for the inversion of this variable substitution is presented in Section VII – allowing us to reconstruct the controller gains for implementation in simulation or real-time feedback.

A. Stabilizing State-Feedback Control

The following Corollary defines an LPI whose solution provides an exponentially stabilizing state-feedback controller for the PDE associated with the PIE $\{\mathcal{T}, \mathcal{A}, \mathcal{B}_2\}$.

Corollary 12. *Suppose there exist some $\alpha > 0$ and $\mathcal{Z}, \mathcal{P} \in \mathbf{\Pi}_4$ such that $\mathcal{P} = \mathcal{P}^* \succeq \eta I$ for some $\eta > 0$, and*

$$(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})\mathcal{T}^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})^* \preceq -2\alpha\mathcal{T}\mathcal{P}\mathcal{T}^*.$$

Then if $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$, the PIE defined by $\{\mathcal{T}, \mathcal{A} + \mathcal{B}_2\mathcal{K}\} \subset \mathbf{\Pi}_4$ is ***Exponentially Stable*** *with decay rate α.*

Proof. Let $\mathcal{P}$, $\mathcal{Z}$, and $\mathcal{K}$ be as defined above. Then, $\mathcal{Z} = \mathcal{K}\mathcal{P}$, and

$$\begin{aligned} &(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})\mathcal{T}^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})^* \\ &\quad = (\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{K}\mathcal{P})\mathcal{T}^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{K}\mathcal{P})^* \\ &\quad = (\mathcal{A} + \mathcal{B}_2\mathcal{K})\mathcal{P}\mathcal{T}^* + \mathcal{T}\mathcal{P}(\mathcal{A} + \mathcal{B}_2\mathcal{K})^* \preceq -2\alpha\mathcal{T}\mathcal{P}\mathcal{T}^*. \end{aligned}$$

Then, from Theorem 9 (statement b), the PIE defined by $\{\mathcal{T}, \mathcal{A} + \mathcal{B}_2\mathcal{K}\}$, is exponentially stable with decay rate α. ☐

B. H_∞-Optimal State-Feedback Control

Next, we provide an LPI to find the H_∞-optimal state-feedback controller, $\mathcal{K}$, for PIEs with inputs and outputs of the form Eq. (17). Here, we use $(\cdot)^*$ notation to represent the symmetric adjoint/transpose completion of block operators.

Corollary 13. *Suppose there exist some $\gamma > 0$ and $\mathcal{Z}, \mathcal{P} \in \mathbf{\Pi}_4$ such that $\mathcal{P} = \mathcal{P}^* \succeq \eta I$ for some $\eta > 0$, and*

$$\begin{bmatrix} -\gamma I & D_1^T & \mathcal{B}_1^* \\ (\cdot)^* & -\gamma I & (\mathcal{C}\mathcal{P} + D_2\mathcal{Z})\mathcal{T}^* \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})^* \end{bmatrix} \preceq 0.$$

Then if $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$, for any $w \in L_2$, if z satisfies

$$\begin{bmatrix} \partial_t(\mathcal{T}\mathbf{x}(t)) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} + \mathcal{B}_2\mathcal{K} & \mathcal{B}_1 \\ \mathcal{C} + D_2\mathcal{K} & D_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ w(t) \end{bmatrix},$$

for some $\mathbf{x}$ with $\mathcal{T}\mathbf{x}(0) = 0$, then $\|z\|_{L_2} \leq \gamma \|w\|_{L_2}$.

Proof. Let $\mathcal{P}$, $\mathcal{Z}$, and $\mathcal{K}$ satisfy the corollary statement. Then, $\mathcal{Z} = \mathcal{K}\mathcal{P}$, and

$$\begin{bmatrix} -\gamma I & D_1^T & \mathcal{B}_1^* \\ (\cdot)^* & -\gamma I & (\mathcal{C}\mathcal{P} + D_2\mathcal{Z})\mathcal{T}^* \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})^* \end{bmatrix}$$

$$= \begin{bmatrix} -\gamma I & D_1^T & \mathcal{B}_1^* \\ (\cdot)^* & -\gamma I & (\mathcal{C}+D_2\mathcal{K})\mathcal{P}\mathcal{T}^* \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + \mathcal{T}\mathcal{P}(\mathcal{A}+\mathcal{B}_2\mathcal{K})^* \end{bmatrix} \preceq 0.$$

Thus, from Theorem 10 (statement b), for $z, w, \mathbf{x}$ as in the corollary statement, we have that $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$. □

Given a PDE with associated PIE defined by $\{\mathcal{T}, \mathcal{A}, \mathcal{B}_i, \mathcal{C}, D_i\}$, Corollary 13 provides a controller gain $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$ such that $u(t) = \mathcal{K}\mathbf{x}(t)$ achieves a closed-loop performance bound of $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$. Note that this controller does not necessarily imply internal exponential stability unless the LPI in Corollary 12 is negative definite in a suitable sense.

C. Optimal Controllers using Extended Lyapunov Functions

In this subsection, we briefly propose alternative optimal state-feedback controller synthesis conditions based on the extended L_2-gain conditions described in Subsection V-C and which allow for Lyapunov/storage functions which include partial derivatives of the state. Specifically, for the problem of H_∞-optimal state-feedback controller synthesis, we have the following alternative formulation of Thm. 11.

Corollary 14. *Suppose there exist some $\gamma > 0$ and $\mathcal{Z}, \mathcal{R}, \mathcal{Q} \in \mathbf{\Pi}_4$ such that $\mathcal{T}\mathcal{Q} = \mathcal{Q}^*\mathcal{T}^* = \mathcal{R} \succeq 0$, $\mathcal{Q}$ is invertible, and*

$$\begin{bmatrix} -\gamma I & D_1^T & \mathcal{B}_1^* \\ (\cdot)^* & -\gamma I & \mathcal{C}\mathcal{Q}^* + D_2\mathcal{Z} \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + (\mathcal{A}\mathcal{Q} + \mathcal{B}_2\mathcal{Z}) \end{bmatrix} \preceq 0.$$

Then if $\mathcal{K} = \mathcal{Z}\mathcal{Q}^{-1}$, for any $w \in L_2$, if z satisfies

$$\begin{bmatrix} \partial_t(\mathcal{T}\mathbf{x}(t)) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}+\mathcal{B}_2\mathcal{K} & \mathcal{B}_1 \\ \mathcal{C}+D_2\mathcal{K} & D_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ w(t) \end{bmatrix},$$

for some $\mathbf{x}$ with $\mathcal{T}\mathbf{x}(0) = 0$, then $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$.

Proof. The proof is similar to the proof of Cor. 13 using the alternative conditions defined in Thm. 11. □

VII. Controller Reconstruction

In this section, we presume that for some given PIE, the LPIs in Corollary 12, 13, or 14 have solution $\mathcal{Z}, \mathcal{P} \in \mathbf{\Pi}_4$ where $\mathcal{P} \succ 0$. In this case, a controller for the PDE (with state $\mathbf{v}$) associated with the given PIE may be constructed as $u(t) = \mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t)$ where $\mathbf{x}$ is the PIE state associated with the PDE. Now suppose $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$ is a PI operator, and $\mathbf{x}(t) \in \mathbb{R}L_2$ is partitioned as $\mathbf{x}(t,s) = [x_1(t), \mathbf{x}_2(t,s)]$. Then such a controller has the form

$$u(t) = K_1 x_1(t) + \int_a^b K_2(s)\mathbf{x}_2(t,s)ds.$$

Real-time estimates of the $\mathbf{x}_2(t)$ may be obtained by measurement and spatial differentiation of the PDE state or through use of an estimator, as described in [12]. However, construction of the gains $K_1, K_2(s)$ requires us to compute the inverse operator $\mathcal{P}^{-1}$.

The question of inverting a PI operator of the form $\mathcal{P} = \mathit{\Pi}\left[\begin{array}{c|c} P & Q \\ \hline Q^T & \{R_i\} \end{array}\right] \in \mathbf{\Pi}_4$ has been considered in [33] under the restriction $R_1 = R_2$ (the case of *separable* kernels) and without the terms P, Q (no ODE states). Unfortunately, the restriction $R_1 = R_2$ often results in suboptimal controllers. In this section, therefore, we lift the restriction $R_1 = R_2$ using a new method for inverting PI operators based on generalization of a result in [18, Chapter IX.2]. This method is presented in stages: first neglecting ODE states and restricting $R_0 = I$ (Lemma 15); then accounting for R_0 (Cor. 16); then accounting for ODE states (Lemma 17).

Because the initial steps of this method do not consider ODE states, Lemma 15 and Cor. 16 utilize the simplified notation

$$\mathit{\Pi}_{\{R_0,R_1,R_2\}} := \mathit{\Pi}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{R_0,R_1,R_2\} \end{array}\right].$$

The following lemma provides a construction for the inverse of a PI operator, $\mathit{\Pi}_{\{I,H_1,H_2\}}$ where separability of parameters H_1, H_2 is implied by separability of polynomials – i.e. any polynomial, H can be written as $H(s,\theta) = F(s)G(\theta)$ for some polynomials F, G.

Lemma 15 (Gohberg [18])**.** *Define $\mathcal{P} := \mathit{\Pi}_{\{I,H_1,H_2\}}$ where $H_i(s,\theta) = -F_i(s)G_i(\theta)$ for some $F_i, G_i \in L_2[a,b]$. Let U and V be the unique solutions to*

$$U(s) = I + \int_a^s B(\theta)C(\theta)U(\theta)d\theta$$

and

$$V(\theta) = I - \int_a^\theta V(s)B(s)C(s)ds$$

where $C(s) = \begin{bmatrix} F_1(s) & F_2(s) \end{bmatrix}$, $B(s) = \begin{bmatrix} G_1(s) \\ -G_2(s) \end{bmatrix}$. Then $V(s)U(s) = U(s)V(s) = I$. Furthermore, if we partition

$$U(b) = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad U_{22} \in \mathbb{R}^{q\times q},$$

where q is the number of columns in F_2, then $\mathcal{P}$ is invertible if and only if U_{22} is invertible and $\mathcal{P}^{-1} = \mathit{\Pi}_{\{I,M_1,M_2\}}$ where

$$P = \begin{bmatrix} 0 & 0 \\ U_{22}^{-1}U_{21} & I \end{bmatrix}, \quad \begin{array}{l} M_1(s,\theta) = C(s)U(s)(I-P)V(\theta)B(\theta), \\ M_2(s,\theta) = -C(s)U(s)PV(\theta)B(\theta). \end{array}$$

Note that, by construction, M_1 and M_2 are *separable* – implying that $\mathcal{P}^{-1}$ is a PI operator, albeit not necessarily with polynomial parameters.

We now extend Lemma 15 to PI operators of the form $\mathcal{P} = \mathit{\Pi}_{\{R_0,R_1,R_2\}}$, where $\mathcal{P} \succ 0$ implies invertibility of R_0.

Corollary 16. *Suppose $R_0^{-1} \in L_2$ and define $H_1(s,\theta) = R_0(s)^{-1}R_1(s,\theta)$ and $H_2(s,\theta) = R_0(s)^{-1}R_2(s,\theta)$. Now let M_1, M_2 be as defined in Lemma 15. Then*

$$\mathit{\Pi}_{\{R_i\}}^{-1} = \mathit{\Pi}_{\{\hat{R}_i\}}$$

where $\hat{R}_0 = R_0^{-1}$, $\hat{R}_1(s,\theta) = M_2(s,\theta)R_0^{-1}(\theta)$, and $\hat{R}_2(s,\theta) = M_1(s,\theta)R_0^{-1}(\theta)$.

Proof. The proof follows immediately from Lemma 15 and operator composition rules [42] as

$$\begin{aligned} \mathit{\Pi}_{\{R_0,R_1,R_2\}}^{-1} &= (\mathit{\Pi}_{\{R_0,0,0\}}\mathit{\Pi}_{\{I,H_1,H_2\}})^{-1} \\ &= \mathit{\Pi}_{\{I,M_1,M_2\}}\mathit{\Pi}_{\{R_0^{-1},0,0\}} = \mathit{\Pi}_{\{\hat{R}_0,\hat{R}_1,\hat{R}_2\}}. \end{aligned}$$

□

Next, we extend Corollary 16 to $\mathcal{P} \in \mathbf{\Pi}_4$ using a generalization of a standard formula for block matrix inversion.

Lemma 17. *Suppose $\mathit{\Pi}_{\{R_i\}}^{-1} = \mathit{\Pi}_{\{\hat{R}_i\}}$. Then $\mathcal{P} := \mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] \in \mathbf{\Pi}_4$ is invertible if and only if the matrix*

$$T = P - \mathit{\Pi}\left[\begin{array}{c|c} \emptyset & Q_1 \\ \hline \emptyset & \{\emptyset\} \end{array}\right] \mathit{\Pi}_{\{\hat{R}_i\}} \mathit{\Pi}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline Q_2 & \{\emptyset\} \end{array}\right]$$

is invertible. Furthermore,

$$\mathcal{P}^{-1} = \mathcal{U}\,\Pi\left[\begin{array}{c|c} T^{-1} & 0 \\ \hline 0 & \{\hat{R}_i\}\end{array}\right]\mathcal{V}$$

where

$$\mathcal{U} = \Pi\left[\begin{array}{c|c} I & 0 \\ \hline 0 & \{\hat{R}_i\}\end{array}\right]\Pi\left[\begin{array}{c|c} I & 0 \\ \hline -Q_2 & \{R_i\}\end{array}\right]$$

$$\mathcal{V} = \Pi\left[\begin{array}{c|c} I & -Q_1 \\ \hline 0 & \{R_i\}\end{array}\right]\Pi\left[\begin{array}{c|c} I & 0 \\ \hline 0 & \{\hat{R}_i\}\end{array}\right].$$

Proof. Let $\hat{R}$ be as specified and define $\mathcal{R} = \Pi\left[\begin{array}{c|c} I & 0 \\ \hline 0 & \{\hat{R}_i\}\end{array}\right]$. Then $\mathcal{P}$ can be decomposed as

$$\mathcal{P} = \overbrace{\Pi\left[\begin{array}{c|c} I & Q_1 \\ \hline 0 & \{R_i\}\end{array}\right]\mathcal{R}}^{\mathcal{M}:=}\overbrace{\Pi\left[\begin{array}{c|c} T & 0 \\ \hline 0 & \{R_i\}\end{array}\right]}^{\mathcal{Q}:=}\overbrace{\mathcal{R}\,\Pi\left[\begin{array}{c|c} I & 0 \\ \hline Q_2 & \{R_i\}\end{array}\right]}^{\mathcal{N}:=}.$$

Clearly, $\mathcal{M}, \mathcal{N}$ are triangular, which implies $\mathcal{N}^{-1} = \mathcal{U}$ and $\mathcal{M}^{-1} = \mathcal{V}$. Hence invertibility of $\mathcal{P}$ is now equivalent to the invertibility of $\mathcal{Q}$. Finally, we have

$$\mathcal{Q}^{-1} = \Pi\left[\begin{array}{c|c} T & 0 \\ \hline 0 & \{R_i\}\end{array}\right]^{-1} = \Pi\left[\begin{array}{c|c} T^{-1} & 0 \\ \hline 0 & \{\hat{R}_i\}\end{array}\right]$$

which completes the proof. □

Given the results in Lemma 15, Cor. 16, and Lemma 17, we now consider the numerical problem of computing the controller gains, K_1, K_2, where

$$\mathcal{K} = \Pi\left[\begin{array}{c|c} K_1 & K_2 \\ \hline \emptyset & \{\emptyset\}\end{array}\right] = \mathcal{Z}\mathcal{P}^{-1}$$

where

$$\mathcal{Z} = \Pi\left[\begin{array}{c|c} Z_1 & Z_2 \\ \hline \emptyset & \{\emptyset\}\end{array}\right] \quad \text{and} \quad \mathcal{P} := \Pi\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]$$

where $\mathcal{Z}$ and $\mathcal{P}$ are obtained from Corollary 12, 13, or 14. Specifically, if $\mathcal{P}^{-1} := \Pi\left[\begin{array}{c|c} \hat{P} & \hat{Q}_1 \\ \hline \hat{Q}_2 & \{\hat{R}_i\}\end{array}\right]$, then

$$K_1 = Z_1\hat{P} + \int_a^b Z_2(s)\hat{Q}_2(s)ds$$

and

$$K_2(s) = Z_1\hat{Q}_1(s) + Z_2(s)\hat{R}_0(s) + \int_a^s Z_2(\theta)\hat{R}_1(\theta,s)d\theta + \int_s^b Z_2(\theta)\hat{R}_2(\theta,s)d\theta.$$

Of course, the parameters $\hat{P}, \hat{Q}_i, \hat{R}_i$ are obtained sequentially by first computing solutions M_1, M_2 to the Volterra-type integral equations (of 2^{nd} kind) in Lemma 15, then applying the analytic formulae in Cor. 16 and Lemma 17. We note, however, that both R_0^{-1} and the solutions M_1, M_2 will be non-polynomial and there are no closed-form analytic expressions for these parameters unless $R_1 = R_2$. Fortunately, there exist convergent series expansions which can be used to approximate these terms arbitrarily well. For example, [18, Chapter IX.2] provides the following result.

Lemma 18. *Let $A : [a,b] \to \mathbb{R}^{n\times n}$ be Lebesgue integrable on $[a,b]$. Then, the series $I_n + \sum_{i=1}^\infty U_k(s)$, where $U_k = \int_a^s A(\theta)U_{k-1}(\theta)d\theta$ and $U_1(s) = \int_a^s A(\theta)d\theta$, converges uniformly on $s \in [a,b]$ to a unique function, $U : [a,b] \to \mathbb{R}^{n\times n}$, that solves $U(s) = I_n + \int_a^s A(\theta)U(\theta)d\theta$. Furthermore, for any $k \in \mathbb{N}$,*

$$\|U_k(s)\| \le \frac{1}{k!}\left(\int_a^b \|A(s)\|\,ds\right)^k, \quad s \in [a,b].$$

As a practical matter, we typically only need to obtain values of the gain parameters at discrete points in space, which also simplifies the problem of inversion of the matrix-valued function $R_0(s)$. A more detailed description of the computation of the inverse operator $\mathcal{P}^{-1}$ as implemented in PIETOOLS is described in Appendix A.

VIII. Numerical Implementation and Verification

In this section, we use the PIETOOLS software package to test the LPI for analysis in Thms. 9, 10, 11, and controller synthesis in Cor. 13. These tests are performed on several PDE models, including heat, wave, transport, and beam equations. For cases of actuation at the boundary, an ODE input filter is added as discussed in Subsec. III-B.

In each case, the PIE representation is constructed by first mapping the parameters of PDE to those of the general class of PDE's defined in [42] and then applying the conversion formulae provided therein. LPI tests are used to either find a primal and dual lower bound on exponential decay rate (in Subsec. VIII-A) or find an H_∞-optimal state-feedback controller (in Subsec. VIII-D). In the case of controller synthesis, the controller gains obtained from the LPI, $\mathcal{K}$, are used to construct a closed-loop PDE for numerical simulation and verification of performance. Numerical simulation is performed using the PIESIM package described in [37].

A. Bounding Exponential Decay Rate (Thm. 9)

We apply the primal and dual LPIs for the exponential decay rate in Thm. 9 to a linear delay-differential equation and a PDE reaction-diffusion equation to obtain the maximum lower bound on the exponential decay rate, α. To maximize α, we observe that the LPIs in Thm. 9 are convex in α for a fixed $\mathcal{P}$ – which implies a bisection search on α can be used to maximize the lower bound on exponential decay rate.

Example 19 (Exponential Stability of a Linear Time-Delay System)**.** *Consider the following autonomous linear delay-differential equation from, e.g. [30].*

$$\dot{x}(t) = \begin{bmatrix}-4 & 1\\ 0 & -4\end{bmatrix}x(t) + \begin{bmatrix}0.1 & 0\\ 4 & 0.1\end{bmatrix}x(t-0.5)$$

Delay differential equations can be represented as a transport equation coupled to an ODE and formulae for conversion of a delay-differential equation to a PIE can be found in [35]. For this example, both the primal and dual LPIs obtained maximum provable lower bounds on the exponential decay rate of $\alpha_p = \alpha_d = 1.1534$. These are similar to the estimate of $\alpha = 1.153$ as reported in [30].

Example 20 (Exponential Stability of a reaction-diffusion PDE)**.** *Consider the following reaction-diffusion equation.*

$$\dot{\mathbf{v}}(t,s) = 2\mathbf{v}(t,s) + \partial_s^2\mathbf{v}(t,s), \quad \mathbf{v}(t,0) = \partial_s\mathbf{v}(t,1) = 0.$$

The PIE representation of this PDE is defined by parameters

$$\mathcal{T} = \Pi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{0,-\theta,-s\}\end{array}\right], \mathcal{A} = \Pi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{1,-\lambda\theta,-\lambda s\}\end{array}\right].$$

Using Thm. 9, the maximum provable primal and dual lower bounds on exponential decay rate are $\alpha_p = \alpha_d = 0.4674$. One can find an analytical solution to the above PDE (using the change of variable $\mathbf{y}(t,s) = e^{-2t}\mathbf{x}(t,s)$) and observe that the largest eigenvalue of this solution is -0.4674 – demonstrating accuracy of the maximal lower bounds obtained from the LPIs.

Example	(E1)	(E2)	(E3)
$err(1.0)$	1.2e-07	-3.2e-06	2.1e-06
$err(2.5)$	-4.7e-07	2.8e-05	-2.3e-05
$err(5.0)$	-2.2e-07	1.6e-05	-2.3e-04

TABLE I: The error in Eqn. (11) at 3 time instances for numerical simulation of the primal and dual PIEs obtained from Examples E1, E2, and E3; subject to zero initial conditions and disturbances $w = \sin(5t)\exp(-2t)$ and $\bar{w} = (t-t^2)\exp(-t)$; and where error is defined as $err(t) = \int_0^t \bar{z}(\theta)^T w(t-\theta)d\theta - \int_0^t \bar{w}(\theta)^T z(t-\theta)d\theta$.

B. Verifying Eqn. (11) through numerical simulation

The key relation between input and outputs of the primal and dual PIE obtained in Eqn. (11) can be verified numerically. To perform this numerical verification, we simulate various primal and dual PIE systems and measure any error in Eqn. (11). For each example, we use zero initial conditions, time interval $t \in [0, 5]$, and arbitrarily chosen L_2-bounded disturbances $w(t) = \sin(5t)e^{-2t}$ (for the primal) and $\bar{w}(t) = (t - t^2)e^{-t}$ (for the dual). The simulated responses z (primal) and $\bar{z}$ (dual) are then used to measure the error in Eqn. (11) where error is defined as

$$err(t) = \int_0^t \bar{z}(\theta)^T w(t-\theta)d\theta - \int_0^t \bar{w}(\theta)^T z(t-\theta)d\theta.$$

The verification is performed for three PIE's obtained from the following three illustrative PDE systems:

(E1) $\dot{\mathbf{v}}(t,s) = \partial_s^2 \mathbf{v}(t,s) + w(t)$, $\mathbf{v}(t,0) = \mathbf{v}(t,1) = 0$, $z(t) = \int_0^1 \mathbf{v}(t,s)ds$.

(E2) $\dot{\mathbf{v}}(t,s) = -\partial_s \mathbf{v}(t,s) + w(t)$, $\mathbf{v}(t,0) = 0$, $z(t) = \mathbf{v}(t,1)$.

(E3) $\dot{\mathbf{v}}(t,s) = 3\mathbf{v}(t,s) + \partial_s^2 \mathbf{v}(t,s) + w(t)$, $\mathbf{v}(t,0) = \partial_s \mathbf{v}(t,1) = 0$, $z(t) = \mathbf{v}(t,1)$.

For each PDE, the formulae in [42, Block 4 and 5]) are used to obtain the parameters $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$. The results are given in Table I and indicate almost no numerical error over the given time interval for all three examples.

Note that Example (E3) is unstable and hence its primal and dual PIE representation are likewise unstable. However, as mentioned in Remark 1, the intertwining relationship in Eqn. (11) does not require stability – an assertion verified by the numerical analysis in Table I.

C. Comparison of L_2-gain Bounds from Thm. 10 and Thm. 11

Next, we examine the practical impact on accuracy of the class of extended storage functions described in Subections V-C and VI-C. For this analysis, we compute the minimum L_2-gain bounds for several PDE's using both the primal and dual LPI conditions in Thm. 11 and the extended primal and dual LPI conditions in Thm. 10. The PDE included in this test are: heat eqn. with Dirichlet BC's (A.1); wave equation (A.2); heat eqn. with mixed BC's (A.3); coupled heat/transfer eqn. (A.4); and coupled heat eqns. (A.5).

$$A.1 \quad \dot{\mathbf{v}}(t,s) = \mathbf{v}_{ss}(t,s) + w(t), \quad \mathbf{v}(t,0) = \mathbf{v}(t,1) = 0, \quad z(t) = \int_0^1 \mathbf{v}(t,s)ds$$

$$A.2 \quad \ddot{\mathbf{v}}(t,s) = \mathbf{v}_{ss}(t,s) + w(t), \quad \mathbf{v}(t,0) = \mathbf{v}(t,1) = 0, \quad z(t) = \mathbf{v}_s(t,1).$$

$$A.3 \quad \dot{\mathbf{v}}(t,s) = \mathbf{v}_{ss}(t,s) + w(t), \quad \mathbf{v}(t,0) = \mathbf{v}_s(t,1) = 0, \quad z(t) = \int_0^1 \mathbf{v}(t,s)ds.$$

$$A.4 \quad \dot{\mathbf{v}}_1(t,s) = \partial_s \mathbf{v}_2(t,s), \quad \dot{\mathbf{v}}_2(t,s) = \partial_s \mathbf{v}_1(t,s) + w(t)$$
$$\mathbf{v}_2(t,0) = \mathbf{v}_1(t,1) + k\mathbf{v}_2(t,1) = 0$$
$$z(t) = \int_0^1 \mathbf{v}_2(t,s)ds$$

$$A.5 \quad \dot{\mathbf{v}}(t,s) = \begin{bmatrix} 1 & 1.5 \\ 5 & 0.2 \end{bmatrix} \mathbf{v}(t,s) + \frac{1}{2.6}\mathbf{v}_{ss}(t,s) + sw(t),$$
$$\mathbf{v}(t,0) = \mathbf{v}(t,1) = 0,$$
$$z(t) = \int_0^1 \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{v}(t,s)ds$$

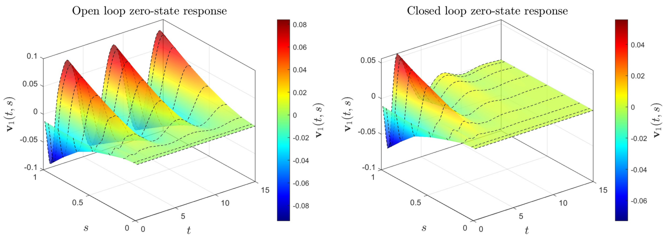
The results are listed in Table II. Accuracy of the L_2-gain bound from Thms. 10 and 11 can be inferred from the gap between primal (Thms. 10a and 11a) and dual LPIs (Thms. 10b and 11b). These results indicate that both the primal and dual LPI's from Thm. 11 are highly accurate. However, it seems that the restriction on the structure of the storage functional in the dual test of Thm. 10 results in conservatism in several cases. Furthermore, in Example A.2, the primal test of Thm. 10 is likewise conservative. These results indicate the importance of using the extended class of storage functionals described in Subections V-C and VI-C.

	Bound on H_∞-norm			
	Thm. 10a	Thm. 10b	Thm. 11a	Thm. 11b
Ex. A.1	0.083	6.89	0.0833	0.0833
Ex. A.2	4.23	8.71	0.5	0.5
Ex. A.3	0.33	3.49	0.33	0.33
Ex. A.4	0.8418	0.8418	0.842	0.8418
Ex. A.5	0.81	3.501	0.81	0.81

TABLE II: Minimum computed bounds on the L_2-gain for PDEs A.1-A.5 using the primal (a) and dual (b) LPIs in Theorems 10 and 11. A gap between primal and dual bounds indicates conservatism in one of the computed bounds and the numerical results illustrate the importance of using extended functionals as described in Subections V-C and VI-C.

D. H_∞-Optimal State-Feedback Control using Cor. 13

We conclude this section by synthesizing optimal state-feedback controllers for three canonical examples of PDE control: 1) the Euler-Bernoulli beam equation with in-domain actuation; 2) a reaction-diffusion PDE with actuation at the boundary; and 3) the wave equation with actuation at the boundary. In each case, motivated by the numerical tests in Subsec. VIII-C, we use the extended class of storage functions for which LPI conditions are given in Cor. 13. Table III summarizes the results for all three examples, providing the minimum achievable closed-loop L_2-gain (H_∞-norm) and computation time as function of the degree ($n = 1, 2, 3, 4$) of the monomial bases as defined in Eq. (12). Computation time is defined as time required to set up the LPI, solve the LPI, and reconstruct the controller gains using a desktop computer



(a) Open loop response (b) Closed loop response

Fig. 2: Numerical simulation of open loop (a) and closed-loop (b) response of $\mathbf{v}_1(t,s)$ for in-domain control of an Euler-Bernoulli beam equation (Ex. 21) with disturbance $w(t) = \sin(3t)e^{-t}$.

with Intel Core *i7-5960X* CPU and 64GB DDR4 RAM.

For each example, we specify the corresponding PIE state and use PIETOOLS to obtain the associated PIE representation. The optimal controller gains (as calculated for order $n = 3$) are obtained using the procedure defined in Sec. VII. For examples 19) and 20), numerical simulation of the closed-loop response is obtained using PIESIM [37] in order to verify that the closed-loop L_2-gain bound is satisfied. Finally, the achievable closed-loop L_2-gains are compared to those achievable using standard LMIs for H_∞-optimal state-feedback as applied to an ODE approximation of the PDE. This ODE was obtained using a simple finite difference approximation scheme where a 2^{nd}-order central difference approximation was used for 2^{nd}-order spatial derivatives and a first-order forward difference was used for time derivatives.

Example 21 (Euler-Bernoulli beam equation)**.** *Recall that in Example 3, we formulated the problem of optimal control of an Euler-Bernoulli (EB) beam model as using the PIE state* $\mathbf{x} = \mathbf{v}_{ss}$*. Solving the LPI in Cor. 13, we find the* H_∞*-optimized state-feedback controller to be*

$$u(t) = \int_0^1 \begin{bmatrix} Q_a(s) & Q_b(s) \end{bmatrix} \mathbf{v}_{ss}(s)ds,$$
$$Q_a(s) = 5.61s^5 - 17.1s^4 + 15.4s^3 - 9.8s^2 + 4.94s - 1.30,$$
$$Q_b(s) = 3.31s^5 - 8.54s^4 + 6.51s^3 - 2.03s^2 + 0.18s + 0.01.$$

The upper bound on the H_∞*-norm of the corresponding closed-loop PDE obtained from the LPI in Cor. 13 is* 0.77*. In Figures 2a, 2b and 3, we plot the system response for a disturbance* $w(t) = \sin(3t)e^{-t}$ *with the zero initial conditions. The* L_2*-gain for this disturbance is* 0.085 *which is less than the predicted worst-case bound of .77. The closed-loop* H_∞*-norm bound using an ODE approximation of the PDE is* 0.1030 *– indicating that the discretized model is not reliable for use in optimal control.*

Example 22 (Reaction-Diffusion Equation)**.** *Consider boundary control of an unstable reaction-diffusion equation:*

$$\dot{\mathbf{v}}(t,s) = 5\mathbf{v}(t,s) + \mathbf{v}_{ss}(t,s) + w(t), \;\; \dot{x}(t) = u(t),$$
$$z(t) = \begin{bmatrix} x(t) \\ \int_0^1 \mathbf{v}(t,s)ds \end{bmatrix}, \;\; \mathbf{v}(t,0) = 0, \;\; \mathbf{v}_s(t,1) = x(t),$$

The corresponding PIE has state $\mathbf{x}(t) = \mathbf{v}_{ss}(t)$ *and parameters*

$$\mathcal{T} = \Pi\left[\begin{array}{c|c} 1 & 0 \\ \hline s & \{0,-\theta,-s\} \end{array}\right], \mathcal{A} = \Pi\left[\begin{array}{c|c} 0 & 0 \\ \hline 5s & \{1,-5\theta,-5s\} \end{array}\right],$$
$$\mathcal{B}_1 = \Pi\left[\begin{array}{c|c} 0 & \emptyset \\ \hline 1 & \{\emptyset\} \end{array}\right], \mathcal{B}_2 = \Pi\left[\begin{array}{c|c} 1 & \emptyset \\ \hline 0 & \{\emptyset\} \end{array}\right],$$
$$\mathcal{C} = \Pi\left[\begin{array}{c|c} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} & \begin{bmatrix} 0 \\ 0.5s^2 - s \end{bmatrix} \\ \hline \emptyset & \{\emptyset\} \end{array}\right].$$

The H_∞*-optimized state-feedback controller is*

$$u(t) = -6.71x(t) + 10^3 \int_0^1 K(s)\mathbf{v}_{ss}(s)ds,$$
$$K(s) = -11.68s^8 + 44.23s^7 - 65.93s^6 + 49.38s^5 - 19.82s^4 + 4.27s^3 - 0.46s^2 + 0.02s - 0.0002.$$

The upper bound on the H_∞*-norm of the corresponding closed-loop PDE obtained from the LPI in Cor. 13 is* 4.99*. The simulated* L_2*-gain under disturbance* $w(t) = \sin(5t)e^{-t}$ *is* 1.8905 *which verifies the bound. The closed-loop* H_∞*-norm bound using an ODE approximation of the PDE is* 3.441*. In Figures 4a, 4b and 5, we plot the system response for a disturbance* $w(t) = \sin(5t)e^{-t}$*.*

Example 23 (Wave equation)**.** *Consider boundary control of a wave equation:*

$$\ddot{\eta}(t,s) = \partial_s^2 \eta(t,s) + w(t), \;\; \dot{x}(t) = u(t)$$

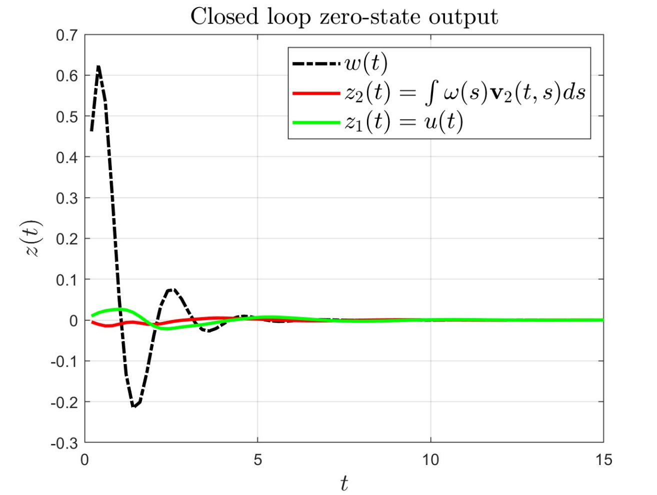
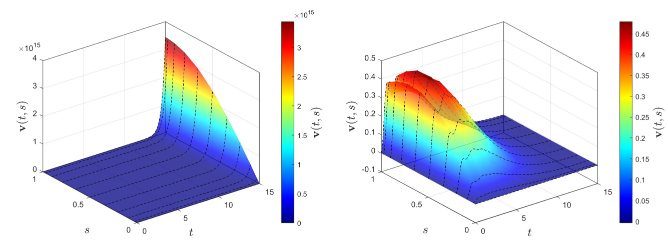


Fig. 3: Closed-loop numerical simulation of regulated outputs $z_1 = x(t)$ and $z_2(t) = \int_0^1 \omega(s)\mathbf{v}_2(t,s)ds$ with disturbance $w(t) = \sin(3t)e^{-t}$ for Ex. 21 where $\omega(s) := (1 - 2s - s^2)/2$.



(a) Open loop response (b) Closed loop response

Fig. 4: Numerical simulation of open loop (a) and closed-loop response (b) for boundary control of the reaction-diffusion equation (Ex. 22) with disturbance $w(t) = \sin(5t)e^{-t}$.

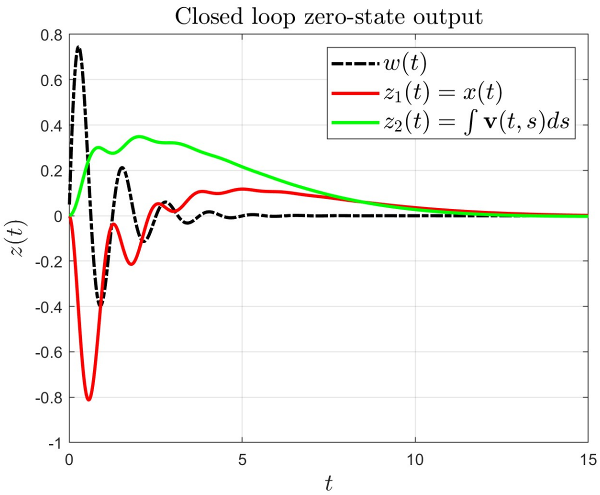


Fig. 5: Closed-loop numerical simulation of regulated outputs $z_1(t) = x(t)$ and $z_2(t) = \int_0^1 \mathbf{v}(t,s)ds$ with disturbance $w(t) = \sin(5t)e^{-t}$ for Ex. 22.

Closed-loop H_∞-norm and (computation time) vs. monomial degree				
Degree, n	1	2	3	4
Ex. 21	3.29 (6.2)	0.89 (15)	0.73 (30.5)	0.66 (67.4)
Ex. 22	7.86 (4.6)	5.11 (5.5)	4.59 (9.5)	4.25 (13.9)
Ex. 23	0.65 (10.9)	0.64 (27.1)	0.64 (49.5)	0.639 (87.5)

TABLE III: Minimum achievable closed-loop bound on H_∞-norm and associated computation time for state-feedback control of PDE examples 21, 22, and 23 in Sec. VIII-D. Closed-loop norm is found by minimizing γ in Cor. 13 wherein variables $\mathcal{P}$ and $\mathcal{Z}$ are parameterized for degree n as $\mathcal{P} = \mathcal{Z}_n^* Q_p \mathcal{Z}_n$, $\mathcal{Z} = Q_z \mathcal{Z}_n$, and $Q_p \geq 0$, Q_z are matrices and where $\mathcal{Z}_n$ is defined in Eq. (12). The values within parentheses correspond to the total CPU runtime, in seconds, for solving the H_∞-optimal state-feedback problem — i.e., time for setting up the LPIs, solving the LPIs, and controller reconstruction.

$$z(t) = \begin{bmatrix} x(t) \\ \int_0^1 \eta(t,s)ds \end{bmatrix}, \; \eta(t,0) = 0, \; \partial_s \eta(t,1) = x(t).$$

To eliminate the second-order time derivative, $\ddot{\eta}$, we define $\mathbf{v} = \begin{bmatrix} \eta & \dot{\eta} \end{bmatrix}^T$ *to obtain*

$$\dot{\mathbf{v}}(t,s) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{v}(t,s) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \partial_s^2 \mathbf{v}(t,s) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t), \; \dot{x}(t) = u(t),$$

$$z(t) = \begin{bmatrix} v(t) \\ \int_0^1 \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{v}(t,s)ds \end{bmatrix}, \; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}(t,0) \\ \mathbf{v}(t,1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t).$$

We convert to a PIE which has state

$$\mathbf{x}(t) = \begin{bmatrix} x(t) & \partial_s^2 \mathbf{v}_1(t) & \mathbf{v}_2(t) \end{bmatrix}^T = \begin{bmatrix} x(t) & \eta_{ss}(t) & \dot{\eta}(t) \end{bmatrix}^T$$

The upper bound on the H_∞-norm of the corresponding closed-loop PDE obtained from the LPI in Cor. 13 is .64. The H_∞-optimized state-feedback controller is

$$u(t) = -0.17x(t) + 10^{-2} \int_0^1 Q_1(s)\eta_{ss}(t,s) + Q_2(s)\dot{\eta}(t,s)ds$$

where

$$Q_1(s) = .5s^8 - 2s^7 + 3s^6 - 2s^5 - 30s^4 + 60s^3 - 70s^2 + 20s - .8,$$
$$Q_2(s) = .2s^8 - .7s^7 + .06s^6 + .6s^5 - 5s^4 - 20s^3 + 80s^2 - 2s - 40.$$

IX. Conclusions

Recent work has shown that a large class of linear PDE systems admit an equivalent state-space representation using Partial Integral Equations (PIEs). However, relatively little is known about the system properties of such PIEs. In this paper, we have presented a series of duality results which establish equivalence in the input-output properties of a PIE and its dual. Specifically, for any given PIE, we have shown how to construct a dual PIE with simple formulae for obtaining the system parameters of this dual. Then, by establishing an intertwining property between the solutions of the dual and primal PIE, we have shown that the primal and dual PIEs have at least three equivalent properties: asymptotic stability, exponential decay rate, and L_2-gain.

A benefit of the PIE representation of PDE systems is the ability to generalize Linear Matrix Inequality (LMI) conditions to Linear PI Inequality (LPI) conditions which may then be solved using convex optimization. Taking this approach, we show that the LPIs for stability, exponential decay rate, and L_2-gain admit equivalent primal and dual formulations. The dual formulations of the stability and L_2-gain LPIs are then used to solve the problem of stabilizing and H_∞-optimal state-feedback controller synthesis. A numerical algorithm is then proposed for reconstructing controller gains and implementation of the controller. Finally, numerical testing is used to verify the theorems and obtain controllers with provable H_∞-norm bounds. The numerical results show no apparent sub-optimality in the resulting controllers or H_∞ bounds.

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Appendix
A

Implementation of operator inversion in PIETOOLS: The inverse $\mathcal{P}^{-1}$ in Lemma 15 (and Corollary 16) is defined in terms of some matrix-valued functions U, V which satisfy a set of Volterra-type integral equations of the 2^{nd} kind. Our approach to constructing U, V is based on the "method of successive approximation" class of algorithms, the convergence of which has been established in Lemma 18. Specifically, we define Algorithm 1 (wherein V is found by solving for its transpose) for computing the solution of U and V at discrete points and then fitting these values to polynomial approximations.

Algorithm 1 Approximating the inverse of $\mathcal{R}\mathbf{x}(s) = \mathbf{x}(s) - \int_a^s F_1(s)G_1(\theta)\mathbf{x}(\theta)d\theta - \int_s^b F_2(s)G_2(\theta)\mathbf{x}(\theta)d\theta$ at

Given: n, ϵ, $[a,b]$, F_i, G_i. Set: $U_0 = V_0 = I, N = 0$.
for $i \in \{0, \cdots, n\}$ **do** $s_i = a + \frac{i(b-a)}{n}$
  $C(s_i) = \begin{bmatrix} F_1(s_i) & F_2(s_i)\end{bmatrix}$
  $B(s_i) = \begin{bmatrix} G_1(s_i) \\ -G_2(s_i)\end{bmatrix}$, $A(s_i) = B(s_i)C(s_i)$
end for
while $\left(\sum_{i=1}^{i=n} \|A(s_i)\|\right)^k \geq \epsilon \cdot k!$ **do** $N = N+1$
  for $i \in \{1, \cdots, n\}$ **do**
    $U_{k+1}(s_i) = \frac{(b-a)}{2n} \sum_{j=1}^{j=i} \begin{bmatrix} A(s_j) & A(s_{j-1})\end{bmatrix} \begin{bmatrix} U_k(s_j) \\ U_k(s_{j-1})\end{bmatrix}$
    $V_{k+1}(s_i) = \frac{(a-b)}{2n} \sum_{j=1}^{j=i} \begin{bmatrix} V_k(s_j) & V_k(s_{j-1})\end{bmatrix} \begin{bmatrix} A(s_j) \\ A(s_{j-1})\end{bmatrix}$
  end for
end while
for $i \in \{0, \cdots, n\}$ **do**
  $U(s_i) = \sum_{i=0}^{N} U_k(s_i) \qquad V(s_i) = \sum_{i=0}^{N} V_k(s_i)$
  $U(s_i) = C(s_i)U(s_i) \qquad V(s_i) = V(s_i)B(s_i)$.
end for
$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22}\end{bmatrix} = U(b) \qquad P = \begin{bmatrix} 0 & 0 \\ U_{22}^{-1}U_{21} & I\end{bmatrix}$
Solve the problem

$$\min_{\alpha,\beta\in\mathbb{R}^{d+1}} \sum_{i=0}^{n} \|U_p(s_i) - U(s_i)\|_2^2 + \|V_p(s_i) - V(s_i)\|_2^2$$
$$s.t. \quad U_p(s) = \alpha\text{col}(1, s, \cdots, s^d),$$
$$V_p(s) = \beta\text{col}(1, s, \cdots, s^d).$$

$M_1(s,t) = U_p(s)(I - P)V_p(t) \qquad M_2(s,t) = -U_p(s)PV_p(t)$ **return** M_1, M_2

Algorithm 1 finds a $\mathbf{\Pi}_2$ approximation of the inverse $\mathit{\Pi}_{\{I,M_1,M_2\}} := \mathit{\Pi}_{\{I,H_1,H_2\}}^{-1}$. The values of the parameters M_1, M_2 are obtained at discerete points $s_i \in [a,b]$ and polynomial approximations of these parameters are found using regression. Given M_1, M_2, Corollary 16 and Lemma 17 are then used to construct the inverse of the full PI operator $\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]$.